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# Approximation and the full Müntz-Szász theorem

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**Abstract:** The genesis of approximation theory is the classic result of Weierstrass about the approximation of continuous functions by polynomials. Later, the Müntz-Szász theorem characterizes the sequences of positive real numbers which define dense span subspaces of monomials on the space of continuous functions. In this paper we present this important and beautiful result and some of its extensions, known as the full Müntz-Szász theorem in Lebesgue spaces $L^p([0,1])$ for $1 \leq p < \infty$ and $C([0,1])$.

**Resumen:** Uno de los orígenes de la teoría de la aproximación es el teorema clásico de Weierstrass sobre aproximación de funciones continuas mediante polinomios. Posteriormente, el teorema de Müntz-Szász caracteriza las sucesiones de números positivos que definen subespacios densos de monomios en el espacio de las funciones continuas. En este artículo presentamos este importante y hermoso resultado y algunas de sus extensiones, conocidas como el teorema completo de Müntz-Szász en los espacios de Lebesgue $L^p([0,1])$ para $1 \leq p < \infty$ y $C([0,1])$.

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# 1. Introduction

The genesis of approximation theory is the classic result of Weierstrass about the approximation of continuous functions by polynomials. As Bertrand Russel said: «All exact science is dominated by the idea of approach». When one calculates, one *approximates*.

The first work on this subject is attributed to Leonhard Euler, who was trying to solve the problem of drawing a map of the Russian Empire whose latitudes were accurate. In 1777, he published the work where he gave the best approximation in relation to the altitudes and latitudes considering all the points of a meridian between the given latitudes, that is, over the entire interval. Given the vast area occupied by the Russian Empire, all projections had a large number of errors on the edge of the map, which is why Euler's approach was a great contribution.

A problem shown by Laplace shared the same character. A paragraph from a famous work, first published in 1799, dealt with the question of determining the best ellipsoidal approximation of the Earth's surface. Here, it was relevant to obtain the least possible error at each point on the Earth's surface. Euler solved his problem in the total domain; on the contrary, Laplace assumed a finite amount of points considerably greater than the number of parameters of the problem. Fourier generalized the results of Laplace in his work *Analyse des équations determinées*. It dealt with the problem of solving, through an approximation method, linear systems of equations with a greater number of equations than of parameters. His method was to minimize the error of each equation.

In 1853, Chebyshev was the first to unify all these considerations in a work under the title of 'Theory of functions that became as little as possible of zero'. A well-known problem of that time was the so-called Watts parallelogram, which studied the determination of the parameters of a steam engine mechanism, so that the conversion of rectilinear movement into a circular movement was as accurate as possible. This led to the general problem of the approximation of a real analytical function by a polynomial of any degree. The first objective that Chebyshev achieved was the determination of the degree $n$ polynomial with the first coefficient given whose zero deviation is the smallest possible over the $[-1, 1]$ interval. Today, this polynomial is known as the first species Chebyshev polynomial.

In 1857, Chebyshev presented a work entitled "Sur les questions de minima qui se rattachent à la représentation approximative des fonctions", in which he pays attention to the following problem: determining the value of the parameters $\lambda_1, \lambda_2, \dots, \lambda_n$ which solve

$$\min_{\lambda_1,\dots,\lambda_n} \max_{x\in[a,b]} |f(x, \lambda_1, \dots, \lambda_n)|,$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a given function. He also proved that, under certain hypotheses about partial derivatives, it was possible to demonstrate a necessary condition for the solution of the previous problem.

The objective of this contribution was to find the polynomial that deviates uniformly as little as possible from zero for any given number of coefficients. This goal mainly determined his early contributions at the St. Petersburg Mathematical School in the area of approximation theory.

The Mathematical School of St. Petersburg was characterized by its tendency to solve specific problems with the intention of obtaining an explicit formula or, failing this, an algorithm that suited its purposes adequately. Consequently, all the contributions of the members of this school were oriented towards classical mathematics, due to the exclusive use of algebraic methods. It is relevant to highlight the articles by Zolotarev and the Markov brothers, who emphasized special problems in the field of uniform approximation theory. One of the most striking results of the Markov brothers is the following inequality, which states, that if $P : [-1, 1] \to \mathbb{R}$ is a polynomial of degree at most $n$, then

$$\max_{x\in[-1,1]} |P^{(k)}(x)| \le \frac{n^2\left(n^2-1\right)\left(n^2-2\right)\cdot\left(n^2-(k-1)^2\right)}{1\cdot 2\cdot 5\cdots(2k-1)} \max_{x\in[-1,1]} |P(x)|, \qquad k \le n.$$

The equality is achieved in the first species Chebyshev polynomials.

Sergei Natanovich Bernstein made use of this result to prove one of his theorems. However, due to the nature of the task, their investigations rested, again, on algebraic methods. The last contribution to the early approximation theory of the St. Petersburg Mathematical School, which came from the hand of Andrey Markov, took place in 1906.

Throughout this time, it is remarkable that the results achieved by western mathematicians were not cited. Not even the renowned Weierstrass theorem from 1885 was cited in any of these publications.

Outside of Russia, approximation theory was born in a different way. It had been preferred to address a theory of the more theoretical approach, due to the great interest in some basic questions generated at the end of the 18th century for the problem of the oscillating rope. The interest in defining the most important concept of modern analysis, the concept of *continuous function*, played a fundamental role in the consequences derived from Weierstrass's approach theorem. He defined the objective, and therefore, it was time to explicitly find sequences of algebraic or trigonometric polynomials that converged to a given continuous function. Finally, they tried to determine the speed of convergence with which these sequences could converge, that is, how quickly the approximation error decreased. Such were the objectives of a large series of alternative tests that quickly emerged after Weierstrass's original work.

**Theorem 1** (Weierstrass, 1885). *Every continuous function defined in a compact of the real line is uniform limit of polynomials.*

Some proofs of the Weierstrass theorem have been provided by great mathematicians: Lipót Fejèr used harmonic analysis techniques; the proof of Edmund Landau is based on basic tools of real analysis in one variable, and Sergei Bernstein applied a probabilistic method.

## 2. The Müntz-Szász theorem on $C([0,1])$

In 1912, at the Cambridge *International Congress of Mathematicians*, Bernstein posed a problem from Weierstrass's result. He asked about the conditions under which a set of positive numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ verifies that the set of finite linear combinations of $\{t^{\lambda_n}\}_{n\in\mathbb{N}}$ is dense in $C([0,1])$. Bernstein himself gave some partial results and he guessed rightly that the harmonic sum $\sum_{n\in\mathbb{N}} 1/\lambda_n$ would be crucial. Only two years later, in 1914, Müntz confirmed the conjecture and demonstrated what went down in history as the Müntz-Százs theorem.

**Theorem 2** (Müntz-Szász theorem). *Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence of positive real numbers. Then, the subspace of finite linear combinations of* $1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots$, i. e.*, the space* $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots\}$*, is dense in* $C[0,1]$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty.$$

In 1916, Szász published an article where he completed the proof, further improving and simplifying it. Müntz's demonstration uses real variable techniques and is based on estimating the distance between any continuous function to certain finite subspaces of polynomials, which can be made as small as desired. Szász's proof makes use of complex variable techniques combined with some arguments of functional analysis. The proof we present here can be found in Rudin's book [7] and follows Szász's ideas.

The first step towards proving theorem 2 is to present a more practical and complete version which implies the Müntz-Szász theorem.

**Theorem 3.** *Let* $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ *and*

$$X = \overline{\operatorname{span}}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots\}.$$

(i) *If* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty$*, then* $X = C[0,1]$.

(ii) *If* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < +\infty$ *and* $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$, $\lambda \neq 0$*, then* $t^{\lambda} \notin X$.

To prove this theorem, we will use the following lemma.

**Lemma 4.** *Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ be a sequence such that $\sum_n 1/\lambda_n = \infty$, and let $\mu$ be a complex Borel measure on $I = (0, 1]$ such that $T \in C(I)^* \cong M(I)$, where $C(I)^*$ is the continuous dual of $C(I)$, is a linear and bounded functional associated to $\mu$ with*

$$T(t^{\lambda_n}) = \int_0^1 t^{\lambda_n}\, \mathrm{d}\mu(t) = 0, \qquad n = 1, 2, 3, \dots \tag{1}$$

*Then,*

$$T(t^k) = \int_0^1 t^k\, \mathrm{d}\mu(t) = 0, \qquad k = 1, 2, 3, \dots$$

*Proof.* Suppose that the condition (1) holds. We may assume that the measure $\mu$ is concentrated on $I = (0, 1]$. We consider the function

$$f(z) = \int_0^1 t^z\, \mathrm{d}\mu(t) = \int_0^1 \mathrm{e}^{z \log t}\, \mathrm{d}\mu(t),$$

which is well-defined and bounded in the right half plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$:

$$|f(z)| \le \int_0^1 |\mathrm{e}^{z \log t}|\, \mathrm{d}|\mu|(t) = \int_0^1 \mathrm{e}^{\Re(z) \log t}\, \mathrm{d}|\mu|(t) = \int_0^1 t^{\Re(z)}\, \mathrm{d}|\mu|(t) \le \|\mu\| < +\infty.$$

Now, we check that the function $f$ is continuous. Let $\varepsilon > 0$. Since the map $t \mapsto t^z$ is uniformly continuous in the compact $[0, 1]$, there exists $\delta(\varepsilon) > 0$ such that

$$|f(z) - f(z_0)| \le \int_0^1 |t^z - t^{z_0}|\, \mathrm{d}|\mu|(t) \le \varepsilon \int_0^1 \mathrm{d}|\mu|(t) = \varepsilon \|\mu\|,$$

for $|z - z_0| < \delta$.

Let $\gamma$ be a regular closed path on $\mathbb{C}^+$. By Fubini's theorem, we obtain that

$$\oint_\gamma f(z)\, \mathrm{d}z = \oint_\gamma \int_0^1 t^z\, \mathrm{d}\mu(t)\, \mathrm{d}z = \int_0^1 \oint_\gamma t^z\, \mathrm{d}z\, \mathrm{d}\mu(t) = 0,$$

and we conclude that $f$ is a bounded analytic function on $\mathbb{C}^+$.

We define the function

$$g(z) := f\left(\frac{1+z}{1-z}\right), \qquad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Note that $g \in H^\infty$, *i. e.*, it is a bounded analytic function on the disc. By the hypothesis (1) we conclude that $g(\alpha_n) = 0$, where

$$\alpha_n := \frac{\lambda_n - 1}{\lambda_n + 1}.$$

We claim that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty \implies \sum_{n=1}^{\infty} 1 - |\alpha_n| = +\infty.$$

Note that

$$\sum_{n=1}^{\infty} 1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right| = \sum_{n=1}^{\infty} \frac{\lambda_n + 1 - |\lambda_n - 1|}{\lambda_n + 1}.$$

We split in two different cases.

- If $0 < \lambda_n < 1$ for all $n \in \mathbb{N}$, then $\lambda_n + 1 - |\lambda_n - 1| = 2\lambda_n$ and

$$\sum_{n=1}^{\infty} 1 - |\alpha_n| = \sum_{n=1}^{\infty} \frac{2\lambda_n}{\lambda_n + 1} = +\infty,$$

  due to the fact that $\dfrac{2\lambda_n}{\lambda_n + 1} \nrightarrow 0$ when $n \to \infty$.

- If there exists $m \in \mathbb{N}$ such that $\lambda_n \geq 1$ for all $n \geq m$, then $\lambda_n + 1 - |\lambda_n - 1| = 2$ and

$$\sum_{n=1}^{\infty} 1 - |\alpha_n| \geq \sum_{n=m}^{\infty} \frac{2}{\lambda_n + 1} = +\infty.$$

By the Riesz representation theorem, we conclude that $g(z) = 0$, for $z \in \mathbb{D}$. In particular,

$$T(t^k) = \int_I t^k \, \mathrm{d}\mu(t) = f(k) = g\left(\frac{k-1}{k+1}\right) = 0, \qquad k = 1, 2, \ldots,$$

and we conclude the proof. ∎

Now we present the proof of theorem 3.

*Proof of theorem 3.* To show part (i), it is enough to show that $X$ contains all the functions $t^k$, for $k = 1, 2, 3, \ldots$, and apply the Weierstrass approximation theorem. Suppose that there exists $k_0 \in \mathbb{N}$ such that $t^{k_0} \notin X$. By the Hahn-Banach theorem, there exists a bounded and linear functional $T\colon C[0,1] \to \mathbb{R}$ such that

$$T(t^{k_0}) \neq 0 \qquad \text{and} \qquad T|_{\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}} \equiv 0.$$

By lemma 4, we conclude that $T(t^{k_0}) = 0$, which contradicts our hypothesis.

In order to prove part (ii), assume that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty.$$

Our objective is to give a bounded linear functional $T = \langle \cdot, \mu \rangle \in C[0,1]^*$ such that $T(t^{\lambda_n}) = 0$ for all $n \in \mathbb{N} \cup \{0\}$ ($\lambda_0 = 0$), but $T(t^{\lambda}) \neq 0$ for $\lambda \notin \{\lambda_n\}_{n \geq 1}$. By the Hahn-Banach theorem, we shall conclude that $t^{\lambda} \notin \overline{\operatorname{span}}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots\}$ for $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$.

To get this, we would need to obtain a complex Borel measure $\mu$ on $[0,1]$ such that the analytic function $f$, given by

$$z \mapsto \int_0^1 t^z \, \mathrm{d}\mu(t),$$

defines a bounded function on the half plane $\mathbb{C}_{-1} := \{z \in \mathbb{C} : \Re(z) > -1\}$, and whose zeros are precisely the sequence $\{\lambda_n\}_{n=1}^{\infty}$. In this case, we shall take $T := \langle \cdot, \mu \rangle$.

We consider the function $f$ given by

$$f(z) := \frac{z}{(2+z)^3} \prod_{n=1}^{\infty} \frac{\lambda_n - z}{2 + \lambda_n + z}, \qquad z \in \mathbb{C} \setminus \{-2 - \lambda_n\}.$$

First, we check that the function $f$ is a meromorphic function whose poles are the set $\{-2-\lambda_n : n \in \mathbb{N}\}$ and whose zero set is $\{\lambda_n\}_{n=1}^{\infty}$. To do this, it is enough to show the uniform convergence of the infinite product on compacts contained in $\mathbb{C} \setminus \{-2 - \lambda_n\}$. This convergence is equivalent to the uniform convergence of the following series:

$$\sum_{n=1}^{\infty} \left| 1 - \frac{\lambda_n - z}{2 + \lambda_n + z} \right| = \sum_{n=1}^{\infty} \left| \frac{2z + 2}{2 + \lambda_n + z} \right|. \tag{2}$$

Fixed $K$ a compact set, there exists $\alpha > 0$ such that $K \subset \mathbb{C}_{-\alpha} = \{z \in \mathbb{C} : \Re(z) > -\alpha\}$. As the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$ is convergent, there exist $N \in \mathbb{N}$ and $C > 0$, which only depends on the compact set $K$, such that

$$\left| \frac{2z + 2}{2 + \lambda_n + z} \right| \leq \frac{C}{2 + \lambda_n - \alpha},$$

for $n > N$. By the Weierstrass M-test, we obtain the uniform convergence of series (2).

Now we claim that the function $f$ is bounded on $\mathbb{C}_{-1}$. Every factor in the infinite product and the fraction $\frac{z}{2+z}$ are Möbius transform from $\mathbb{C}_{-1}$ into the disc. Finally, as $\frac{1}{(2+z)^2} \leq 1$ for $z \in \mathbb{C}_{-1}$, we conclude that $f$ is bounded on $\mathbb{C}_{-1}$.

Note that $f \in L^1(\{z \in \mathbb{C} : \Re(z) = -1\})$, due to the fact that

$$\int_{\mathbb{R}} |f(-1+\mathrm{i}t)|\,\mathrm{d}t \leq C \int_{\mathbb{R}} \frac{1}{1+t^2}\,\mathrm{d}t = C\pi.$$

By Cauchy's theorem, given $z_0 \in \mathbb{C}_{-1}$, we have that

$$f(z_0) = \int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z,$$

where $\gamma$ is the path formed by the semicircle with center $-1$ and radius $R > 1 + |z_0|$, from $-1 - \mathrm{i}R$ to $-1 + \mathrm{i}R$ and the interval $[-1 - \mathrm{i}R, -1 + \mathrm{i}R]$. Then, we obtain that

$$f(z_0) = \frac{1}{2\pi} \int_{-R}^{R} \frac{f(-1+\mathrm{i}s)}{1 - \mathrm{i}s + z_0}\,\mathrm{d}s + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{f(-1+R\mathrm{e}^{\mathrm{i}\theta})}{-1 + R\mathrm{e}^{\mathrm{i}\theta} - z_0} R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta.$$

Since $|f(z)| \leq \left|\frac{z}{(2+z)^3}\right|$, the second summand tends to 0 when $R \to \infty$. We get the equality

$$f(z) = \int_{\mathbb{R}} \frac{f(-1+\mathrm{i}s)}{1 - \mathrm{i}s + z}\,\mathrm{d}s, \qquad z \in \mathbb{C}_{-1}.$$

We apply the identity

$$\frac{1}{1 - \mathrm{i}s + z} = \int_0^1 t^z \mathrm{e}^{-\mathrm{i}s \log t}\,\mathrm{d}t$$

and Fubini's theorem to get that

$$f(z) = \int_0^1 t^z \left( \frac{1}{2\pi} \int_{\mathbb{R}} f(-1+\mathrm{i}s) \mathrm{e}^{-\mathrm{i}s \log t}\,\mathrm{d}s \right) \mathrm{d}t, \qquad z \in \mathbb{C}_{-1}. \tag{3}$$

We define $g(s) = f(-1 + \mathrm{i}s)$. Note that the inner integral in (3) is equal to $\hat{g}(\log t)$, where $\hat{g}$ is the Fourier transform of $g$. Since $\hat{g}$ is continuous and bounded, we define

$$\mathrm{d}\mu = \frac{1}{2\pi} \hat{g}(\log t)\,\mathrm{d}t$$

to conclude that $\mu$ is a complex Borel measure on $[0, 1]$. Finally, we obtain the following representation of $f$:

$$f(z) = \int_0^1 t^z\,\mathrm{d}\mu(t),$$

and the proof is completed. ∎

## 3. The full Müntz-Szász theorem on $L^2([0,1])$

Now that we have demonstrated the classical Müntz-Szász theorem, it is worth asking if we can extend the result for other functional spaces, such as Lebesgue spaces $L^p([0,1])$, or if it is really necessary that the sequence of exponents $\{\lambda_n\}_{n\in\mathbb{N}}$ is monotone increasing. These questions were partially answered by the mathematicians Borwein and Erdélyi in an article published in 1996 [1] where they proved the Müntz-Szász theorem for spaces $L^1([0,1])$, $L^2([0,1])$ and $C([0,1])$ without the monotonicity assumption. We will present here the proof of these facts following the original article by Borwein and Erdélyi.

We start with the proof for $L^2([0,1])$, since, being a Hilbert space, we have a richer structure to rely on.

**Theorem 5.** *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of different real numbers greater than $-1/2$. Then, the set* $\operatorname{span}\{t^{\lambda_n} : n \geq 0\}$ *is dense in* $L^2[0,1]$ *if and only if*

$$\sum_{n=0}^{\infty} \frac{2\lambda_n + 1}{(2\lambda_n + 1)^2 + 1} = +\infty.$$

*Proof.* Our aim is to show that

$$\sum_{n=0}^{\infty} \frac{2\lambda_n + 1}{(2\lambda_n + 1)^2 + 1} = +\infty \iff t^m \in \overline{\operatorname{span}}\{t^{\lambda_n} : n \in \mathbb{N} \cup \{0\}\}, \quad \forall m \in \mathbb{N} \cup \{0\}.$$

We will obtain the result as a consequence of the Weierstrass approximation theorem. Let $m \in \mathbb{N} \cup \{0\}$ be such that $m \notin \{\lambda_n\}_{n=0}^{\infty}$.

Observe $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots, t^{\lambda_n}\}$ is a closed subspace of $L_2([0,1])$, so we can consider the orthogonal projection of $t^m$ onto $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots, t^{\lambda_n}\}$. The expression of this projection is given by $\sum_{i=0}^{n} \langle t^m, t^{\lambda_i}\rangle t^{\lambda_i}$. Observe this function satisfies

$$\min_{a_i \in \mathbb{C}} \left\| t^m - \sum_{i=0}^{n} a_i t^{\lambda_i} \right\|_{L_2([0,1])} = \left\| t^m - \sum_{i=0}^{n} \langle t^m, t^{\lambda_i}\rangle t^{\lambda_i} \right\|_{L_2([0,1])} = \frac{1}{\sqrt{2m+1}} \prod_{i=0}^{n} \left| \frac{m - \lambda_i}{m + \lambda_i + 1} \right|,$$

where the last equality arises from a direct computation of the norm. Then ,

$$t^m \in \overline{\operatorname{span}}\{t^{\lambda_n} : n \in \mathbb{N} \cup \{0\}\} \iff \limsup_{n\to\infty} \prod_{i=0}^{n} \left| \frac{m - \lambda_i}{m + \lambda_i + 1} \right| = 0. \tag{4}$$

We divide into two cases, $\lambda_i > m$ and $\lambda_i < m$, to get

$$\prod_{i=0}^{n} \left| \frac{m - \lambda_i}{m + \lambda_i + 1} \right| = \prod_{i=0, \lambda_i < m}^{n} \left(1 - \frac{2\lambda_i + 1}{m + \lambda_i + 1}\right) \prod_{i=0, m < \lambda_i}^{n} \left(1 - \frac{2m + 1}{m + \lambda_i + 1}\right).$$

Then, the condition (4) is equivalent to one of these two following conditions:

$$\limsup_{n\to\infty} \prod_{i=0, \lambda_i < m}^{n} \left(1 - \frac{2\lambda_i + 1}{m + \lambda_i + 1}\right) = 0 \qquad \text{or} \qquad \limsup_{n\to\infty} \prod_{i=0, \lambda_i > m}^{n} \left(1 - \frac{2m + 1}{m + \lambda_i + 1}\right) = 0.$$

By Theorem 15.5 from Rudin's book [7], each condition is equivalent to the divergence of one of these two series:

$$\sum_{i=0, \lambda_i < m}^{\infty} \frac{2\lambda_i + 1}{m + \lambda_i + 1} \qquad \text{and} \qquad \sum_{i=0, \lambda_i > m}^{\infty} \frac{2m + 1}{m + \lambda_i + 1}.$$

By comparison, the divergence of these two series is equivalent to the divergence of

$$\sum_{i=0, \lambda_i < m}^{\infty} (2\lambda_i + 1) \qquad \text{and} \qquad \sum_{i=0, \lambda_i > m}^{\infty} \left(\frac{1}{2\lambda_i + 1}\right).$$

Finally, the divergence of these two series is equivalent to

$$\sum_{i=0}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = +\infty,$$

and the proof is finished. ∎

## 4. The full Müntz-Szász theorem on $C([0,1])$

Now we will show the full Müntz-Szász theorem on $C[0,1]$. To do so, we need some preliminary results about Newman's inequality and Chebyshev polynomials.

**Theorem 6**. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be different positive real numbers. Then, for $p \in \operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots, t^{\lambda_n}\}$, the following inequality holds:*

$$\|t\, p'(t)\|_{C[0,1]} \leq 11 \left(\sum_{i=1}^{n} \lambda_i\right) \|p(t)\|_{C[0,1]}.$$

It is interesting to point out that the optimal constant on the Newman inequality is conjectured to be 4, see the paper of Borwein and Erdélyi [2]. A modification of this inequality allows to control the derivative of any polynomial $p \in \operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$. You may find this result in work of Borwein and Erdélyi [1, Theorem 3.4].

**Theorem 7**. *Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\lambda_n \geq 1$ for $n = 1, 2, \dots$, and $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$. Take $\varepsilon \in (0, 1)$. Then, there exists $c$ which depends uniquely on $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\varepsilon$ such that*

$$\|p'\|_{C[0,1-\varepsilon]} \leq c \|p\|_{C[0,1]}$$

*for any $p \in \operatorname{span}\{1, t^{\lambda_1}, \dots\}$.*

The theory of approximation using Chebyshev polynomials and the following theorem may be found, for example, in Cheney's book [3].

**Theorem 8** (Chebyshev polynomials). *Let $A$ be a compact subset contained in $[0, +\infty)$ with at least $n + 1$ points, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be $n$ different positive real numbers. Then, there exists a unique Chebyshev polynomial $T_n$ such that*

$$T_n(t) = c\left(t^{\lambda_n} - \sum_{i=1}^{n-1} a_i t^{\lambda_i}\right),$$

*where the coefficients $a_i$ minimize the norm*

$$\left\| t^{\lambda_n} - \sum_{i=1}^{n-1} a_i t^{\lambda_i} \right\|_{C(A)},$$

*the constant $c$ is such that $\|T_n\|_{C(A)} = 1$, and $T_n(\max A) > 0$.*

*The Chebyshev polynomial $T_n \in \operatorname{span}\{t^{\lambda_1}, t^{\lambda_2}, \dots, t^{\lambda_n}\}$ is uniquely characterized by the existence of an alternation set*

$$\{t_0 < t_1 < \dots < t_n\} \subset A$$

*such that*

$$T_n(t_j) = (-1)^{n-j} = (-1)^{n-j} \|T_n\|_{C(A)}, \qquad 0 \leq j \leq n.$$

Now, we present the full Müntz-Szász theorem in $C[0, 1]$.

**Theorem 9**. *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of different positive real numbers. Then,*

$$\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\}$$

*is dense in $C[0, 1]$ if and only if*

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = +\infty.$$

*Proof*. We consider the following four cases depending on the sequence $\{\lambda_n\}$.

1. $\inf_{n \in \mathbb{N}} \lambda_n > 0$.
2. $\lim_{n \to +\infty} \lambda_n = 0$.
3. $\{\lambda_n\} = \{\alpha_n\} \cup \{\beta_n\}$, with $\alpha_n \to 0$ and $\beta_n \to +\infty$.
4. $\{\lambda_n\}$ has a cluster point on $(0, +\infty)$.

Note that every positive sequence $\{\lambda_i\}_{i=1}^{\infty}$ or one of its rearrangements fits in one of these cases.

We consider the first case. We may suppose with no loss of generality that $\lambda_i \geq 1$, for all $i \in \mathbb{N}$. Given $m \in \mathbb{N}$, we have that

$$\left|t^m - \sum_{i=1}^{n} a_i t^{\lambda_i}\right| = \left|\int_0^1 \left(m x^{m-1} - \sum_{i=1}^{n} a_i \lambda_i x^{(\lambda_i - 1)}\right) \mathrm{d}x\right| \leq \int_0^1 \left|m x^{m-1} - \sum_{i=1}^{n} a_i \lambda_i x^{(\lambda_i - 1)}\right| \mathrm{d}x$$
$$\leq \left(\int_0^1 \left(m x^{m-1} - \sum_{i=1}^{n} a_i \lambda_i x^{(\lambda_i - 1)}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

and we get the inequality

$$\min_{a_i \in \mathbb{C}} \left\|t^m - \sum_{i=1}^{n} a_i t^{\lambda_i}\right\|_{C[0,1]} \leq m\left(\min_{b_i \in \mathbb{C}} \left\|t^{m-1} - \sum_{i=1}^{n} b_i t^{\lambda_i - 1}\right\|_{L^2[0,1]}\right). \tag{5}$$

Suppose that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty$. Since $\lambda_i \geq 1$, we also conclude that

$$\sum_{i=1}^{\infty} \frac{2(\lambda_i - 1) + 1}{2((\lambda_i - 1) + 1)^2 + 1} = +\infty.$$

By theorem 5, the set $\operatorname{span}\{1, t^{\lambda_1 - 1}, t^{\lambda_2 - 1}, \ldots\}$ is dense in $L^2[0,1]$, and the inequality (5) shows that, in fact, $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is dense on $C[0,1]$.

Conversely suppose that $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is dense in $C[0,1]$. As the space $C[0,1]$ is dense in $L^2[0,1]$, the set $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is also dense in $L^2[0,1]$, and

$$\sum_{i=1}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = +\infty,$$

by theorem 5. By comparing the sums, we conclude that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = +\infty$.

Now we suppose that the sequence $\{\lambda_i\}_{i \geq 1}$ converges to 0. Then,

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = +\infty \iff \sum_{i=1}^{\infty} \lambda_i = +\infty.$$

If $\sum_{i=1}^{\infty} \lambda_i = +\infty$, then we conclude that

$$\sum_{i=1}^{\infty} \left(1 - \left|\frac{\lambda_i - 1}{\lambda_i + 1}\right|\right) = \infty.$$

We follow the same ideas we used in the proof of lemma 4 to conclude that $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is dense in $C([0,1])$.

On the other hand, if $\eta = \sum_{i=1}^{\infty} \lambda_i < +\infty$, we apply Newman's inequality to get that

$$\|t p'(t)\|_{C[0,1]} \leq 11\eta \|p(t)\|_{C[0,1]}, \tag{6}$$

for any $p \in \operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$. We claim that this last inequality implies that the set $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is not dense in $C([0,1])$. Indeed, suppose that this set is dense in $C([0,1])$. Given the function $f(t) = \sqrt{1-t}$ and $m \in \mathbb{N}$, there exists $p \in \operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ such that $\|p - f\|_{C([0,1])} < 1/m^2$. Then,

$$\left|p\left(1 - \frac{1}{m^2}\right) - p(1)\right| \geq \left|f\left(1 - \frac{1}{m^2}\right) - \frac{1}{m^2} - \left(f(1) + \frac{1}{m^2}\right)\right| = \frac{1}{m} - \frac{2}{m^2}.$$

By the mean value theorem, there exists $\xi \in (1 - 1/m^2, 1)$ such that

$$|\xi p'(\xi)| = \xi \frac{\left|p\left(1 - \frac{1}{m^2}\right) - p(1)\right|}{\frac{1}{m^2}} \geq \frac{m-2}{2},$$

which gives a contradiction with the inequality (6).

Now we consider the third case. We split the sequence $\{\lambda_i\}$ into two sequences $\{\lambda_i : i \in \mathbb{N}\} = \{\alpha_i : i \in \mathbb{N}\} \cup \{\beta_i : i \in \mathbb{N}\}$ such that $\alpha_n \to 0$ and $\beta_n \to +\infty$. Note that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty$ is equivalent to

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \frac{1}{\beta_i} = \infty. \tag{7}$$

If the condition (7) holds, then $\sum \alpha_i = \infty$ or $\sum \frac{1}{\beta_i} = \infty$. Then, we may apply cases 1 and 2 to conclude that $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is dense in $C([0,1])$.

Conversely, if the condition (7) does not hold, then

$$\sum_{i=1}^{\infty} \alpha_i < \infty \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty.$$

Given a sequence $\{w_1, \ldots, w_n\}$ of $n$ different positive real numbers, we denote by $T_n\{1, t^{w_1}, \ldots, t^{w_n}\}$ the associated Chebychev polynomial in the compact $[0,1]$ given by theorem 8. With this notation, we define

$$\begin{aligned} T_{n,\alpha} &:= T_n\{1, t^{\alpha_1}, \ldots, t^{\alpha_n}\}, \\ T_{n,\beta} &:= T_n\{1, t^{\beta_1}, \ldots, t^{\beta_n}\}, \\ T_{2n,\alpha,\beta} &:= T_{2n}\{1, t^{\alpha_1}, \ldots, t^{\alpha_n}, t^{\beta_1}, \ldots, t^{\beta_n}\}. \end{aligned}$$

Now our objective is to count and localize the zeros of these polynomials. It follows from Newman's inequality (theorem 6) and the mean value theorem that for every $\varepsilon > 0$ there exists a $k_1(\varepsilon) \in \mathbb{N}$ depending only on $\{\alpha_n\}_{n=1}^{\infty}$ and $\varepsilon$ (and not on $n$) so that $T_{n,\alpha}$ has at most $k_1(\varepsilon)$ zeros in $[\varepsilon, 1)$ and at least $n - k_1(\varepsilon)$ zeros in $(0, \varepsilon)$.

Analogously, it follows from theorem 7 and the mean value theorem that for every $\varepsilon > 0$ there exists a $k_2(\varepsilon) \in \mathbb{N}$ depending only on $\{\beta_n\}_{n=1}^{\infty}$ and $\varepsilon$ (and not on $n$) so that $T_{n,\beta}$ has at most $k_2(\varepsilon)$ zeros in $(0, 1-\varepsilon]$ and at least $n - k_1(\varepsilon)$ zeros in $(1-\varepsilon, 1)$.

Now, counting the zeros of $T_{n,\alpha} - T_{2n,\alpha,\beta}$ and $T_{n,\beta} - T_{2n,\alpha,\beta}$, we can deduce that for every $\varepsilon > 0$ there exists a $k(\varepsilon) \in \mathbb{N}$ depending only on $\{\lambda_n\}_{n=1}^{\infty}$ and $\varepsilon > 0$ (and not on $n$) so that $T_{2n,\alpha,\beta}$ has at most $k(\varepsilon)$ zeros in $[\varepsilon, 1-\varepsilon]$.

Take a fixed $\varepsilon = \frac{1}{4}$ and $k := k\left(\frac{1}{4}\right)$. We pick $k + 4$ points such that

$$\frac{1}{4} < \eta_0 < \eta_1 < \ldots < \eta_{k+3} < \frac{3}{4},$$

and a function $f \in C([0,1])$ such that$f(t) = 0$ for every $t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, and

$$f(\eta_i) := 2(-1)^i, \qquad i = 0, 1, \ldots, k+3.$$

Suppose that there exists a polynomial $p \in \operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ such that

$$\|f - p\|_{C([0,1])} < 1.$$

Then, $p - T_{2n,\alpha,\beta}$ has at least $2n + 1$ zeros in $(0,1)$. However, for sufficiently large n,

$$p - T_{2n,\alpha,\beta} \in \operatorname{span}\{1, t^{\lambda_1}, \ldots, t^{\lambda_{2n}}\},$$

which can have at most $2n$ zeros in $[0,+\infty)$. This contradiction shows that $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is not dense in $C([0,1])$.

Finally we consider the case when the sequence $\{\lambda_n\}$ has a cluster point in $(0,\infty)$. In this case there, exists a subsequence $\{\lambda_{n_k}\}$ such that $\inf_{k\in\mathbb{N}} \lambda_{n_k} > 0$ and $\sum_{k=1}^{\infty} \frac{\lambda_{n_k}}{\lambda_{n_k}^2+1} = \infty$, where we may apply the case 1. ■

## 5. The full Müntz-Szász theorem on $L^1([0,1])$

Now we present the full Müntz-Szász theorem on $L^1([0,1])$.

**Theorem 10**. *Suppose that $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of real numbers greater than $-1$. Then,*

$$\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}$$

*is dense in $L^1([0,1])$ if and only if*

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i+1)^2+1} = \infty.$$

*Proof*. Suppose that the set $\operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N}\cup\{0\}\}$ is dense in $L^1([0,1])$. We fix a non-negative integer $m$. For $\varepsilon > 0$, we choose $p \in \operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}$ such that

$$\|t^m - p\|_{L^1([0,1])} < \varepsilon. \tag{8}$$

We define the function

$$q(t) := \int_0^t p(s)\,\mathrm{d}s \in \operatorname{span}\{t^{\lambda_0+1}, t^{\lambda_1+1}, \ldots\}.$$

By the inequality (8), we have that

$$\left\| \frac{t^{m+1}}{m+1} - q \right\|_{C([0,1])} < \varepsilon.$$

We apply the Weierstrass approximation theorem to conclude that the set

$$\operatorname{span}\{1, t^{\lambda_0+1}, t^{\lambda_1+1}, \ldots\}$$

is dense in $C([0,1])$. We apply the full Müntz-Szász theorem in $C([0,1])$ (theorem 9) to conclude that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i+1)^2+1} = +\infty.$$

Conversely, now we suppose that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i+1)^2+1} = +\infty.$$

By the Hahn-Banach theorem and the Riesz representation theorem, the set $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}$ is not dense in $L^1([0,1])$ if and only if there exists a function $0 \neq h \in L^\infty([0,1])$ such that

$$\int_0^1 t^{\lambda_i} h(t)\,\mathrm{d}t = 0; \qquad i = 0, 1, \ldots.$$

Given this function $h$, we define

$$f(z) := \int_0^1 t^z h(t)\,\mathrm{d}t,$$

and then

$$g(z) := f\left(\frac{1+z}{1-z} - 1\right).$$

Note that the function $g$ is bounded and analytic in the open unit disc and

$$g\left(\frac{\lambda_n}{\lambda_n+2}\right) = f(\lambda_n) = 0.$$

Note that $\sum_{i=0}^{\infty} \frac{\lambda_i+1}{(\lambda_i+1)^2+1} = +\infty$ implies that

$$\sum_{n=1}^{\infty}\left(1-\left|\frac{\lambda_n}{\lambda_n+2}\right|\right) = +\infty.$$

We consider the Blaschke product [7, Theorem 15.23] to conclude that $g = 0$ on the open unit disc, and $f(z) = 0$ in the half plane $\Re(z) > -1$. In particular, we have that

$$f(n) = \int_0^1 t^n h(t)\,\mathrm{d}t = 0; \qquad n = 0, 1, \ldots$$

We apply the Weierstrass approximation theorem to get

$$\int_0^1 u(t)h(t)\,\mathrm{d}t = 0,$$

for every $u \in C([0,1])$. Finally, we conclude that $h = 0$ and that $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}$ is dense in $L^1([0,1])$. ∎

## 6. The full Müntz-Szász theorem on $L^p([0,1])$ for $1 \leq p < \infty$

Once we have shown the full Müntz-Szász theorem on $L^1([0,1])$ and $L^2([0,1])$, it is natural to ask about the full Müntz-Szász theorem on $L^p([0,1])$ for $1 \leq p < \infty$. This question was posed by Borwein and Erdélyi [1] and solved by Operstein [6, Theorem 1].

**Theorem 11.** *Let $1 < p < \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of different real numbers greater than $-1/p$. Then, the set $\operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $L^p[0,1]$ if and only if*

$$\sum_{n=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i+1/p)^2+1} = +\infty. \tag{9}$$

To show this theorem, we need the following lemma.

**Lemma 12.** *Let $\{\mu_i\}_{i=0}^{\infty}$ be a sequence of positive real numbers such that the set $\operatorname{span}\{t^{\mu_i - 1/r} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $L^r([0,1])$. Then, the space $\operatorname{span}\{t^{\mu_i-1/s} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $L^s([0,1])$ for $s > r$, and $\operatorname{span}\{1, t^{\mu_0}, t^{\mu_1}, \ldots\}$ is dense in $C([0,1])$.*

*Proof.* We consider the spaces $X = L^r([0,1])$, $Y = L^s([0,1])$ and $A = \operatorname{span}\{t^{\mu_i - 1/r} : i \in \mathbb{N} \cup \{0\}\}$. Our aim is to define a linear bounded operator $J$ between the spaces $X$ and $Y$ such that $J(X)$ is dense in $Y$. Note that this fact implies that $J(A)$ is dense in $Y$. We consider the operator $J\colon L^r([0,1]) \to L^s([0,1])$ defined by

$$(J\varphi)(t) = t^{-(1/r'+1/s)} \int_0^t \varphi(u)\,\mathrm{d}u, \qquad t \in [0,1],\ \varphi \in L^r([0,1]),$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. By the generalized Hardy inequality [5, Theorem 329], this operator $J$ is bounded.

For every $n \in \mathbb{N}$, we define the function $\psi_n(t) := (n + 1/r' + 1/s)t^{n+1/s-1/r}$ for $t \in [0,1]$. Note that $\psi_n \in L^r([0,1])$ and $(J\psi_n)(t) = t^n$ for $n \in \mathbb{N}$. By the Weierstrass approximation theorem, we conclude that $J(X)$ is dense in $Y$ and the set $J(A) = \operatorname{span}\{t^{\mu_i - 1/s} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $L^s([0,1])$.

To show the second part, we consider the linear bounded operator $J\colon L^r([0,1]) \to C([0,1])$ defined by

$$(J\varphi)(t) = t^{-1/r'} \int_0^t \varphi(u)\,\mathrm{d}u, \qquad t \in (0,1], \qquad (J\varphi)(0) = 0,$$

for $\varphi \in L^r([0,1])$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Using similar ideas to the ones in the first part, we conclude that the set $\operatorname{span}\{1, t^{\mu_0}, t^{\mu_1}, \ldots\}$ is dense in $C([0,1])$. ∎

With the help of this lemma and the full Müntz-Szász theorem on $L^1([0,1])$ and $C([0,1])$, we prove theorem 11.

*Proof of theorem 11.* We take a sequence $\{\lambda_i\}_{i=0}^{\infty}$ satisfying condition (9). Now we consider the sequence $\{\upsilon_i\}_{i=0}^{\infty}$, where $\upsilon_i = \lambda_i - 1/p'$ for $i \geq 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$. By hypothesis, we have that

$$\sum_{i=0}^{\infty} \frac{\upsilon_i + 1}{(\upsilon_i + 1)^2 + 1} = \sum_{n=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = +\infty.$$

We apply theorem 10 to conclude that the space

$$\operatorname{span}\{t^{\upsilon_i} : i \in \mathbb{N} \cup \{0\}\} = \operatorname{span}\{t^{\lambda_i - 1/q} : i \in \mathbb{N} \cup \{0\}\}$$

is dense in $L^1([0,1])$. We take $\mu_i = \lambda_i + 1/p$ for $i \in \mathbb{N} \cup \{0\}$ and we apply lemma 12 to get that

$$\operatorname{span}\{t^{\mu_i - 1/p} : i \in \mathbb{N} \cup \{0\}\} = \operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N} \cup \{0\}\}$$

is dense in $L^p([0,1])$ for $p > 1$.

Conversely, we suppose that the space $\operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $L^p([0,1])$. We write $\mu_i = \lambda_i + 1/p$, for $i \in \mathbb{N} \cup \{0\}$, to obtain that

$$\operatorname{span}\{t^{\mu_i - 1/p} : i \in \mathbb{N} \cup \{0\}\}$$

is dense in $L^p([0,1])$. By lemma 12, the space $\operatorname{span}\{1, t^{\mu_i} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $C([0,1])$. Now we apply theorem 9 to obtain that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \sum_{i=0}^{\infty} \frac{\mu_i}{\mu_i^2 + 1} = +\infty. \qquad \blacksquare$$

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