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Hardy spaces and holomorphic functions of infinitely many variables

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Abstract: We take a classical result on the interplay between complex and Fourier analysis in one variable (that the space of bounded holomorphic functions on the unit disc and the Hardy space on the torus are isomorphic), and extend it to functions in infinitely many variables. This will present several difficulties that we sort out.

Resumen: Tomamos un resultado clásico sobre la interacción del análisis complejo y de Fourier de una variable (que el espacio de funciones holomorfas y acotadas en el disco y el espacio de Hardy en el toro son isomorfos) y lo extendemos a funciones en infinitas variables. Esto presenta varias dificultades que deberemos sortear.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ its boundary (that we call the *torus*). We denote by $H_\infty(\mathbb{D})$ the space of bounded holomorphic functions on the disc and by $H_\infty(\mathbb{T})$ the Hardy space on the torus (precise definitions are given in section 2). A classical result in analysis states that

$$H_\infty(\mathbb{D}) = H_\infty(\mathbb{T});$$

that is: both spaces are isometrically isomorphic as Banach spaces. Our aim in this note is to present an analogous result for functions of infinitely many variables (or, to be more precise, defined on subsets of infinite dimensional spaces, see theorem 43). This was done for the first time by Cole and Gamelin [4]. We follow here a different approach, based in the one given by Defant et al. [5] using results from Rudin [6]. We do it in several steps. First we are going to analyse the proof of the 1-dimensional case, so that we can transfer it to functions of several complex variables and, finally to infinite dimensional spaces. In order to achieve this goal we have to face several issues: to find proper analogues to $\mathbb{D}$ and $\mathbb{T}$ for several and infinitely many variables, find a good definition of holomorphy in this setting, and to find a device that allows to extend the results from the finite to the infinite dimensional case. We assume some knowledge of the basic concepts of complex, harmonic and functional analysis.

2. The 1-dimensional case

As we explained before, we are going to look at the interplay between complex and harmonic analysis, and all the time we will keep one foot in each side. We begin by defining the spaces we will be dealing with. First of all, the space of holomorphic functions on $\mathbb{D}$ is denoted by $H(\mathbb{D})$. We consider the following subspace.

Definition 1. We define the space $H_\infty(\mathbb{D}) = \{f : \mathbb{D} \longrightarrow \mathbb{C} : f \text{ is bounded and holomorphic}\}$. ◂

Theorem 2. *$H_\infty(\mathbb{D})$ with the norm*

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$$

is a Banach space.

Proof. Let $\{f_n\}_{n=0}^{\infty}$ be a Cauchy sequence in $H_\infty(\mathbb{D})$. The space $C_\infty(\mathbb{D})$ of bounded continuous functions on the disc (that obviously contains $H_\infty(\mathbb{D})$) is Banach (see, e.g., Cerdà's book [2, Chapter 2]). Then, the sequence converges uniformly on $\mathbb{D}$ to some bounded continuous $f : \mathbb{D} \to \mathbb{C}$. But then $\{f_n\}_{n=0}^{\infty}$ converges uniformly on the compact subsets of $\mathbb{D}$ to the function f, and a straightforward application of Morera's theorem (see Stein and Shakarchi's book [8, Theorem 5.2]) shows that f is holomorphic. ■

Remark 3. If U is an open subset of $\mathbb{C}$, then a function $f : U \to \mathbb{C}$ is analytic on U if, for every point $z_0 \in U$, there exist $r > 0$ and a sequence $\{c_n\}_{n=0}^{\infty} \subset \mathbb{C}$ which depend on z_0 such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \text{ for every } z \in (z_0 + r\mathbb{D}) \subset U,$$

where $z_0 + r\mathbb{D} = \{z \in \mathbb{C} : |z - z_0| < r\}$. This is, f admits a power series expansion in a neighbourhood of each point $z_0 \in U$. One of the key results (probably one of the most important ones in complex analysis) is that every holomorphic function is analytic [8, Theorem 4.4]. In our particular case, that is, on the disc $\mathbb{D}$, it is known that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic if and only if there exist coefficients $\{c_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

for every $z \in \mathbb{D}$. This convergence is, moreover, absolute on $\mathbb{D}$ and uniform on $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$ for every $0 < r < 1$. ◂

This is our main object in the side of complex analysis. Let us explore now the side of harmonic analysis. On $\mathbb{T}$ we consider the normalised Lebesgue measure, and the corresponding space $L_1(\mathbb{T})$. This means that, for each f, the integral has to be understood in the following sense:

$$\int_{\mathbb{T}} f(w)\,\mathrm{d}w = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t.$$

If $f \in L_1(\mathbb{T})$, then $w \in \mathbb{T} \mapsto f(w)w^{-n}$ is again in $L_1(\mathbb{T})$ for every $n \in \mathbb{Z}$ (because $|w^{-n}| = 1$). Then, we can define the Fourier coefficients of f in the following way.

Definition 4. Given $f \in L_1(\mathbb{T})$ and $n \in \mathbb{Z}$, the n-th Fourier coefficient is defined as

$$\hat{f}(n) = \int_{\mathbb{T}} f(w)w^{-n}\,\mathrm{d}w. \qquad ◂$$

Let us note that

(1) $$|\hat{f}(n)| \leq \int_{\mathbb{T}} |f(w)w^{-n}|\,\mathrm{d}w = \|f\|_1,$$

and the operator $L_1(\mathbb{T}) \to \mathbb{C}$ defined by $f \mapsto \hat{f}(n)$ is continuous. We can now define the second space we are going to be dealing with.

Definition 5. The Hardy space on the circumference is defined as

$$H_\infty(\mathbb{T}) = \{f \in L_\infty(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0\}. \qquad ◂$$

Theorem 6. *$H_\infty(\mathbb{T})$ is a closed subspace of $L_\infty(\mathbb{T})$, hence Banach.*

Proof. The result follows as a straightforward consequence of the fact that the operator $f \mapsto \hat{f}(n)$ is continuous. ■

Thus, the goal of this section is to prove that

(2) $$H_\infty(\mathbb{T}) = H_\infty(\mathbb{D})$$

as Banach spaces. That is: there is an isometric isomorphism between these two spaces.

Remark 7. Let us give the first step towards the proof. Each $f \in H_\infty(\mathbb{T})$ defines a family of Fourier coefficients $\{\hat{f}(n)\}_{n=0}^\infty$, and we may consider the (in principle only formal) power series given by $\sum_{n=0}^\infty \hat{f}(n)z^n$. Note that (recall (1) and the fact that $\|f\|_1 \leq \|f\|_\infty$)

$$\sum_{n=0}^\infty |\hat{f}(n)|\,|z^n| \leq \|f\|_\infty \sum_{n=0}^\infty |z|^n < \infty \iff |z| < 1.$$

Then, the function $g : \mathbb{D} \to \mathbb{C}$ given by $g(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$ is well defined and, by remark 3, is holomorphic. In other words, the operator $H_\infty(\mathbb{T}) \to H(\mathbb{D})$ given by $f \mapsto g(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$ is well defined. It is an easy exercise to check that it is linear and injective. The main problem now is to show that in fact it takes values in $H_\infty(\mathbb{D})$ (that is, the function g defined in this way is bounded) and is surjective. ◂

Our first concern is to show that the function defined by the power series is indeed bounded on $\mathbb{D}$. To do this, we will reformulate the function in more convenient terms. We bring now our tool for this purpose.

Definition 8. The Poisson kernel $p : \mathbb{D} \times \mathbb{T} \to \mathbb{C}$ is defined as

(3) $$p(z, w) = \sum_{n \in \mathbb{Z}} w^{-n} r^{|n|} u^n,$$

for $z \in \mathbb{D}$ and $w \in \mathbb{T}$, where $z = ru$ with $u = z/|z| \in \mathbb{T}$ and $r = |z|$. ◂

Remark 9. Let us note that, since $|w| = |u| = 1$ and $0 \leq r < 1$, we have

$$\sum_{n\in\mathbb{Z}} |w^{-n} r^{|n|} u^n| \leq \sum_{n\in\mathbb{Z}} r^{|n|} = 1 + 2\sum_{n=1}^{\infty} r^n < \infty\,.$$

Hence, the series in (3) converges (even absolutely) for each fixed $w \in \mathbb{T}$ and $z \in \mathbb{D}$ and p is well defined. Moreover, by the Weierstrass M-test (see Rudin's book [6, Theorem 7.10]), the series converges uniformly on $r\mathbb{D} \times \mathbb{T}$ for every $0 < r < 1$. ◂

Proposition 10. *The following statements hold:*

1. $p(z, w) = \dfrac{|w|^2 - |z|^2}{|w - z|^2} > 0$ *for all* $w \in \mathbb{T}$ *and* $z \in \mathbb{D}$*;*
2. $\displaystyle\int_{\mathbb{T}} p(z, w)\,\mathrm{d}w = 1$ *for every fixed* $z \in \mathbb{D}$.

Proof.

1. Let $w \in \mathbb{T}$ and $z \in \mathbb{D}$. Observe that

$$\begin{aligned} p(z,w) = \sum_{n\in\mathbb{Z}} w^{-n} r^{|n|} u^n &= \sum_{n=1}^{\infty}\left(\frac{wr}{u}\right)^n + \sum_{n=0}^{\infty}\left(\frac{ru}{w}\right)^n \\ &= \frac{wr}{u - wr} + \frac{w}{w - ru} = \frac{wu(1-r^2)}{uw - ru^2 - rw^2 + r^2wu} \\ &= \frac{1-r^2}{1 - r\frac{u}{w} - r\frac{w}{u} + r^2} = \frac{1-r^2}{1 - ru\overline{w} - r\overline{u}w + r^2} \\ &= \frac{|w|^2 - |z|^2}{|w|^2 - z\overline{w} - \overline{z}w + |z|^2} = \frac{|w|^2 - |z|^2}{|w-z|^2} > 0. \end{aligned}$$

2. If we fix $z \in \mathbb{D}$, the series in (3) converges uniformly on $\mathbb{T}$. Then, we may change the sum and the integral as follows:

$$\int_{\mathbb{T}} p(z,w) = \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} w^{-n} r^{|n|} u^n \,\mathrm{d}w = \sum_{n\in\mathbb{Z}} r^{|n|} u^n \int_{\mathbb{T}} w^{-n}\,\mathrm{d}w.$$

   A straightforward computation shows that

$$\int_{\mathbb{T}} w^{-n}\,\mathrm{d}w = \int_0^{2\pi} \mathrm{e}^{-in\pi}\frac{\mathrm{d}w}{2\pi} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases} \tag{4}$$ ■

A direct consequence of proposition 10.1 is that $p(z, w)$ is bounded for each fixed $z \in \mathbb{D}$. Then, for every $f \in L_1(\mathbb{T})$, the function given by $w \mapsto p(z, w)f(w)$ again belongs to $L_1(\mathbb{T})$ and the function $P[f] : \mathbb{D} \to \mathbb{C}$ given by

$$P[f](z) = \int_{\mathbb{T}} p(z,w) f(w)\,\mathrm{d}w$$

is well defined. In fact, since the series defining p is uniformly convergent on $\mathbb{T}$ (for fixed $z \in \mathbb{D}$), then

$$\begin{aligned} \int_{\mathbb{T}} p(z,w) f(w)\,\mathrm{d}w &= \int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} w^{-n} r^{|n|} u^n\right) f(w)\,\mathrm{d}w \\ &= \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{T}} f(w) w^{-n}\,\mathrm{d}w\right) r^{|n|} u^n \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} u^n \end{aligned}$$

(note that this series converges because $|\hat{f}(n)| \leq \|f\|$, $0 \leq r \leq 1$ and $|u| = 1$). In this way, we may define an operator P (that we call *Poisson operator*) acting on $L_1(\mathbb{T})$ by doing $f \mapsto P[f]$.

Remark 11. If $f \in H_\infty(\mathbb{T})$, then $\hat{f}(n) = 0$ for $n < 0$ and, for $z = ru \in \mathbb{D}$, we have

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} u^n = \sum_{n=0}^{\infty} \hat{f}(n) r^{|n|} u^n = \sum_{n=0}^{\infty} \hat{f}(n) r^n u^n = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

Then, P (restricted to $H_\infty(\mathbb{T})$) is exactly the operator that we already considered in remark 7. We have

$$|P[f](z)| \leq \int_{\mathbb{T}} |p(z,w)||f(w)|\,\mathrm{d}w \leq \|f\|_\infty \int_{\mathbb{T}} |p(z,w)|\,\mathrm{d}w = \|f\|_\infty$$

and, hence,

$$\sup_{z\in\mathbb{D}} |P[f](z)| \leq \|f\|_\infty. \tag{5}$$

This shows that $P[f]$ is bounded or, in other words, $P\colon H_\infty(\mathbb{T}) \to H_\infty(\mathbb{D})$ is well defined and continuous. ◂

Roughly speaking, what the operator P does is to "extend" functions on $\mathbb{T}$ to $\mathbb{D}$. Let us see how this operator acts on some particularly nice functions.

Remark 12. We begin by considering *trigonometric polynomials*. There are functions $Q\colon \mathbb{T} \to \mathbb{C}$ that can be written as

$$Q(w) = \sum_{n=N}^{M} c_n w^n,$$

where $a_n \in \mathbb{C}$, $N < M \in \mathbb{Z}$. First of all, for each $n \in \mathbb{Z}$, taking (4) we have

$$\hat{Q}(n) = \int_{\mathbb{T}} \sum_{k=N}^{M} c_k w^k w^{-n}\,\mathrm{d}w = \sum_{k=N}^{M} c_k \int_{\mathbb{T}} w^{k-n}\,\mathrm{d}w = \begin{cases} c_n & \text{if } N \leq n \leq M, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$P[Q](z) = \sum_{n\in\mathbb{Z}} \hat{Q}(n) r^{|n|} u^n = \sum_{n=N}^{M} c_n r^{|n|} u^n.$$

This function is continuous on all $\overline{\mathbb{D}}$ and coincides with Q on $\mathbb{T}$. ◂

This, in fact, also happens to every continuous function.

Proposition 13. *Let $f \in C(\mathbb{T})$. Then, $P[f]$ extends to a continuous function on $\overline{\mathbb{D}}$ which is equal to f on $\mathbb{T}$.*

Proof. By the Stone-Weierstrass theorem [6, Chapter 7], there is a sequence of trigonometric polynomials $\{Q_n\}_{n=0}^\infty$ converging to f with the supremum norm $\|\cdot\|_\infty$. As we already observed in remark 12, each $P[Q_n]$ is continuous on $\overline{\mathbb{D}}$. On the other hand, (5) gives

$$\|P[Q_n] - P[Q_m]\|_\infty = \|P[Q_n - Q_m]\|_\infty \leq \|Q_n - Q_m\|_\infty$$

for every n and m. This implies that $\{P[Q_n]\}_{n=0}^\infty$ is a Cauchy sequence in $C(\overline{\mathbb{D}})$ and, hence, converges uniformly to some continuous function F on $\overline{\mathbb{D}}$. For each $w \in \mathbb{T}$ we have

$$F(w) = \lim_{n\to\infty} P[Q_n](w) = \lim_{n\to\infty} Q_n(w) = f(w).$$ ■

One would expect that, whenever a function is defined on $\overline{\mathbb{D}}$ and is restricted to $\mathbb{T}$, then "extending" it to $\mathbb{D}$ with P would give us the original function. We have that, at least if the function is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, then this is the case. This shows that, when restricted to this space, the operator P is surjective.

Proposition 14. *Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ be holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Let $f_{|\mathbb{T}}$ denote the restriction of f to $\mathbb{T}$. Then, $P[f_{|\mathbb{T}}] = f$ on $\mathbb{D}$.*

The last tools we need to prove (2) are some basic concepts of functional analysis related with the weak topologies. We just recall here what we are going to need later. For a deeper study, the reader is referred to Brezis's book [1]. Given a Banach space E, we denote its topological dual by E^*, and given $x \in E$ and $x \in E^*$, we write $\langle x, x^* \rangle := x^*(x)$. For each $x^* \in E^*$, we consider the function $\varphi_{x^*} : E \to \mathbb{C}$, defined by $\varphi_{x^*}(x) = \langle x, x^* \rangle$. Then, the weak topology $\sigma(E, E^*)$ on E is the finest topology that makes all the maps $(\varphi_{x^*})_{x^* \in E^*}$ continuous. A sequence $\{x_n\}_{n=0}^{\infty}$ in E converges to x in the weak topology if and only if $\{\langle x_n, x^* \rangle\}_{n=0}^{\infty}$ converges to $\langle x, x^* \rangle$ for all $x^* \in E^*$. Similarly, for each $x \in E$ we may consider the function $\psi_x : E^* \to \mathbb{C}$ defined by $\varphi_x(x^*) = \langle x, x^* \rangle$, and the weak-star (or weak*) topology $\sigma(E^*, E)$ is defined as the finest topology on E^* making all the maps $(\psi_x)_{x \in E}$ continuous. A sequence $\{x_n^*\}_{n=0}^{\infty}$ in E^* converges to x^* in the weak* topology if and only if $\{\langle x, x_n^* \rangle\}_{n=0}^{\infty}$ converges to $\langle x, x^* \rangle$ for all $x \in E$.

Remark 15. A key fact when dealing with weak topologies is the Banach-Alaoglu theorem [1, Theorem 3.16], by which the closed ball $B_{E^*} = \{f \in E^* : \|f\| \leq 1\}$ is compact in the weak* topology.

Let us also recall that $(L_1(\mathbb{T}))^* = L_\infty(\mathbb{T})$, and the duality is given by

$$\langle f, g \rangle = \int_{\mathbb{T}} f(w) g(w) \, \mathrm{d}w$$

for $f \in L_1(\mathbb{T})$ and $g \in L_\infty(\mathbb{T})$. An important fact for us is that, since $L_1(\mathbb{T})$ is separable, the closed unit ball of $L_\infty(\mathbb{T})$ is metrizable in the $\sigma(L_\infty, L_1)$-topology [1, Theorem 3.28]. These two facts imply that every bounded sequence in $L_\infty(\mathbb{T})$ has a subsequence that converges in the $\sigma(L_\infty, L_1)$-topology. ◂

We are finally ready to state and prove the main result of this section. Given a function $g \in H_\infty(\mathbb{D})$, we will denote by $c_n(g)$ the n-th coefficient of its power series expansion centered on 0.

Theorem 16. *The Poisson operator $P : H_\infty(\mathbb{T}) \to H_\infty(\mathbb{D})$ defined as $f \mapsto P[f]$ is an isometric isomorphism so that $c_n(P[f]) = \hat{f}(n)$ for all $n \in \mathbb{N}_0$.*

Proof. From remarks 7 and 11 we already know that it is well defined, continuous and injective. It is only left, then, to see that it is onto. Let $g \in H_\infty(\mathbb{D})$, and consider its power series expansion (recall remark 3)

$$g(z) = \sum_{n=0}^{\infty} c_n(g) z^n, \tag{6}$$

which converges absolutely and uniformly on $r\mathbb{D}$ for every $0 < r < 1$. Now, for each $n \in \mathbb{N}$ we consider the function $f_n : \mathbb{T} \to \mathbb{C}$ given by $f_n(w) = g\left((1 - 1/n)w\right)$. Note that

$$\|f_n\|_\infty = \sup_{w \in \mathbb{T}} |f_n(w)| = \sup_{w \in \mathbb{T}} |g\left((1 - 1/n)w\right)| \leq \sup_{z \in \mathbb{D}} |g(z)| = \|g\|_\infty.$$

Then, $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $L_\infty(\mathbb{T})$ that, in view of remark 15, has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges to some $f \in L_\infty(\mathbb{T})$. Notice that $\|f\|_\infty \leq \|g\|_\infty$. Our aim now is to see that, in fact, $P[f] = g$. First, if $n \in \mathbb{Z}$, the weak* convergence implies

$$\hat{f}(n) = \int_{\mathbb{T}} f(w) w^{-n} \, \mathrm{d}w = \langle f, w^{-n} \rangle = \lim_{k \to \infty} \langle f_{n_k}, w^n \rangle = \lim_{k \to \infty} \widehat{f_{n_k}}(n).$$

But, since the series in (6) converges uniformly on $\mathbb{T}$, we have (recall again (4))

$$\begin{aligned} \widehat{f_{n_k}}(n) &= \int_{\mathbb{T}} f_{n_k}(w) w^{-n} \, \mathrm{d}w = \int_{\mathbb{T}} \sum_{m=0}^{\infty} c_m(g) \left(1 - \tfrac{1}{n_k}\right)^m w^m w^{-n} \, \mathrm{d}w \\ &= \sum_{m=0}^{\infty} c_m(g) \left(1 - \tfrac{1}{n_k}\right)^m \int_{\mathbb{T}} w^m w^{-n} \, \mathrm{d}w = \begin{cases} \left(1 - \tfrac{1}{n_k}\right)^n c_n(g) & \text{if } n \geq 0, \\ 0 & \text{if } 0 > n. \end{cases} \end{aligned}$$

Hence,

$$\hat{f}(n) = \begin{cases} c_n(g) & \text{if } n \geq 0, \\ 0 & \text{if } 0 > n, \end{cases}$$

thus $f \in H_\infty(\mathbb{T})$. Moreover, for $z \in \mathbb{D}$,

$$P[f](z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n = \sum_{n=0}^{\infty} c_n(g) z^n = g(z),$$

and by remark 11 we have $\|g\|_\infty \leq \|f\|_\infty$. ∎

We have the result for functions of one variable. Our aim is to extend this to an analogous result in infinitely many variables. As an intermediate step we have to look at it for functions of several (but finitely many) variables. We do this in the following section.

3. The N-dimensional case

We want to reproduce in this section the program that we presented in section 2. As there, we have to keep a foot in the world of holomorphic functions and another foot in the world of Fourier analysis. But before we proceed we have to define the concepts and spaces that we are going to work with.

Definition 17. Let U be open in $\mathbb{C}^N$. A function $f : U \to \mathbb{C}$ is holomorphic if for every $z \in U$ there exists a unique $\nabla f(z) \in \mathbb{C}^N$ so that

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - \langle \nabla f(z), h \rangle}{\|h\|} = 0,$$

where $\langle x, y \rangle = \sum_{i=1}^{N} x_i y_i$ for $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in \mathbb{C}^N$, and $h \to 0$ with the usual topology on $\mathbb{C}^N$. ◂

Remark 18. Given a holomorphic function $f : U \subset \mathbb{C}^N \to \mathbb{C}$ and fixed $N-1$ coordinates, that is, $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_N \in \mathbb{C}$, then the restricted function $g(z) = f(a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_N)$ is holomorphic in its domain of definition and $g'(z) = \big(\nabla f(a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_N)\big)_j$. Because then

$$\lim_{h \to 0} \frac{g(z+h) - g(z) - g'(z)}{|h|} = \lim_{h \to 0} \frac{f(\overline{z} + \overline{h}) - f(\overline{z}) - \langle \nabla f(\overline{z}), \overline{h} \rangle}{\|\overline{h}\|},$$

where $\overline{z} = (a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_N)$ and $\overline{h} = (0, \dots, 0, h, 0, \dots, 0)$ with h in the j-th coordinate, and $\langle \nabla f(\overline{z}), \overline{h} \rangle$ as in the previous definition (not a scalar product). This limit equals 0 since f is holomorphic. ◂

A key fact of the bounded holomorphic functions space is the following version of the Weierstrass theorem [5, Theorem 2.4].

Theorem 19. *Let $(f_n)_n$ be a sequence of holomorphic functions on $r\mathbb{D}^N$ that converges uniformly on all compact subsets of $r\mathbb{D}^N$ to some $f : r\mathbb{D}^N \to \mathbb{C}$. Then, f is holomorphic.*

Following exactly the same proof as in the one-variable case (theorem 2) we have that it is a Banach space.

Theorem 20. *$H_\infty(\mathbb{D}^N) = \{f : \mathbb{D}^N \to \mathbb{C} : f$ is bounded and holomorphic$\}$ is a Banach space with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}^N} |f(z)|$.*

In the one variable case we had that every holormorphic function is also analytic. This is also true for finitely many variables, as we shall see in theorem 23. But before we get into that we have to make clear what it means that a series converges in this context.

Remark 21. If $(c_i)_{i \in I}$ is a family of scalars with I being a countable family of indexes, we say that $(c_i)_{i \in I}$ is summable if there exists $s \in \mathbb{C}$ (which we call the sum of $(c_i)_{i \in I}$ and write $s = \sum_{i \in I} c_i$) such that for all $\varepsilon > 0$ there exists a finite set $F_0 \subseteq I$ such that $|s - \sum_{i \in F} c_i| < \varepsilon$ for all finite sets $F_0 \subseteq F \subseteq I$. This is equivalent to the following three statements:

1. (Cauchy's criterion) for all $\varepsilon > 0$ there exists a finite set $F_0 \subseteq I$ such that $|\sum_{i \in F} c_i| < \varepsilon$ for all finite sets $F \subseteq I \setminus F_0$;

2. (absolute summability) $(|c_i|)_{i\in I}$ is summable, which equivalently means that $\sup_{F\subseteq I \text{ finite}} \sum_{i\in F} |c_i| < \infty$; in this case $\sum_{i\in I} |c_i| = \sup_{F\subseteq I \text{ finite}} \sum_{i\in F} |c_i|$;
3. (unconditional summability) for every bijection $\sigma : \mathbb{N} \to I$, $\sum_{n=1}^{\infty} c_{\sigma(n)}$ converges; in this case $\sum_{i\in I} c_i = \sum_{n=1}^{\infty} c_{\sigma(n)}$. ◂

Let us fix some notation. A *multi-index* is an N-tuple $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$. Given such a multi-index and some $z = (z_1, \dots, z_N) \in \mathbb{C}^N$, we denote

$$z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N} .$$

A *monomial* is any mapping of the form $z \mapsto z^\alpha$.

Example 22. The first example of power series in N variables is

$$\sum_{\alpha\in\mathbb{N}_0^N} z^\alpha ,$$

that converges if and only if $|z_j| < 1$ for every $j = 1, \dots, N$ and, in this case,

$$\sum_{\alpha\in\mathbb{N}_0^N} z^\alpha = \prod_{j=1}^{N} \frac{1}{1-z_j}.$$

This follows immediately from the fact that every finite subset of $\mathbb{N}_0^N$ is contained in $\{0, 1, \dots, M\}^N$ for some M, that

$$\sum_{\alpha\in\{0,1,\dots,M\}^N} |z|^\alpha = \Big(\sum_{k=0}^{M} |z_1|^k\Big) \cdots \Big(\sum_{k=0}^{M} |z_N|^k\Big)$$

and the formula of the geometric series. ◂

Theorem 23. *Let $f : \mathbb{D}^N \to \mathbb{C}$. The following two statements are equivalent:*

1. *f is holomorphic;*
2. *there exist coefficients $\big(c_\alpha(f)\big)_{\alpha\in\mathbb{N}_0^N} \subset \mathbb{C}$ so that*

$$f(z) = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f) z^\alpha,$$

for every $z \in \mathbb{D}^N$.

Moreover, in this case, the convergence is absolute and uniform on each compact set of $\mathbb{D}^N$, and the coefficients are unique and can be computed as

$$c_\alpha(f) = \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \dots \int_{|\zeta_N|=\rho_N} \frac{f(\zeta_1, \dots, \zeta_N)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1$$

for any $0 < \rho_i < 1$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$.

Proof. Suppose first that f is holomorphic. Let $z = (z_1, \dots, z_N) \in \mathbb{D}^N$ and choose $= (\rho_1, \dots, \rho_N)$ such that $|z_j| < \rho_j < 1$ for all $1 \leq j \leq N$. Then, applying N times the Cauchy integral formula for one one variable, we obtain

$$f(z) = \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \dots \int_{|\zeta_N|=\rho_N} \frac{f(\zeta_1, \dots, \zeta_N)}{(\zeta_1 - z_1) \dots (\zeta_N - z_N)} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1.$$

But $1/(\zeta_j - z_j) = \sum_{k_j=0}^{\infty} z_j^{k_j} / \zeta_j^{k_j+1}$, so this can be rewritten as

$$f(z) = \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \dots \int_{|\zeta_N|=\rho_N} \Big(\sum_{k_1=0}^{\infty} \frac{z_1^{k_1}}{\zeta_1^{k_1+1}}\Big) \dots \Big(\sum_{k_N=0}^{\infty} \frac{z_N^{k_N}}{\zeta_N^{k_N+1}}\Big) f(\zeta_1, \dots, \zeta_N) \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1.$$

Since these series converge uniformly on compact subsets, they commute with the integration and, therefore,

$$f(z)=\sum_{k_1=0}^{\infty}\dots\sum_{k_N=0}^{\infty}\left(\frac{1}{(2\pi \mathrm{i})^N}\int_{|\zeta_1|=\rho_1}\dots\int_{|\zeta_N|=\rho_N}\frac{f(\zeta_1,\dots,\zeta_N)}{\zeta_1^{k_1+1}\dots\zeta_N^{k_N+1}}\,\mathrm{d}\zeta_N\cdots\mathrm{d}\zeta_1\right)z_1^{k_1}\dots z_N^{k_N}.$$

So for each $\alpha=(k_1,\dots,k_N)$ we can define

(7)
$$c_\alpha(f)=\frac{1}{(2\pi \mathrm{i})^N}\int_{|\zeta_1|=\rho_1}\dots\int_{|\zeta_N|=\rho_N}\frac{f(\zeta_1,\dots,\zeta_N)}{\zeta_1^{k_1+1}\dots\zeta_N^{k_N+1}}\,\mathrm{d}\zeta_N\cdots\mathrm{d}\zeta_1$$

and notice that it does not depend on the choice of . With this, we get that f is analytic as

$$f(z)=\sum_{\alpha\in\mathbb{N}_0^N}c_\alpha(f)z^\alpha\quad\text{for all } z\in\mathbb{D}^N$$

and the convergence is absolute (recall remark 21). Let us see that the series converges uniformly in every K, compact subset of $\mathbb{D}^N$. Since $\mathbb{D}^N$ is open, there exists $\epsilon>0$ such that $(1+\epsilon)K\subset\mathbb{D}$. Now, given $z=(z_1,\dots,z_N)\in K$, define $s(z)=\big((1+\epsilon)|z_1|,\dots,(1+\epsilon)|z_N|\big)\in[0,1)^N$, so $z\in s(z)\mathbb{D}^N$ and, therefore, $\{s(z)\mathbb{D}^N:\ z\in K\}$ is an open cover of K. Hence, there exist $z_1,\dots,z_n\in K$ such that $K\subset\cup_{j=1}^n s(z_j)\mathbb{D}^N$. Then it is enough to check the uniform convergence on subsets of the form $s\mathbb{D}^N$ with $s\in(0,1)^N$. To see this we choose $=(\rho_1,\dots,\rho_N)$ such that $s_j<\rho_j<1$ for all j and use (7) to get

$$|c_\alpha(f)|s^\alpha\le\frac{s^\alpha}{\rho^\alpha}\|f\|_\infty.$$

Since $\sum_{\alpha\in\mathbb{N}_0^N}s^\alpha/\rho^\alpha$ converges (recall example 22), the Weierstrass M-test [6, Theorem 7.10] implies that $\sum_{\alpha\in\mathbb{N}_0^N}c_\alpha(f)z^\alpha$ converges uniformly in $\overline{s\mathbb{D}^N}$. Let us assume now that $f(z)=\sum_{\alpha\in\mathbb{N}_0^N}b_\alpha z^\alpha$ (pointwise) for every $z\in\mathbb{D}^N$. Pick some $0<r<1$ and note that $|b_\alpha z^\alpha|<|b_\alpha r^{\alpha_1}\cdots r^{\alpha_N}|$ for every $z\in r\mathbb{D}^N$. But the series $\sum_\alpha|b_\alpha r^{\alpha_1}\cdots r^{\alpha_N}|$ converges (by assumption) and, by the Weierstrass M-test, the series $\sum_\alpha b_\alpha z^\alpha$ converges uniformly on $s\mathbb{D}^N$ for every $0<s<r$. In particular, the polynomials given by

$$\sum_{\alpha\in\{0,1,\dots,k\}^N}b_\alpha z^\alpha$$

converge (as $k\to\infty$) uniformly to f on $s\mathbb{D}^N$ for every $0<s<r$. By theorem 19 we have that f is holomorphic and bounded on $s\mathbb{D}^N$ and, since s is arbitrary, f is holomorphic in $\mathbb{D}^N$. In particular,

$$\begin{aligned}c_\alpha(f)=&\frac{1}{(2\pi \mathrm{i})^N}\int_{|\zeta_1|=\rho_1}\dots\int_{|\zeta_N|=\rho_N}\frac{f(\zeta_1,\dots,\zeta_N)}{\zeta_1^{\alpha_1+1}\dots\zeta_N^{\alpha_N+1}}\,\mathrm{d}\zeta_N\cdots\mathrm{d}\zeta_1=\\ =&\sum_{\beta\in\mathbb{N}_0^N}b_\beta\frac{1}{(2\pi \mathrm{i})^N}\int_{|\zeta_1|=\rho_1}\dots\int_{|\zeta_N|=\rho_N}\frac{\zeta^\beta}{\zeta_1^{\alpha_1+1}\dots\zeta_N^{\alpha_N+1}}\,\mathrm{d}\zeta_N\cdots\mathrm{d}\zeta_1=b_\alpha.\end{aligned}$$ ■

Now we move to the Fourier side. Analogously to the one dimensional case, we have to define the Fourier coefficients and then the Hardy space which we are going to work with.

Definition 24. Given $f\in L_1(\mathbb{T}^N)$, for each $\alpha\in\mathbb{Z}^N$ the α-th Fourier coefficient is defined as

$$\hat f(\alpha)=\int_{\mathbb{T}^N}f(w)w^{-\alpha}\,\mathrm{d}w.$$ ◂

Just as in (1), we have $|\hat f(\alpha)|\le\|f\|_1$ and the operator $L_1(\mathbb{T}^N)\to\mathbb{C}$ given by $f\mapsto\hat f(\alpha)$ is continuous. This immediately gives the following.

Theorem 25. *The Hardy space*

$$H_\infty(\mathbb{T}^N)=\{f\in L_\infty(\mathbb{T}^N):\ \hat f(\alpha)=0\ \textit{for}\ \alpha\notin\mathbb{N}_0^N\}$$

is a closed subspace of $L_\infty(\mathbb{T}^N)$ *and, therefore, it is a Banach space.*

The goal now is to show that there exists an isometric isomorphism between these two spaces, the space of holomorphic functions and the Hardy space. For that purpose we define the analogous tool to the one we used in the one dimensional case:

Definition 26. The N-dimensional Poisson kernel $p_N : \mathbb{D}^N \times \mathbb{T}^N \to \mathbb{C}$ is defined as

$$p_N(z, w) = \prod_{j=1}^{N} p(z_j, w_j)$$

for $z \in \mathbb{D}^N$ and $w \in \mathbb{T}^N$, where $u \in \mathbb{T}^N$ is given by $u_j = z_j/|z_j|$ and $r = (r_1, \dots, r_N)$ by $r_j = |z_j|$, and we write $r^{|\alpha|} = r_1^{|\alpha_1|} \cdots r_N^{|\alpha_N|}$ and $z = ru$. ◂

The absolute convergence of the series defining $p(z, w)$ in (4) gives

$$p_N(z, w) = \sum_{\alpha \in \mathbb{Z}^N} w^{-\alpha} r^{|\alpha|} u^{\alpha},$$

for every $z \in \mathbb{D}^N$ and $w \in \mathbb{T}^N$. Also, the series converges uniformly on $r\mathbb{D}^N$ for every $0 < r < 1$.

Proposition 27. *The following statements hold:*

1. $p_N(z, w) > 0$ *for every* $w \in \mathbb{T}^N$ *and* $z \in \mathbb{D}^N$*;*
2. $\displaystyle\int_{\mathbb{T}^N} p_N(z, w) dw = 1$ *for every fixed* $z \in \mathbb{D}^N$.

Proof. Both statements follow from the one-dimensional case (proposition 10) and Fubini's theorem. ∎

Definition 28. Given $f \in L_1(\mathbb{T}^N)$, we define the N-dimensional Poisson operator for f by the function $P_N[f] : \mathbb{D}^N \to \mathbb{C}$ given by

$$P_N[f](z) = \int_{\mathbb{T}^N} p_N(z, w) f(w)\, dw = \sum_{\alpha \in \mathbb{Z}^N} \hat{f}(\alpha) r^{|\alpha|} u^{\alpha}.$$ ◂

$P_N[f]$ is well defined for every $f \in L_1(\mathbb{T}^N)$ since $|\hat{f}(\alpha)| \le \|f\|_1$, $r \in [0, 1]^N$ and $u \in \mathbb{T}^N$.

Theorem 29. *For each* $N \in \mathbb{N}$*, the* N*-dimensional Poisson operator*

$$P_N : H_\infty(\mathbb{T}^N) \to H_\infty(\mathbb{D}^N)$$

is an isometric isomorphism such that $c_\alpha(P_N[f]) = \hat{f}(\alpha)$ *for all* $\alpha \in \mathbb{N}_0^N$.

Proof. On the one hand, observe that, for $f \in H_\infty(\mathbb{T}^N)$,

$$P_N[f](z) = \sum_{\alpha \in \mathbb{N}_0^N} \hat{f}(\alpha) z^{\alpha} \tag{8}$$

for every $z \in \mathbb{D}^N$ and (recall theorem 23) $P_N[f]$ is holomorphic on $\mathbb{D}^N$. Moreover, using the properties of the N-dimensional Poisson kernel (proposition 27), we have that

$$\|P_N[f]\|_\infty \le \sup_{z \in \mathbb{D}^N} \int_{\mathbb{T}^N} |p_N(z, w) f(w)|\, dw \le \|f\|_\infty \sup_{z \in \mathbb{D}^N} \int_{\mathbb{T}^N} p_N(z, w)\, dw = \|f\|_\infty.$$

Therefore, $P_N[f] \in H_\infty(\mathbb{D}^N)$ and the operator P_N is well defined and continuous such that $\hat{f}(\alpha) = c_\alpha(P_N[f])$ for every $f \in H_\infty(\mathbb{T}^N)$ and $\alpha \in \mathbb{N}_0^N$. The uniqueness of the coefficients shows that the operator defined is injective. It only remains to see that P_N is surjective. We take some $g \in H_\infty(\mathbb{D}^N)$ and, for each $n \in \mathbb{N}$, consider the function defined by $f_n(u) = g((1 - 1/n)u)$ for every $u \in \mathbb{T}^N$, which is in $H_\infty(\mathbb{T}^N)$ and has Fourier coefficients $\hat{f}_n(\alpha) = c_\alpha(g)(1 - 1/n)^{|\alpha|}$. Indeed, since the monomial series expansion of g converges uniformly on $r\mathbb{T}^N$ for every $0 < r < 1$, we have that

$$\hat{f}_n(\alpha) = \int_{\mathbb{T}^N} f_n(w) w^{-\alpha}\, dw = \begin{cases} c_\alpha(g)(1 - 1/n)^{|\alpha|} & \text{for } \alpha \in \mathbb{N}_0^N, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\|f_n\|_\infty \leq \|g\|_\infty$ for every $n \in \mathbb{N}$. With exactly the same argument as in remark 15 we can find a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which $\sigma(L_\infty, L_1)$-converges to some $f \in B_{L_\infty(\mathbb{T}^N)}(0, \|g\|_\infty)$. As a consequence of the weak convergence,

$$\hat{f}(\alpha) = \int_{\mathbb{T}^N} f(w)w^{-\alpha}\,\mathrm{d}w = \lim_{k\to\infty} \int \mathbb{T}^N f_{n_k}(w)w^{-\alpha}\,\mathrm{d}w = \begin{cases} c_\alpha(g) & \text{for } \alpha \in \mathbb{N}_0^N, \\ 0 & \text{otherwise.} \end{cases}$$

This implies $f \in H_\infty(\mathbb{T}^N)$. Moreover, by (8) we get $P_N[f](z) = g$, and then $\|f\|_\infty = \|g\|_\infty$. ∎

4. The infinite-dimensional case

We jump now from finitely to infinitely many variables. To do so, we will restrict the problem to finite variables, we will apply the finite-dimensional theorem and, using some powerful tools, we will go back to infinitely many variables. We have, then, to face two problems:

- to define a proper setting for our problem in the infinite dimensional setting,
- to find tools that allow us to jump from the finite to the infinite dimensional case.

We begin by tackling the first issue: to translate our problem to the setting of infinite dimensions. We start by defining the main components of our result. Firstly, we need to find a proper substitute for $\mathbb{D}^N$. Then, we need to understand the concept of holomophic function in infinite dimensions. Finally, we define the Fourier coefficients for infinitely many variables.

As substitute for $\mathbb{D}^N$ we could think of the unit ball of the Banach space ℓ_∞ (the space of bounded sequences with the supremum norm $\|\cdot\|$). However, this candidate presents some problems, since the space c_{00} (of sequences with only finitely many non-zero elements) is not dense in $(\ell_\infty, \|\cdot\|)$. So, we may choose a "smaller" space: this is going to be the Banach space $c_0 = \{\{z_n\}_{n=1}^\infty \subset \mathbb{C} : \lim_{n\to\infty} z_n = 0\}$, and we will consider its unit ball B_{c_0} as analogous to $\mathbb{D}^N$. Recall that the dual space of c_0 is the space $(c_0)^* = \ell_1 = \{\{z_n\}_{n=1}^\infty \subset \mathbb{C} : \sum_{n=1}^\infty |z_n| < \infty\}$.

The analogue to $\mathbb{T}^N$ will be the infinite dimensional torus $\mathbb{T}^\infty = \{\{w_n\}_{n=1}^\infty : w_n \in \mathbb{T} \text{ for each } n \in \mathbb{N}\}$, which is a compact space by Tychonoff's theorem. Also, since $\mathbb{T}^\infty$ is a grup with the product coordinate to coordinate, we are able to work with the Haar measure (see Cohn's book [3, Chapter 9]).

The definition of holomorphic functions on B_{c_0} is a particular case of the Fréchet differentiability, which is valid for any normed space and any open subset.

Definition 30. Let U be an open subset of a normed space X. A function $f: U \to \mathbb{C}$ is said to be holomorphic if it is Fréchet differentiable at every $x \in U$, that is, if there exists a continuous linear functional $x^* \in X^*$ such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - x^*(h)}{\|h\|} = 0.$$

In that case we denote the unique x^* by $\mathrm{d}f(x)$, and call it the differential of f at x. ◂

Remark 31. The restriction of every holomorphic function to finite dimensional subspaces is again holomorphic. More precisely, given a holomorphic function $f: U \to \mathbb{C}$ and M an N dimensional subspace of X with basis $e_1, \dots, e_N$, then we just take the inclusion $i_M : M \to X$, $e_i \mapsto i_M(e_i) = b_i$ and consider, for each $z_0 \in U \cap M$, the vector $\nabla(f \circ i_M)(z_0) = \big([\mathrm{d}f(z_0)](i_M(e_k))\big)_{k=1}^N = \big([\mathrm{d}f(z_0)](b_k)\big)_{k=1}^N$, which is the differential of $f_{|U\cap M}$. Indeed,

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - \langle\nabla(f\circ i_M)(z_0), h\rangle}{\|h\|} = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - \sum_k [\mathrm{d}f(z_0)](b_k)h_k}{\|h\|} = 0.$$

In particular, if $f: B_{c_0} \to \mathbb{C}$ is a holomorphic function and $N \in \mathbb{N}$, the restriction $f_N : \mathbb{D}^N \to \mathbb{C}$ defined by $f_N(z_1, \dots, z_N) = f(z_1, \dots, z_N, 0, 0, \dots)$ is holomorphic. ◂

Remark 32. Given a holomorphic function $f: \mathbb{D}^N \to \mathbb{C}$, we may see it as a function on B_{c_0} (let us denote it by $\tilde{f}$) just by adding zeros $\tilde{f}(z) = f(z_1, \dots, z_N, 0, \dots)$, which is holomorphic. ◂

Theorem 33. *The space* $H_\infty(B_{c_0}) = \{f : B_{c_0} \to \mathbb{C} : f$ *is holomorphic and bounded*$\}$ *with the norm* $\|f\|_\infty = \sup_{z\in B_{c_0}} |f(z)|$ *is a Banach space.*

This fundamental fact is a consequence of the following simplified Weierstrass type theorem, a proof of which can be found in the book of Defant et al. [5, Theorem 2.13].

Theorem 34. *Let X be a normed space, and (f_n) a bounded sequence in $H_\infty(B_X)$ that converges to $f : B_X \to \mathbb{C}$ uniformly on each compact subset of B_X (i.e., with respect to the compact-open topology). Then, $f \in H_\infty(B_X)$ and $\|f\|_\infty \leq \sup_n \|f_n\|_\infty$.*

Before introducing Taylor coefficients, let us fix some notation that will be used frequently throughout this section. We will write

$$\mathbb{N}_0^{(\mathbb{N})} = \bigcup_{N\in\mathbb{N}} \mathbb{N}_0^N \quad \text{and} \quad \mathbb{Z}^{(\mathbb{N})} = \bigcup_{N\in\mathbb{Z}} \mathbb{Z}^N.$$

When convenient, we will also identify $(\alpha_1, \alpha_2, \dots, \alpha_N)$ with $(\alpha_1, \alpha_2, \dots, \alpha_N, 0, 0, \dots)$. Given $f \in H_\infty(B_{c_0})$ and N, let $f_N : \mathbb{D}^N \to \mathbb{C}$ be its restriction, that is, $f_N(z_1, \dots, z_N) = f(z_1, \dots, z_N, 0, 0, \dots)$. This is a holomorphic function on $\mathbb{D}^N$ (see remark 31) and, by theorem 23, can be expanded as a power series

$$f_N(z) = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f_N) z^\alpha,$$

for every $z \in \mathbb{D}^N$. This, in principle, provides us a set of coefficients $\{c_\alpha(f_N)\}_{\alpha\in\mathbb{N}_0^N}$ for each N. But, as a matter of fact, when we increase the dimension we only add "new" coefficients for the "new" dimensions. Let us be more precise. If $M \geq N$ and $\alpha \in \mathbb{N}_0^N$ (that we identify with $(\alpha_1, \dots, \alpha_N, 0, \dots, 0) \in \mathbb{N}_0^M$ and call again α), then

$$c_\alpha(f_N) = c_\alpha(f_M).$$

Indeed, by theorem 23 we have

$$c_{(\alpha,0)}(f_{N+1}) = \frac{1}{(2\pi \mathrm{i})^{N+1}} \int_{\rho_1\mathbb{T}} \cdots \int_{\rho_{N+1}\mathbb{T}} \frac{f(\zeta_1, \dots, \zeta_{N+1}, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1} \zeta_{N+1}} \,\mathrm{d}\zeta_{N+1} \cdots \mathrm{d}\zeta_1,$$

and using Cauchy's integral formula,

$$\begin{aligned} c_{(\alpha,0)}(f_{N+1}) &= \frac{2\pi \mathrm{i}}{(2\pi \mathrm{i})^{N+1}} \int_{\rho_1\mathbb{T}} \cdots \int_{\rho_{N+1}\mathbb{T}} \frac{f(\zeta_1, \dots, \zeta_N, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \,\mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 \\ &= \frac{1}{(2\pi \mathrm{i})^N} \int_{\rho_1\mathbb{T}} \cdots \int_{\rho_N\mathbb{T}} \frac{f(\zeta_1, \dots, \zeta_N, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \,\mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 = c_\alpha(f_N). \end{aligned}$$

Then, each function $f \in H_\infty(B_{c_0})$ defines a unique family of coefficients $\{c_\alpha(f)\}_{\alpha\mathbb{N}_0^{(\mathbb{N})}}$, that we call *Taylor coefficients*. In other words, each function f defines a **formal** power series

$$f \sim \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha(f) z^\alpha.$$

The problem now is that $f(z)$ may not coincide with $\sum c_\alpha(f) z^\alpha$. Toeplitz [9] gave an example of a function $f \in H_\infty(B_{c_0})$ and a point $z \in B_{c_0}$ for which $\sum c_\alpha(f) z^\alpha$ does not converge. In other words, when we deal with functions of infinitely many variables, *holomorphic* and *analytic* are no longer equivalent. This is a problem for us, since the proof of the isometry between the spaces, both in the case of one and several variables (recall theorems 16 and 29) depends heavily on the fact that a holomorphic function has a representation as a power series.

We now move on to the Fourier part, as we did in the previous sections. Given $f \in L_1(\mathbb{T}^\infty)$ and $\alpha \in \mathbb{Z}^{(\mathbb{N})}$, we define the α-th Fourier coefficient as

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha} \,\mathrm{d}w$$

and the Hardy space

$$H_\infty(\mathbb{T}^\infty) = \{f \in L_\infty(\mathbb{T}^\infty) : \hat{f}(\alpha) = 0 \text{ if } \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})}\}.$$

Once again, we have that $|\hat{f}(\alpha)| \le \|f\|_1$ for every α. We are also going to use the following two facts, the proof of which can be found, for example, in the book of Defant et al. [5, Chapter 5].

Proposition 35. *If $f_1, f_2 \in L_1(\mathbb{T}^\infty)$ are such that $\hat{f}_1(\alpha) = \hat{f}_2(\alpha)$ for every α, then $f_1 = f_2$.*

Definition 36. An **analytic trigonometric polynomial** is a function $Q \in L_1(\mathbb{T}^\infty)$ of the form

$$Q = \sum_{\substack{\alpha \in F \subseteq \mathbb{N}_0^{(\mathbb{N})} \\ F \text{ finite}}} a_\alpha w^\alpha.$$

Proposition 37. *The set of analytic trigonometric polynomials is dense in $L_1(\mathbb{T}^\infty)$.*

With this we have accomplished the first goal that we stated at the very beginning of this section: to define a proper setting on which to formulate our problem. So, our goal now is to show that

$$H_\infty(\mathbb{T}^\infty) = H_\infty(B_{c_0})$$

isometrically as Banach spaces. Since we have lost the equivalence between holomorphy and analiticity, we cannot adapt the proof of the finite-dimensional case, and we have to go a different way. What we are going to do is to go "down" in each side (holomorphic and harmonic) to N variables, apply the result that we already know (theorem 29) and then "climb up" again to the infinite dimensional setting.

$$\begin{tikzcd}
H_\infty(\mathbb{T}^\infty) \arrow[r] \arrow[d, dashed, leftrightarrow] & H_\infty(B_{c_0}) \arrow[d, dashed, leftrightarrow] \\
H_\infty(\mathbb{T}^N) \arrow[r] & H_\infty(\mathbb{D}^N)
\end{tikzcd}$$

So, we now need to find tools that allow us to go "down" and "up" in each side (this was the second goal that we stated at the beginning of the section). We start with the holomorphic part. Given $f \in H_\infty(B_{c_0})$ and $N \in \mathbb{N}$, by remark 31, $f_N : \mathbb{D}^N \to \mathbb{C}$, where $f_N(z_1, \ldots, z_N) = f(z_1, \ldots, z_N, 0, 0, \ldots)$, is a holomorphic function on $\mathbb{D}^N$. This is the way to go from infinite to finite dimensions. We will use the next theorem (taken from the book of Defant et al. [5, Theorem 2.21], sometimes known as "Hilbert's criterion") to go the opposite way (that is, to "jump" from finitely to infinitely many variables). But before we need a tiny observation.

Remark **38.** If $g : \mathbb{D}^N \to \mathbb{C}$ is a holomorphic function with $g(0) = 0$ and $|g(u)| < C$ for every $u \in \mathbb{D}^N$, then

$$|g(u)| \le C \max_{1 \le n \le N} |u_n|, \tag{9}$$

for each such u. Indeed, for $0 \ne u \in \mathbb{D}^N$, define $h : \mathbb{D} \to \mathbb{D}$ by $h(\zeta) = 1/Cg(\zeta \cdot \frac{u}{\max_n |u_n|})$. Then, the classical Schwarz' lemma (see Rudin's book [7, Theorem 12.2]) yields $|h(\zeta)| \le |\zeta|$ for all $\zeta \in \mathbb{D}$, which for $\zeta = \max_n |u_n|$ gives our claim.

Theorem 39. *Let $(c_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \subset \mathbb{C}$ be so that*

$$\sum_{\alpha \in \mathbb{N}_0^N} |c_\alpha z^\alpha| < \infty \quad \text{for every } z \in \mathbb{D}^N \text{ and every } N \in \mathbb{N}, \text{ and} \tag{10}$$

$$\sup_{N \in \mathbb{N}} \sup_{z \in \mathbb{D}^N} \Big| \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha \Big| < \infty. \tag{11}$$

Then, there exists a unique $f \in H_\infty(B_{c_0})$ such that $c_\alpha(f) = c_\alpha$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$. Moreover, $\|f\|_\infty$ equals the supremum in (11).

Proof. For each $N \in \mathbb{N}$ we define the function $f_N : \mathbb{D}^N \to \mathbb{C}$ by

$$f_N(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha.$$

By (10) and (11), this function is in $H_\infty(\mathbb{D}^N)$ (every analytic function on $\mathbb{D}^N$ is holomorphic by theorem 23). Moreover, $|f_N(z)| \leq \eta$ for all $z \in \mathbb{D}^N$ and all N (where η is the supremum in (11)); in other words, $\|f_N\|_\infty \leq \eta$ for every N. We look now at these functions as defined on B_{c_0} (recall remark 32), and our aim is to show that $(f_N)_N$ converges uniformly on every compact subset $K \subset B_{c_0}$. We choose then some compact $K \subset B_{c_0}$ and we want to see that $(f_N)_N$ is uniformly Cauchy on K. We fix $z \in K$ and define for $1 \leq N < M$ the holomorphic function (remember that we look to the functions f_N as defined on B_{c_0})

$$f_{N,M} : \prod_{n=N+1}^{M} \mathbb{D} \to \mathbb{C} \quad \text{by} \quad f_{N,M}(u) = f_N(z_1, \dots, z_N, 0, 0, \dots) - f_M(z_1, \dots, z_N, u, 0, 0, \dots).$$

Then, $f_{N,M}(0) = 0$ and by (11) we have $|f_{N,M}(u)| < 3\eta$ for $u \in \prod_{n=N+1}^{M}$, and hence by (9), for these u,

$$|f_{N,M}(u)| \leq 3\eta \max_{N+1 \leq n \leq M} |u_n|.$$

Now we pick $r \in c_0$ such that $K \subset \{x \in c_0 : |z_j| \leq |r_j| \text{ for all } j \in \mathbb{N}\}$ (take $r_j := \sup_{z \in K} \sup_{k \geq j} |z_k|$) and then, taking $u = (z_{N+1}, \dots, z_M)$,

$$|f_N(z) - f_M(z)| = |f_{N,M}(z_{N+1}, \dots, z_M)| \leq 3\eta \max_{N+1 \leq n \leq M} |r_n|.$$

Using the fact that $r \in c_0$, we obtain that $\{f_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $H_\infty(B_{c_0})$ with respect to the uniform convergence on compact subsets, and then converges to a certain function f that, by theorem 34, belongs to $H_\infty(B_{c_0})$ and satisfies $\|f\|_\infty \leq \eta$. Let us see that $c_\alpha(f) = c_\alpha$ for all α. Take $\alpha \in \mathbb{N}_0^M$ and $0 < r < 1$. Then (note that $c_\alpha(f) = c_\alpha$ for all $N \geq M$), if we take $N \geq M$,

$$\begin{aligned} c_\alpha = c_\alpha(f) = \lim_N c_\alpha(f_N) &= \lim_N \frac{1}{(2\pi \mathrm{i})^M} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_M|=r} \frac{f_N(\zeta_1, \dots, \zeta_M, 0, \dots)}{\zeta_1^{\alpha_1+1} \cdots \zeta_M^{\alpha_M+1}} \, \mathrm{d}\zeta_M \cdots \mathrm{d}\zeta_1 \\ &= \frac{1}{(2\pi \mathrm{i})^M} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_M|=r} \frac{f(\zeta_1, \dots, \zeta_M, 0, \dots)}{\zeta_1^{\alpha_1+1} \cdots \zeta_M^{\alpha_M+1}} \, \mathrm{d}\zeta_M \cdots \mathrm{d}\zeta_1 = c_\alpha(f). \end{aligned}$$

Finally, we have $\eta \leq \|f\|_\infty$ since

$$\eta = \sup_N \sup_{z \in \mathbb{D}^N} \Big| \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha \Big| = \sup_N \sup_{z \in \mathbb{D}^N} |f_N(z)| = \|f_N\|_\infty \leq \|f\|_\infty. \qquad \blacksquare$$

We move now to the side of Fourier analysis. To begin with, we need a way to go from $\mathbb{T}^\infty$ to $\mathbb{T}^N$ in a reasonable way. Given $f \in L_1(\mathbb{T}^\infty)$ and $N \in \mathbb{N}$, we define

$$f_{[N]}(w) = \int_{\mathbb{T}^\infty} f(w, z) \, \mathrm{d}z.$$

Recall that $H_\infty(\mathbb{T}^\infty) \subset L_\infty(\mathbb{T}^\infty) \subset L_1(\mathbb{T}^\infty)$. Let us see that with this definition everything works fine.

Lemma 40. *Given $f \in L_p(\mathbb{T}^\infty)$ with $p = 1, \infty$, and $N \in \mathbb{N}$, we have the following:*

(i) *$f_{[N]} \in L_p(\mathbb{T}^N)$, and $\|f_{[N]}\|_p \leq \|f\|_p$;*

(ii) *$\hat{f}_{[N]}(\alpha) = \hat{f}(\alpha)$ for every $\alpha \in \mathbb{Z}_0^N$;*

(iii) *if $p = 1$, then $f_{[N]} \to f$ in $L_1(\mathbb{T}^\infty)$; if $p = \infty$, then $f_{[N]} \to f$ in the $w(L_\infty, L_1)$-topology.*

Proof. Let us first look at (i). If $p = 1$, using the monotonicity of the integral together with Fubini's theorem we have

$$\begin{aligned} \|f_{[N]}\|_1 = \int_{\mathbb{T}^N} |f_{[N]}(w)| \, \mathrm{d}w &= \int_{\mathbb{T}^N} \Big| \int_{\mathbb{T}^\infty} f(w, u) \, \mathrm{d}u \Big| \, \mathrm{d}w \leq \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}^\infty} |f(w, u)| \, \mathrm{d}u \Big) \mathrm{d}w \\ &= \int_{\mathbb{T}^N \times \mathbb{T}^\infty} |f(w, u)| \, \mathrm{d}(w, u) = \int_{\mathbb{T}^\infty} |f(z)| \, \mathrm{d}z = \|f\|_1. \end{aligned}$$

If $p = \infty$, recall that

$$\begin{aligned}|f_{[N]}(w)| &= \left|\int_{\mathbb{T}^\infty} f(w,u)\,\mathrm{d}u\right| \leq \int_{\mathbb{T}^\infty} |f(w,u)|\,\mathrm{d}u \\ &\leq \int_{\mathbb{T}^\infty} \|f\|_\infty\,\mathrm{d}u \leq \|f\|_\infty \int_{\mathbb{T}^\infty} \mathrm{d}u = \|f\|_\infty, \text{ almost everywhere,}\end{aligned}$$

so we have $\|f_{[N]}\|_\infty \leq \|f\|_\infty$.

For the proof of (ii) take $\alpha \in \mathbb{Z}^N$ and $f \in L_1(\mathbb{T}^\infty)$. Then, again by Fubini's theorem,

$$\hat{f}_{[N]}(\alpha) = \int_{\mathbb{T}^N} \left(\int_{\mathbb{T}^\infty} f(w,u)\,\mathrm{d}u\right) w^{-\alpha}\,\mathrm{d}w = \int_{\mathbb{T}^N\times\mathbb{T}^\infty} f(w,u)(w,u)^{-\alpha}\,\mathrm{d}(w,u) = \hat{f}(\alpha).$$

We begin the proof of (iii) by considering $L_1(\mathbb{T}^\infty)$. Let us suppose first that $f \in L_1(\mathbb{T}^k)$ for some k. Then, a straightforward calculation shows that $f_{[N]} = f$ for every $N \geq k$. In particular, $f_{[N]} \to f$ for every $f \in \bigcup_{k\in\mathbb{N}} L_1(\mathbb{T}^k)$, and as an immediate consequence of the density of trigonometric polynomials on $L_1(\mathbb{T}^\infty)$ (proposition 37), these functions are dense in $L_1(\mathbb{T}^\infty)$. Now, by (i) for $p = 1$, the projection $L_1(T^\infty) \to L_1(T^N)$ given by $f \mapsto f_{[N]}$ is a contraction. Given $f \in L_1(\mathbb{T}^\infty)$ and $\varepsilon > 0$, we can take $g \in \bigcup_k L_1(\mathbb{T}^k)$ such that $\|g - f\|_1 < \varepsilon/3$ and, by the previous coment, $\|f_{[N]} - g_{[N]}\|_1 < \varepsilon/3$. Since $g_N \to g$ in $L_1(\mathbb{T}^\infty)$, there exists $N_0 \in \mathbb{N}$ such that $\|g_{[N]} - g\|_1 < \varepsilon/3$ for every $N \geq N_0$. Then, for every $N \geq N_0$ we have

$$\|f_{[N]} - f\|_1 \leq \|f_{[N]} - g_{[N]}\|_1 + \|g_{[N]} - g\|_1 + \|g - f\|_1 < \varepsilon.$$

It is only left to show (iii) for $p = \infty$. Given $f \in L_\infty(\mathbb{T}^\infty)$, we have to show that $\langle f_{[N]}, \cdot\rangle \to \langle f, \cdot\rangle$ pointwise on $L_1(\mathbb{T}^\infty)$. Using (iii) for $L_1(\mathbb{T}^\infty)$, this holds true on the dense subspace $L_\infty(\mathbb{T}^\infty)$ of $L_1(\mathbb{T}^\infty)$, and using (i) for $L_\infty(\mathbb{T}^\infty)$, all functionals $\langle f_{[N]}, \cdot\rangle$ are uniformly bounded on $L_1(\mathbb{T}^\infty)$; that is, for every $h \in L_1(\mathbb{T}^\infty)$, $|\langle f_{[N]}, h\rangle| \leq \|f\|_\infty \|h\|_1$.

Then, given $h \in L_1(\mathbb{T}^\infty)$ and $\varepsilon > 0$, we can take $g \in L_\infty(\mathbb{T}^\infty)$ such that $\|h - g\|_1 < \frac{\varepsilon}{4\|f\|_\infty}$. Since $\langle f_{[N]}, g\rangle \to \langle f, g\rangle$, there exists $N_0 \in \mathbb{N}$ such that $|\langle f_{[N]} - f, g\rangle| < \varepsilon/2$ for every $N \geq N_0$. Then, for every $N \geq N_0$ we have

$$\begin{aligned}|\langle f_{[N]}, h\rangle - \langle f, h\rangle| &\leq |\langle f_{[N]}, h - g\rangle| + |\langle f, h - g\rangle| + |\langle f_{[N]}, g\rangle - \langle f, g\rangle| \\ &\leq 2\|f\|_\infty \|h - g\|_1 + |\langle f_{[N]} - f, g\rangle| < \varepsilon.\end{aligned}$$ ■

Proposition 41. *Given $f \in H_\infty(\mathbb{T}^\infty)$ and $N \in \mathbb{N}$, we have the following:*

(i) $f_{[N]} \in H_\infty(\mathbb{T}^N)$, *and* $\|f_{[N]}\|_\infty \leq \|f\|_\infty$;

(ii) $\hat{f}_{[N]}(\alpha) = \hat{f}(\alpha)$ *for every* $\alpha \in \mathbb{N}_0^N$;

(iii) $f_{[N]} \to f$ *in* $H_\infty(\mathbb{T}^\infty)$ *in the* $w(L_\infty, L_1)$*-topology.*

Proof. It is a consequence of lemma 40 and the fact that $H_\infty(\mathbb{T}^\infty)$ is a closed subspace of $L_\infty(\mathbb{T}^\infty)$. ■

Theorem 42. *Let $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} \subset \mathbb{C}$. The following are equivalent:*

(i) *there exists $f \in H_\infty(\mathbb{T}^\infty)$ so that $\hat{f}(\alpha) = c_\alpha$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$;*

(ii) *for each $N \in \mathbb{N}$, there exists $f_N \in H_\infty(\mathbb{T}^N)$ so that $\hat{f}_N(\alpha) = c_\alpha$ for every $\alpha \in \mathbb{N}_0^N$ satisfying $\sup\limits_{N\in\mathbb{N}} \|f_N\|_\infty < \infty$.*

Moreover, in this case $\|f\|_\infty = \sup_N \|f_N\|_\infty$.

Proof. Taking $f_N = f_{[N]}$ in proposition 41 immediately gives that (i) implies (ii) and $\sup_{N\in\mathbb{N}} \|f_N\|_\infty = \sup_{N\in\mathbb{N}} \|f_{[N]}\|_\infty \leq \|f\|_\infty$.

Assume that (ii) holds, consider the sequence $\{f_N\}_{N\in\mathbb{N}}$ as a bounded sequence in $L_\infty(\mathbb{T}^\infty)$, and let $K = \sup_{N\in\mathbb{N}} \|f_N\|_\infty$. Using remark 15 we can find a subsequence $\{f_{N_k}\}_{k\in\mathbb{N}}$ that $\sigma(L_\infty, L_1)$-converges to some $f \in L_\infty(\mathbb{T}^\infty)$ with $\|f\| \leq \sup_{N\in\mathbb{N}} \|f_N\|$. Take now $\alpha \in \mathbb{Z}^{(\mathbb{N})}$ and find $L \geq 1$ such that $\alpha = (\alpha_1, \dots, \alpha_L, 0, 0 \dots)$

with $\alpha_L \neq 0$, and some k_0 such that for all $k \geq k_0$ we have $N_k \geq L$. As a consequence, we have $\hat{f_{N_k}}(\alpha) = c_\alpha$ for all $k \geq k_0$, and therefore

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha}\, \mathrm{d}w = \lim_{k\to\infty} \int_{\mathbb{T}^\infty} f_{N_k}(w_1, \dots, w_{N_k}) w^{-\alpha}\, \mathrm{d}w = \lim_{k\to\infty} \hat{f}_{N_k}(\alpha) = \begin{cases} c_\alpha & \text{if } \alpha \in \mathbb{N}_0^{(\mathbb{N})}, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the uniqueness of the Fourier coeffients (proposition 35) shows that $f_N = f_{[N]}$ and completes the proof of the equivalence. ∎

We finally have at hand everything we need to prove the result we are aiming for.

Theorem 43. *There exists a (unique) isometric isomorphism*

$$P_\infty : H_\infty(\mathbb{T}^\infty) \to H_\infty(B_{c_0})$$

so that $c_\alpha(P_\infty[f]) = \hat{f}(\alpha)$ for every $f \in H_\infty(\mathbb{T}^\infty)$ and all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$.

Proof. Our aim now is to define

$$P_\infty : H_\infty(\mathbb{T}^\infty) \to H_\infty(B_{c_0}),$$

satisfying our requests. First of all, given $f \in H_\infty(\mathbb{T}^\infty)$ we consider $f_{[N]} \in H_\infty(\mathbb{T}^N)$ defined in (41), that satisfies $\hat{f}(\alpha) = \hat{f}_{[N]}(\alpha)$ for every $\alpha \in \mathbb{N}_0^N$. Theorem 29 provides us with $g_N = P[f_{[N]}] \in H_\infty(\mathbb{D}^N)$ satisfying $\hat{f}_{[N]}(\alpha) = c_\alpha(g_N)$ for all $\alpha \in \mathbb{N}_0^N$ and $\|g_N\|_\infty = \|f_{[N]}\|_\infty \leq \|f\|_\infty$. Now, for each $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ we consider $c_\alpha = \hat{f}(\alpha)$ and, by theorem 29, the family $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ satisfies (10) and (11). Then, by theorem 39, we can find $g \in H_\infty(B_{c_0})$ with $\hat{f}(\alpha) = c_\alpha(g)$ for all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and $\|g\|_\infty \leq \|f\|_\infty$. In this way, P_∞ is well defined and a contraction.

Conversely, given $g \in H_\infty(B_{c_0})$, define for each N the function $g_N \in H_\infty(\mathbb{D}_N)$ as the restriction of g to the first N variables. Then, again using theorem 29, look at $f_N = P_N^{-1}[g_N] \in H_\infty(\mathbb{T}^N)$. Considering this time $c_\alpha = c_\alpha(g)$ for each $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and using theorem 42 we obtain $f \in H_\infty(\mathbb{T}^\infty)$ with $\|f\|_\infty \leq \|g\|_\infty$. Finally, the uniqueness of the Fourier coeffients shows that $f_{[N]} = f_N$ for every N and, therefore, $P_\infty[f] = g$. ∎

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