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Four different approaches to the fractional Laplacian

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Abstract: In this paper we introduce four different definitions for the fractional Laplacian operator. First, we begin by giving the definition through the Fourier transform, motivated by the problem of finding an inverse operator for the Riesz potential. Next, we introduce a second definition given by a pointwise integral formula with a probabilistic motivation. The last two definitions come from functional analysis and partial differential equations: one is given in terms of the heat semigroup and the other one is given by the extension problem, which allows us to study properties of a nonlocal operator by means of local methods. We prove the equivalence between these four definitions and also give some of the properties of the fractional Laplacian.

Resumen: En este artículo introducimos cuatro definiciones distintas del operador laplaciano fraccionario. En primer lugar comenzamos dando la definición con la transformada de Fourier, que viene motivada por la búsqueda de un operador inverso del potencial de Riesz. A continuación se introduce una segunda definición como operador dado por una fórmula integral puntual a partir de una motivación de naturaleza probabilística. Las dos últimas definiciones provienen del análisis funcional y las ecuaciones diferenciales: una de ellas se da en términos del semigrupo del calor y la otra a partir del conocido como problema de extensión, que permite estudiar las propiedades de un operador no local mediante métodos locales. Se prueba la equivalencia de las cuatro definiciones y se muestran algunas de las propiedades del laplaciano fraccionario.

Keywords: fractional Laplacian, Fourier transform, pointwise formula, semigroup, extension problem.

MSC2010: 26A33, 60G22.

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1. Introduction

Fractional operators, and in particular the fractional Laplacian, are well known from the point of view of functional analysis. However, they also appear in other areas of mathematics. Some of the settings in which this operator arises are the theory of Banach spaces [10], potential theory [8], Lévy proceses [1], the theory of partial differential equations [4], and scattering theory in conformal geometry [6]. Bibliography in this topic is extensive and the above are just some examples.

The goal of this paper is to introduce four of the different definitions of the fractional Laplacian which appear in the literature. There are other definitions that we will not consider here, see for instance the paper of Kwaśnicki [7] for a nice exposition on ten different definitions of the fractional Laplacian.

The function space we are going to work with is the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, which is the space of functions $f \in C^\infty(\mathbb{R}^n)$ satisfying

$$\|f\|_p := \sup_{|\alpha| \le p} \sup_{x \in \mathbb{R}^n} (1+|x|^2)^{p/2} |\partial^\alpha f(x)| < \infty, \qquad p \in \mathbb{N} \cup \{0\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ denotes a multi-index in $(\mathbb{N} \cup \{0\})^n$ and $\partial^\alpha f$ denotes the derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$. With the metric given by

$$d(f,g) = \sum_{p=0}^{\infty} 2^{-p} \frac{\|f-g\|_p}{1+\|f-g\|_p}, \qquad f, g \in \mathscr{S}(\mathbb{R}^n),$$

the Schwartz space is a Fréchet space.

In section 2 we introduce the fractional Laplacian operator through the Fourier transform, as the inverse operator of the Riesz potential. In section 3 we introduce the fractional Laplacian from a probabilistic motivation, obtaining a pointwise integral formula. Section 4 is devoted to the study of the fractional Laplacian through the heat semigroup. Also in this section, we will prove the equivalence of the previous definitions. Finally, in section 5 we will study the fractional Laplacian as a «Dirichlet-to-Neumann» operator for a harmonic extension problem.

2. First definition: Fourier transform

The first time the fractional Laplacian appeared in the literature is in the paper by M. Riesz [10]. The usual Laplacian, given by $-\Delta f = -\sum_{k=1}^n \frac{\partial^2 f}{\partial x_k^2}$ for $f \in C^2(\mathbb{R}^n)$, can be understood as the inverse operator of the Newton potential I_2, which is defined as

$$I_2 f(x) := c_{n,2} |\cdot|^{-n+2} * f(x) = c_{n,2} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} \, \mathrm{d}y, \qquad n \ge 3, \ f \in \mathscr{S}(\mathbb{R}^n),$$

where $c_{n,2} = \frac{1}{4\pi^{n/2}} \Gamma\left(\frac{n-2}{2}\right)$.
Riesz generalized the concept of Newton potential by defining the fractional integral operator (or Riesz potential) of order $0 < \alpha < n$, $n \in \mathbb{N}$, as

$$I_\alpha f(x) := c_{n,\alpha} |\cdot|^{-n+\alpha} * f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}y, \qquad f \in \mathscr{S}(\mathbb{R}^n),$$

with $c_{n,\alpha} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2}} \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}$.
In the same way we have that the relation

$$I_2 \circ (-\Delta) = \mathrm{Id} = (-\Delta) \circ I_2 \tag{1}$$

holds in $\mathscr{S}(\mathbb{R}^n)$, Riesz posed the natural question of the existence of an operator $(-\Delta)^{\alpha/2}$ that would satisfy, in $\mathscr{S}(\mathbb{R}^n)$, the analogous relation

$$I_\alpha \circ (-\Delta)^{\alpha/2} = \mathrm{Id} = (-\Delta)^{\alpha/2} \circ I_\alpha. \tag{2}$$

If we understand $(-\Delta)^{\alpha/2}$ as a fractional version of the differential operator $(-\Delta)$, expression (2) somehow represents a fractional version of the fundamental theorem of calculus (thus the name «fractional integral» for I_α).

In order to find an explicit expression for an operator satisfying (2) we will use the Fourier transform. For a function $f \in \mathscr{S}(\mathbb{R}^n)$, its *Fourier transform* $\mathcal{F}[f]$ is defined to be the function

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) \coloneqq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}\xi\cdot x} \,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n.$$

This defines an invertible operator in $\mathscr{S}(\mathbb{R}^n)$ whose inverse is given by

$$\mathcal{F}^{-1}[f](x) \coloneqq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi) \mathrm{e}^{\mathrm{i}x\cdot \xi} \,\mathrm{d}\xi, \qquad x \in \mathbb{R}^n.$$

By using the well-known properties of $\mathcal{F}$ with respect to derivatives, it follows easily that $\mathcal{F}[(-\Delta)f](\xi) = |\xi|^2 \hat{f}(\xi)$. Next, we are going to obtain an analogous expression for the operator $(-\Delta)^{\alpha/2}$ defined above.

As one can see in Stein's book [11, Ch. V], the following identity

$$c_{n,\alpha}\mathcal{F}(|\cdot|^{-n+\alpha})(\xi) = |\xi|^{-\alpha}, \qquad \xi \in \mathbb{R}^n,$$

holds for each $n \in \mathbb{N}$ and each $0 < \alpha < n$. Hence, as the Fourier transform takes the convolution of two functions to the product of their Fourier transforms (convolution theorem), we have that

$$\mathcal{F}[I_\alpha f](\xi) = \mathcal{F}[c_{n,\alpha}|\cdot|^{-n+\alpha} * f](\xi) = c_{n,\alpha}\mathcal{F}(|\cdot|^{-n+\alpha})\hat{f}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi).$$

Thus, by taking into account that we want to define $(-\Delta)^{\alpha/2}$ as the inverse of I_α, and by writing $\alpha/2 \mapsto s$, we can define the operator $(-\Delta)^s$ as follows.

Definition 1. Given $0 < s < 1$, we define the fractional Laplacian as the operator $(-\Delta)^s$ satisfying

$$\mathcal{F}[(-\Delta)^s f](\xi) = |\xi|^{2s}\hat{f}(\xi), \qquad \xi \in \mathbb{R}^n,\ f \in \mathscr{S}(\mathbb{R}^n).$$ ◂

With this definition, we have

$$\mathcal{F}[I_{2s} \circ (-\Delta)^s f](\xi) = |\xi|^{-2s}\mathcal{F}[(-\Delta)^s f](\xi) = |\xi|^{-2s}|\xi|^{2s}\hat{f}(\xi) = \hat{f}(\xi),$$

so by taking the Fourier transform we have that, indeed, $(-\Delta)^s$ and I_{2s} are inverse operators in $\mathscr{S}(\mathbb{R}^n)$.

It is a well-known fact that the usual Laplacian satisfies

$$\Delta(u \circ T) = \Delta u \circ T, \qquad T \in \mathbb{O}(n), \tag{3}$$

i. e., the Laplacian commutes with elements of the orthogonal group. The same property can be easily obtained for the fractional Laplacian $(-\Delta)^s$ thanks to the invariance of the Fourier transform with respect to these transformations. Indeed, since $\mathcal{F}(f \circ T) = \mathcal{F}(f) \circ T$ and $\mathcal{F}^{-1}(f \circ T) = \mathcal{F}^{-1}(f) \circ T$ for every $T \in \mathbb{O}(n)$ and every $f \in \mathscr{S}(\mathbb{R}^n)$, we can write, for each $T \in \mathbb{O}(n)$ and each $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}(-\Delta)^s(u \circ T)(x) &= \mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(u \circ T)(\cdot)](x) = \mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(u) \circ T(\cdot)](x) \\ &= \mathcal{F}^{-1}[|T(\cdot)|^{2s}\mathcal{F}(u) \circ T(\cdot)](x) = \mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(u)(\cdot)] \circ T(x) = (-\Delta)^s u \circ T(x),\end{aligned}$$

where we have used that $|x| = |T(x)| = |T^{-1}(x)|$ for each $x \in \mathbb{R}^n$. This in particular proves that, if f has radial symmetry (*i. e.* if $f \circ T = f$ for every $T \in \mathbb{O}(n)$), then $(-\Delta)^s f$ also has radial symmetry. Note that, for the Fourier transform, the invariance with respect to elements of the orthogonal group and the fact that $\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x)$ imply that the Fourier transform and the inverse Fourier transform of a function with radial symmetry coincide.

3. Second definition: pointwise formula

In this section we will describe a probabilistic situation in which the fractional Laplacian appears in a natural way. In this situation, the operator represents the rate of change in time of the probability of a particle to be in a particular position at a particular moment if it moves according to a certain process. This situation is depicted in Bucur and Valdinoci's book [2].

We are going to discretize the movement of the particle in such a way that $\tau > 0$ is the discrete step in time and $h > 0$ is the discrete space step. We will use the time step $\tau = h^{2s}$ for a fixed space step h. Let us denote by $u(x,t)$ the probability of finding the particle in position x at time t.

Let us define, for each $0 < s < 1$, the following probability on $\mathbb{N}$. For each subset I of natural numbers, we define

$$P(I) = c_s \sum_{k\in I} \frac{1}{k^{1+2s}},$$

where $c_s^{-1} \coloneqq \sum_{k\in\mathbb{N}} \frac{1}{k^{1+2s}}$.

The particle under study moves according the following probability law: in each time step τ, the particle chooses a direction $v \in \mathbb{S}^{n-1}$ randomly according to a uniform distribution on the unit sphere $\mathbb{S}^{n-1}$, and a natural number $k \in \mathbb{N}$ according to the probability law P depicted above, and then it performs a translation by the vector khv. Note that, in this motion, large jumps are allowed, but their probability is very low.

According to this, a particle in position x_0 at time t, after a time step τ (*i. e.*, in time $t+\tau$), will be placed in position $x_0 + hkv$ for some $k \in \mathbb{N}$ and some $v \in \mathbb{S}^{n-1}$. Then, given $x \in \mathbb{R}^n$ and $t, \tau > 0$, the probability that the particle is in position x after a time step τ from the initial time t, $u(x, t+\tau)$, is the sum of the probabilities of finding the particle in position $x + hkv$ in the previous time for some $k \in \mathbb{N}$ and some $v \in \mathbb{S}^{n-1}$ multiplied by the probability of having chosen that direction v and that natural number k, *i. e.*,

$$u(x, t+\tau) = \frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv, t)}{k^{1+2s}} \, \mathrm{d}\sigma(v),$$

where σ is the surface measure of the sphere.

On one hand we can write

$$u(x, t+\tau) - u(x,t) = \frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv, t) - u(x,t)}{k^{1+2s}} \, \mathrm{d}\sigma(v),$$

and, on the other hand, by the radial symmetry of the process,

$$u(x, t+\tau) - u(x,t) = \frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x-hkv, t) - u(x,t)}{k^{1+2s}} \, \mathrm{d}\sigma(v).$$

If we add the above and divide by 2,

$$u(x, t+\tau) - u(x,t) = \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv, t) + u(x-hkv,t) - 2u(x,t)}{k^{1+2s}} \, \mathrm{d}\sigma(v).$$

Now, dividing by $\tau = h^{2s}$ on both sides of the above inequality, we arrive at

$$\frac{u(x, t+\tau) - u(x,t)}{\tau} = \frac{h}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv, t) + u(x-hkv,t) - 2u(x,t)}{(hk)^{1+2s}} \, \mathrm{d}\sigma(v).$$

Here we recognize a Riemann sum. By writing $hk \mapsto r$, and taking into account that $\tau = h^{2s}$, we can take the (formal) limit when h goes to 0 on both sides to obtain

$$\begin{aligned}
\partial_t u(x,t) &= \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{u(x+rv, t) + u(x-rv,t) - 2u(x,t)}{r^{1+2s}} \, \mathrm{d}\sigma(v) \, \mathrm{d}r \\
&= \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \int_0^\infty r^{n-1} \int_{\mathbb{S}^{n-1}} \frac{u(x+rv, t) + u(x-rv,t) - 2u(x,t)}{r^{n+2s}} \, \mathrm{d}\sigma(v) \, \mathrm{d}r \\
&= \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{R}^n} \frac{u(x+y, t) - 2u(x,t) + u(x-y,t)}{|y|^{n+2s}} \, \mathrm{d}y,
\end{aligned}$$

where we have used polar coordinates.

Thus, the operator L_1^s given by

$$L_1^s u(x) := -\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{n+2s}} \, \mathrm{d}y,$$

where $c_{n,s}$ is a positive constant (which we will choose later), depicts (up to a constant) the rate of change in time of the probability of the particle being in a certain position in a certain moment according to the laws we have just described. As we will see in section 4 below, this operator L_1^s coincides with the fractional Laplacian $(-\Delta)^s$, so we have got a probabilistic interpretation for this operator. Note that, according to the obtained expression, L_1^s is a nonlocal operator, as in order to obtain its value at a point, we need to know the value of the original function in the whole space.

Even though the above computations are just formal, the definition of L_1^s makes sense as, for functions in $\mathscr{S}(\mathbb{R}^n)$, the integral there converges. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\quad = \int_{|y|\le 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y + \int_{|y|\ge 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\quad = I + II.
\end{aligned}$$

Since $u \in \mathscr{S}(\mathbb{R}^n)$, by using the Taylor expansion, we know that, if $|y| \le 1$, then $2u(x) - u(x+y) - u(x-y) = -2\langle \nabla^2 u(x) y, y\rangle + o(|y|^2)$ for each $x \in \mathbb{R}^n$, so

$$\begin{aligned}
I &= \int_{|y|\le 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&= \int_{|y|\le 1} \frac{|2\langle \nabla^2 u(x) y, y\rangle + o(|y|^2)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\le \int_{|y|\le 1} \frac{2|\nabla^2 u(x)||y|^2 + |o(|y|^2)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\le C_x \int_{|y|\le 1} \frac{1}{|y|^{n-2(1-s)}} \, \mathrm{d}y < \infty,
\end{aligned} \tag{4}$$

as $0 < s < 1$.

The boundedness of II is simpler:

$$II = \int_{|y|\ge 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \le 4\|u\|_{L_\infty(\mathbb{R}^n)} \int_{|y|\ge 1} \frac{1}{|y|^{n+2s}} \, \mathrm{d}y < \infty.$$

Remark 2. Observe that in the above argument, we only use that u is a bounded $C^2(\mathbb{R}^n)$ function. Thus, L_1^s makes sense in a bigger class than $\mathscr{S}(\mathbb{R}^n)$. ◂

This pointwise formula allows us to prove in a very simple way the behavior of this operator with respect to translations $\tau_h f(x) = f(x+h)$, $h \in \mathbb{R}^n$; dilations $\Delta_\lambda f(x) = f(\lambda x)$, and transformations of the orthogonal group $\mathbb{O}(n)$ (*i. e.*, the isometries of $\mathbb{R}^n$ which fix the origin, as, for instance, rotations or reflections).

Proposition 3. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. For each $h \in \mathbb{R}^n$ and each $\lambda > 0$, we have that*

$$L_1^s(\tau_h u) = \tau_h(L_1^s u) \quad \text{and} \quad L_1^s(\Delta_\lambda u) = \lambda^{2s} \Delta_\lambda [L_1^s u]. \tag{5}$$

Proof. The behavior of L_1^s with respect to translations is easily obtained by direct computations using the definition of the operator. The behavior with respect to dilations is straightforward to obtain as well.

Let us see the proof of the behavior with respect to dilations in order to illustrate the simplicity of these computations. Let us fix $x \in \mathbb{R}^n$. By using the change of variables $z = \lambda y$, we have

$$\begin{aligned}
L_1^s(\Delta_\lambda u)(x) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2\Delta_\lambda u(x) - \Delta_\lambda u(x+y) - \Delta_\lambda u(x-y)}{|y|^{n+2s}}\,\mathrm{d}y \\
&= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2u(\lambda x) - u(\lambda x+\lambda y) - u(\lambda x-\lambda y)}{|y|^{n+2s}}\,\mathrm{d}y \\
&= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2u(\lambda x) - u(\lambda x+z) - u(\lambda x-z)}{\left(\frac{|z|}{\lambda}\right)^{n+2s}}\,\frac{\mathrm{d}z}{\lambda^n} \\
&= \lambda^{2s}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2u(\lambda x) - u(\lambda x+z) - u(\lambda x-z)}{|z|^{n+2s}}\,\mathrm{d}z \\
&= \lambda^{2s}L_1^s u(\lambda x) = \lambda^{2s}\Delta_\lambda[L_1^s u](x). \qquad \blacksquare
\end{aligned}$$

In the following two sections we will give two new definitions of the fractional Laplacian and we will prove that they are equivalent to the definition of L_1^s and the one of $(-\Delta)^s$. First, we will rewrite L_1^s.

Theorem 4. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. For each $x \in \mathbb{R}^n$ we have*

$$L_1^s u(x) = c_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y, \tag{6}$$

where PV *is the Cauchy principal value,* i. e.*,*

$$\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y = \lim_{\epsilon\to 0^+}\int_{|x-y|>\epsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y \tag{7}$$

and $c_{n,s}$ is a constant that depends on the dimension n and the order of the fractional Laplacian s.

Proof. Let $x \in \mathbb{R}^n$. As the integral defining L_1^s converges, we can rewrite it as follows:

$$\begin{aligned}
L_1^s u(x) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,\mathrm{d}y \\
&= \frac{c_{n,s}}{2}\lim_{\epsilon\to 0^+}\int_{|y|>\epsilon} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,\mathrm{d}y \\
&= \frac{c_{n,s}}{2}\lim_{\epsilon\to 0^+}\int_{|y|>\epsilon} \frac{u(x)-u(x+y)}{|y|^{n+2s}}\,\mathrm{d}y + \frac{c_{n,s}}{2}\lim_{\epsilon\to 0^+}\int_{|y|>\epsilon} \frac{u(x)-u(x-y)}{|y|^{n+2s}}\,\mathrm{d}y.
\end{aligned}$$

If we use the changes of variables $z = x+y$ and $z = x-y$, respectively, we can write the previous sum as

$$\frac{c_{n,s}}{2}\lim_{\epsilon\to 0^+}\int_{|z-x|>\epsilon} \frac{u(x)-u(z)}{|z-x|^{n+2s}}\,\mathrm{d}z + \frac{c_{n,s}}{2}\lim_{\epsilon\to 0^+}\int_{|x-z|>\epsilon} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z.$$

By grouping all the terms in one integral, we finally obtain the desired result:

$$L_1^s u(x) = c_{n,s}\lim_{\epsilon\to 0^+}\int_{|x-z|>\epsilon} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z. \qquad \blacksquare$$

Remark 5. Note that when we first defined the operator L_1^s, we proved that this definition made sense in the Schwartz class by using that, in the Taylor expansion of $u(x)-u(x+y)-u(x-y)$ used in (4), the linear term disappears, while in the formulation of theorem 4 we do not have this property at hand; hence we need to plug in the principal value. ◂

4. Third definition: the heat semigroup. Equivalence of definitions

In this section we will give a new definition of the fractional Laplacian in terms of the heat semigroup and then we will prove that this and the ones introduced above through the Fourier transform and the pointwise formula are equivalent and define the same operator.

Let us take a positive second order differential operator $L = L_x$ acting on functions in the spatial variable x defined on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Let us consider the problem

$$\begin{cases} v_t(x,t) + Lv(x,t) = 0, & \text{for } (x,t) \in \Omega \times (0,\infty), \\ v(x,0) = u(x), & \text{for } x \in \Omega. \end{cases}$$

Inspired by the form of solutions of linear first order differential equations, given an initial data u for the previous problem, we define the operator e^{-tL} by

$$e^{-tL}u(x) = v(x,t), \qquad x \in \Omega, t \geq 0, \tag{8}$$

where v is the solution to the previous problem corresponding to the initial data u.

If we now think of the family $\{e^{-tL},\ t \geq 0\}$, it turns out that it is a *semigroup of class* (C_0) [14, Def. 1, Ch. 9]. Indeed, if v is the solution to the previous problem with initial data u and we consider $t_2 \in (0,\infty)$, then the function $v^{t_2}(x,t) \coloneqq v(x, t+t_2)$ satisfies the differential equation and also verifies $v^{t_2}(x,0) = v(x,t_2) = e^{-t_2 L}u(x)$. Then,

$$e^{-t_1 L}(e^{-t_2 L}u(x)) = v^{t_2}(x,t_1) = v(x,t_1+t_2) = e^{-(t_1+t_2)L}u(x), \qquad x \in \Omega,\ t_1, t_2 \in (0,\infty).$$

If we take $L = -\Delta$ acting in the spatial variable x, we have defined the operator $e^{t\Delta}$ in the way we just depicted in (8). In this case, a more specific expression can be given by solving the initial value problem

$$\begin{cases} v_t(x,t) = \Delta v(x,t), & \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) = u(x), & \text{for } x \in \mathbb{R}^n. \end{cases}$$

By applying the Fourier transform with respect to the variable x, we can rewrite the previous problem as

$$\begin{cases} \hat{v}_t(\xi,t) = |\xi|^2 \hat{v}(\xi,t), & \text{for } (\xi,t) \in \mathbb{R}^n \times (0,\infty), \\ \hat{v}(\xi,0) = \hat{u}(\xi), & \text{for } \xi \in \mathbb{R}^n. \end{cases} \tag{9}$$

The resulting problem is a Cauchy problem associated to a homogeneous linear first order differential equation with initial value $\hat{u}$. Its solution is $\hat{v}(\xi,t) = e^{-t|\xi|^2}\hat{u}(\xi)$. By applying the inverse Fourier transform,

$$\begin{aligned} e^{t\Delta}u(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-t|\xi|^2}\hat{u}(\xi) e^{ix\cdot\xi}\, d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-t|\xi|^2} \left(\int_{\mathbb{R}^n} u(z) e^{-i\xi\cdot z}\, dz \right) e^{ix\cdot\xi}\, d\xi \\ &= \int_{\mathbb{R}^n} u(z) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-t|\xi|^2} e^{i(x-z)\cdot\xi}\, d\xi \right) dz = \int_{\mathbb{R}^n} u(z) W_t(x-z)\, dz, \end{aligned}$$

where

$$W_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

is the Gauss kernel.

Once we got an explicit expression for $e^{t\Delta}$, and inspired by the following numerical identity,

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}},$$

which is valid for any $0 < s < 1$ and $\lambda > 0$, we can give the following alternative definition for the fractional Laplacian.

Definition 6. For $0 < s < 1$ and $u \in \mathscr{S}(\mathbb{R}^n)$, we define the operator L_2^s as

$$L_2^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta}u(x) - u(x)) \frac{dt}{t^{1+s}}.$$ ◂

This definition can be justified by means of the spectral theorem, and can be found for instance in Stinga's thesis [13].

At this point, we prove the equivalence of the operators L_1^s and L_2^s to the fractional Laplacian $(-\Delta)^s$ which we defined in section 2. The following result can be found in Stinga's thesis [13, Lemma 2.1].

Theorem 7. *Let* $0<s<1$ *and* $u\in\mathscr{S}(\mathbb{R}^n)$*. Then,* $(-\Delta)^s u = L_1^s u = L_2^s u$*, i. e.,*

$$(-\Delta)^s u(x) = -\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{u(x+y)-2u(x)+u(x-y)}{|y|^{n+2s}}\,\mathrm{d}y = \frac{1}{\Gamma(-s)}\int_0^\infty (\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}}, \qquad x\in\mathbb{R}^n,$$

where

$$c_{n,s} = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)}.$$

Proof. We will prove that the last expression coincides with the other ones. Let us see first the equivalence between $(-\Delta)^s u$ and $L_2^s u$. By the Fourier inversion theorem,

$$\mathrm{e}^{t\Delta}u(x)-u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}(\mathrm{e}^{-t|\xi|^2}-1)\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi.$$

By using this and the change of variables $w=t|\xi|^2$, we obtain

$$\begin{aligned}\int_0^\infty |\mathrm{e}^{t\Delta}u(x)-u(x)|\,\frac{\mathrm{d}t}{t^{1+s}} &\le C_n\int_0^\infty\int_{\mathbb{R}^n}|\mathrm{e}^{-t|\xi|^2}-1||\hat{u}(\xi)|\,\mathrm{d}\xi\,\frac{\mathrm{d}t}{t^{1+s}}\\ &= C_n\int_{\mathbb{R}^n}\int_0^\infty|\mathrm{e}^{-w}-1|\,\frac{\mathrm{d}w}{w^{1+s}}|\xi|^{2s}|\hat{u}(\xi)|\,\mathrm{d}\xi\\ &= C_{n,s}\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{u}(\xi)|\,\mathrm{d}\xi<\infty,\end{aligned}$$

as $u\in\mathscr{S}(\mathbb{R}^n)$. Hence, by Fubini's theorem

$$\begin{aligned}\frac{1}{\Gamma(-s)}\int_0^\infty(\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}} &= \frac{1}{\Gamma(-s)}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_0^\infty(\mathrm{e}^{-t|\xi|^2}-1)\,\frac{\mathrm{d}t}{t^{1+s}}\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi\\ &= \frac{1}{\Gamma(-s)}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_0^\infty(\mathrm{e}^{-w}-1)\,\frac{\mathrm{d}w}{w^{1+s}}|\xi|^{2s}\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi = \mathcal{F}^{-1}\left(|\cdot|^{2s}\hat{u}\right)(x).\end{aligned}$$

Now, we see that $L_1^s u$ and $L_2^s u$ coincide for a suitable choice of $c_{n,s}$. More precisely, we will prove that

$$\frac{1}{\Gamma(-s)}\int_0^\infty(\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}} = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)}\mathrm{PV}\int_{\mathbb{R}^n}\frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z, \qquad x\in\mathbb{R}^n.$$

Let $\varepsilon>0$. By using the fact that $\|W_t(x-\cdot)\|_{L^1(\mathbb{R}^n)}=1$ for every $x\in\mathbb{R}^n$ and Fubini's theorem,

$$\begin{aligned}\int_0^\infty(\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}} &= \int_0^\infty\int_{\mathbb{R}^n}W_t(x-z)(u(z)-u(x))\,\mathrm{d}z\,\frac{\mathrm{d}t}{t^{1+s}}\\ &= \int_{\mathbb{R}^n}\int_0^\infty W_t(x-z)(u(z)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}z = I_\varepsilon+II_\varepsilon.\end{aligned}$$

On one hand,

$$\begin{aligned}I_\varepsilon &:= \int_{|x-z|>\varepsilon}\int_0^\infty(4\pi t)^{-n/2}\mathrm{e}^{-\frac{|x-z|^2}{4t}}(u(z)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}z\\ &= \int_{|x-z|>\varepsilon}(u(z)-u(x))\int_0^\infty(4\pi t)^{-n/2}\mathrm{e}^{-\frac{|x-z|^2}{4t}}\,\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}z\\ &= \int_{|x-z|>\varepsilon}(u(x)-u(z))\frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}}\frac{1}{|x-z|^{n+2s}}\,\mathrm{d}z,\end{aligned}$$

where the change of variables $r = \frac{|x-z|^2}{4t}$ is used. Note that, as u is bounded, I_ε is absolutely convergent for any $\varepsilon > 0$. On the other hand, by using polar coordinates,

$$\begin{aligned} II_\varepsilon &:= \int_0^\infty \int_{|x-z|<\varepsilon} W_t(x-z)(u(z)-u(x))\,\mathrm{d}z\,\frac{\mathrm{d}t}{t^{1+s}} \\ &= \int_0^\infty (4\pi t)^{-n/2} \int_0^\varepsilon \mathrm{e}^{-\frac{r^2}{4t}} r^{n-1} \int_{|z'|=1} (u(x+rz')-u(x))\,\mathrm{d}S(z')\,\mathrm{d}r\,\frac{\mathrm{d}t}{t^{1+s}}. \end{aligned}$$

Now, by Taylor's theorem and by using the symmetry of the sphere, we can write

$$\int_{|z'|=1} (u(x+rz')-u(x))\,\mathrm{d}S(z') = K_n r^2 \Delta u(x) + O(r^3),$$

with K_n some constant that we will specify later. Indeed, by Taylor's theorem we can write, for each $z' \in \mathbb{S}^{n-1}$,

$$u(x+rz') = u(x) + r\langle \nabla u(x), z'\rangle + \frac{r^2}{2}\langle \nabla^2 u(x) z', z'\rangle + O(r^3),$$

where $\langle\cdot\,,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. Taking into account the symmetry of the sphere, we know that the integral of an odd function over $\mathbb{S}^{n-1}$ is 0: this means, in our formula above, that the first order terms integrate zero over the sphere. Likewise, when we examine closely the cross derivative terms of $\nabla^2 u(x)$, they also accompany an odd function in the formula above and so they disappear when we take the integral. This computation can be easily done by taking polar coordinates. It can also be noticed that what we are left with is a multiple of $r\Delta u(x)$, as we wrote above.

Hence,

$$|II_\varepsilon| \le K_{n,\Delta u(x)} \int_0^\varepsilon r^{n+1} \int_0^\infty \frac{\mathrm{e}^{-\frac{r^2}{4t}}}{t^{n/2+s}}\,\frac{\mathrm{d}t}{t} = K_{n,\Delta u(x)} \int_0^\varepsilon r^{n+1} K_{n,s} r^{-n-2s}\,\mathrm{d}r = K_{n,\Delta u(x),s}\varepsilon^{2(1-s)}.$$

This proves that $II_\varepsilon \to 0$ when $\varepsilon \to 0$, so

$$\int_0^\infty (\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}} = \lim_{\varepsilon\to 0} I_\varepsilon + II_\varepsilon = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}}\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z. \qquad \blacksquare$$

With computations very similar to the ones we just did, we can prove that, whenever $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we have that $(-\Delta)^s u(x)$ converges to $u(x)$ when $s \to 0^+$, for every $x \in \mathbb{R}^n$. Actually, for a given $x \in \mathbb{R}^n$, to prove the aforementioned convergence, we would only need that u belongs to $C^2(B(x)) \cap L^\infty(\mathbb{R}^n)$, with $B(x)$ the ball of center x and radius 1 [13, Prop. 2.3]. Note also that, when $u \in \mathscr{S}(\mathbb{R}^n)$, the pointwise convergence is obvious by the definition via Fourier transform, in the same way that it is obvious that $(-\Delta)^s \to (-\Delta)$ when $s \to 1^-$.

5. Fourth definition: the extension problem

When one works with nonlocal operators as $(-\Delta)^s$, one of the principal difficulties which appears is the fact that they do not act on functions in the same way that differential operators do, but they are defined by integral formulas. As a consequence, we do not have some of the properties that local operators have. From the point of view of the tools to study these operators, it is desirable to have some procedure which allows us to connect a nonlocal problem with a local one at hand. The bibliography in the topic of differentiable problems is extensive, and then the set of techniques is very rich. With this motivation, we present the trace relation and the extension problem.

Caffarelli and Silvestre [3] introduced a method which allows to transform nonlocal problems in $\mathbb{R}^n$ into other ones in which some differential operator in $\mathbb{R}^{n+1}_+$ appears. The method is described as follows: given $0 < s < 1$ and $u \in \mathscr{S}(\mathbb{R}^n)$, we want to study the solution of the system

$$\begin{cases} L_{1-2s}U(x,y) := \mathrm{div}_{x,y}(y^{1-2s}\nabla_{x,y}U) = 0, & x \in \mathbb{R}^n_+, y > 0, \\ U(x,0) = u(x), \\ U(x,y) \to 0 \text{ when } y \to \infty. \end{cases} \tag{10}$$

By using the definition of the divergence, the system in (10) can be rewritten as

$$\text{(11)} \qquad \begin{cases} -\Delta_x U(x,y) = \left(\partial_{yy} + \frac{1-2s}{y}\partial_y\right) U(x,y), & x \in \mathbb{R}^n_+, y > 0, \\ U(x,0) = u(x), \\ U(x,y) \to 0 \text{ when } y \to \infty, \end{cases}$$

and the solution of (11) is given by the following result.

Theorem 8 (extension theorem). *The solution U of the extension problem (11) is given by the convolution*

$$\text{(12)} \qquad U(x,y) = (P_s(\,\cdot\,,y) * u)(x) = \int_{\mathbb{R}^n} P_s(x-z,y)u(z)\,\mathrm{d}z,$$

where

$$P_s(x,y) = \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{(n+2s)/2}}$$

is the generalized Poisson kernel *for the extension problem in the semispace* $\mathbb{R}^{n+1}_+$*. Moreover, for U defined as in (12) one has*

$$\text{(13)} \qquad (-\Delta)^s u(x) = -\frac{2^{s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y),$$

which is what we call the trace relation.

Remark 9. The extension theorem provides an interesting relation between the operators $(-\Delta)^s$ and ∂_y. This relation allows to obtain properties of the nonlocal operator from the properties of the local one. ◂

Proof of theorem 8. By taking the partial Fourier transform in the variable x in (11), we get the system

$$\text{(14)} \qquad \begin{cases} \partial_{yy}\hat{U}(\xi,y) + \frac{1-2s}{y}\partial_y\hat{U}(\xi,y) - |\xi|^2\hat{U}(\xi,y) = 0, & (\xi,y) \in \mathbb{R}^{n+1}_+, \\ \hat{U}(\xi,0) = \hat{u}(\xi), \quad \hat{U}(\xi,y) \to 0 \text{ when } y \to \infty, & \xi \in \mathbb{R}^n. \end{cases}$$

Now, if we fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and write $Y(y) = Y_\xi(y) := \hat{U}(\xi,y)$, the previous problem can be written as

$$\text{(15)} \qquad \begin{cases} y^2Y''(y) + (1-2s)yY'(y) - |\xi|^2y^2Y(y) = 0, & y \in \mathbb{R}_+, \\ Y(0) = \hat{u}(\xi), \quad Y(y) \to 0 \text{ when } y \to \infty. \end{cases}$$

The equation in the above problem, under some adjustment of the parameters, is known in the literature as the generalised modified Bessel equation (see for instance Lebedev's book [9]), and is given by

$$\text{(16)} \qquad y^2Y'' + (1-2\alpha)yY'(y) + [\beta^2\gamma^2y^{2\gamma} + (\alpha - \nu^2\gamma^2)]Y(y) = 0,$$

where

$$\alpha = s, \quad \gamma = 1, \quad \nu = s, \quad \beta = |\xi|.$$

It is well known (see Lebedev's book [9]) that (16) has two linear independent solutions given by

$$u_1(y) = y^s I_s(|\xi|y) \qquad \text{and} \qquad u_2(y) = y^s K_s(|\xi|y),$$

where I_s and K_s are Bessel functions of second and third kind, respectively:

$$\text{(17)} \qquad \begin{aligned} I_r(z) &= \left(\frac{z}{2}\right)^r \sum_{k=0}^\infty \frac{(z/2)^{2k}}{\Gamma(k+1)\Gamma(k+s+1)}, & |z| < \infty, |\arg(z)| < \pi; \\ K_r(z) &= \frac{\pi}{2}\frac{I_{-r}(z) - I_r(z)}{\sin \pi r}, & |\arg(z)| < \pi. \end{aligned}$$

Thus, for each $\xi \neq 0$, the solution of (15) is

$$\text{(18)} \qquad \hat{U}(\xi,y) = Ay^sI_s(|\xi|y) + By^sK_s(|\xi|y), \qquad (\xi,y) \in \mathbb{R}^{n+1}_+.$$

From the expressions in (17) one can obtain (see [9, formulas (5.11.9) and (5.11.10), p. 123]) the following asymptotics for $z \to \infty$:

$$I_s(z) \approx \mathrm{e}^{z}(2\pi z)^{-1/2}, \qquad K_s(z) \approx \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z}, \tag{19}$$

where we use the symbol $A \approx B$ to say that there exist constants $c, C > 0$ such that $cA \le B \le CA$. Thus, the function $I_s(z)$ diverges, while $K_s(z)$ takes finite values for z sufficiently large. From here, we can deduce that the condition that $\hat{U}(\xi, y) \to 0$ when $y \to \infty$ implies that, in (18), we must have $A = 0$. Therefore,

$$\hat{U}(\xi, y) = By^sK_s(|\xi|y), \qquad (\xi, y) \in \mathbb{R}^{n+1}_+. \tag{20}$$

On the other hand, it can be proved that, when $z \to 0^+$ in (17),

$$I_s(z) \approx \frac{1}{\Gamma(s+1)}\left(\frac{z}{2}\right)^{s} \qquad \text{and} \qquad I_{-s}(z) \approx \frac{1}{\Gamma(1-s)}\left(\frac{z}{2}\right)^{-s} \tag{21}$$

(c.f. [9, formula (5.7.1), p. 108]). Hence, by imposing the condition $\hat{U}(\xi, 0) = \hat{u}(\xi)$, we can fix B in the following way:

$$By^sK_s(|\xi|y) = B\frac{\pi}{2}\frac{y^sI_{-s}(|\xi|y) - y^sI_s(|\xi|y)}{\sin \pi s} \overset{y\to 0^+}{\to} \frac{B\pi 2^{s-1}}{\Gamma(1-s)\sin \pi s}|\xi|^{-s} = B2^{s-1}\Gamma(s)|\xi|^{-s}, \tag{22}$$

where in the last equality we have used the property of the Γ function that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad z \notin \mathbb{Z}. \tag{23}$$

Since $\hat{U}(\xi, 0) = \hat{u}(\xi)$, we have that

$$\hat{U}(\xi, y) = \frac{|\xi|^s\hat{u}(\xi)}{2^{s-1}\Gamma(s)}y^sK_s(|\xi|y). \tag{24}$$

Suppose that U is given by the convolution of u with a kernel $P_s(x, y)$ and let us see the explicit expression for this kernel. With this assumption, from the properties of the Fourier transform with respect to the convolution of functions, we deduce that $\hat{U}(\xi, y) = \hat{P}_s(\xi, y)\hat{u}(\xi)$, so

$$P_s(x, y) = \mathcal{F}^{-1}\left(\frac{\hat{U}(\,\cdot\,, y)}{\hat{u}(\,\cdot\,)}\right)(x) = \mathcal{F}^{-1}\left(\frac{|\cdot|^s}{2^{s-1}\Gamma(s)}y^sK_s(|\cdot|y)\right)(x).$$

Now, as the function $\frac{|\cdot|^s}{2^{s-1}\Gamma(s)}y^sK_s(|\cdot|y)$ is a function with spherical symmetry, we know that its Fourier transform coincides with its inverse Fourier transform, so finding an expression for the kernel $P_s(x, y)$ is equivalent to computing the Fourier transform (in the case in which the Fourier transform is applied to radial functions, it is called the Hankel transform) of the function $\frac{|\cdot|^s}{2^{s-1}\Gamma(s)}y^sK_s(|\cdot|y)$.

For this, we use the fact that the Hankel transform of a radial function $f(\,\cdot\,) = f_0(|\cdot|)$ is (see Stein and Weiss's book [12, Ch. IV, Th. 3.3])

$$\mathcal{F}(f_0)(r) = \frac{1}{(2\pi)^{\frac{n}{2}}r^{\frac{n-2}{2}}}\int_0^\infty f_0(s)J_{\frac{n-2}{2}}(rs)s^{\frac{n}{2}}\,\mathrm{d}s.$$

Here, $J_{\frac{n-2}{2}}$ denotes the Bessel function, defined for k a real number greater than 1/2 by letting

$$J_k(t) := \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma(\frac{1}{2})}\int_{-1}^{1} \mathrm{e}^{its}(1-s^2)^{(2k-1)/2}\,\mathrm{d}s.$$

Thus, by applying this to our function and by using [5, formula 3 in 6.576, p. 684], we get

$$\begin{aligned}
\mathcal{F}\left(\frac{|\cdot|^s}{2^{s-1}\Gamma(s)}y^sK_s(|\cdot|y)\right)(x) &= \mathcal{F}\left(\frac{|\cdot|^s}{2^{s-1}\Gamma(s)}y^sK_s(|\cdot|y)\right)(|x|)\\
&= \frac{y^s}{2^{s-1}\Gamma(s)(2\pi)^{\frac{n}{2}}|x|^{\frac{n-2}{2}}}\int_0^\infty |\xi|^{\frac{n}{2}+s}K_s(|\xi|y)J_{\frac{n-2}{2}}(|x|s)\,\mathrm{d}|\xi|\\
&= \frac{\Gamma(\frac{n}{2}+s)}{\pi^{n/2}}\frac{y^{2s}}{(y^2+|x|^2)^{(n+2s)/2}}.
\end{aligned}$$

Only (13) is left, *i. e.*, we just have to check the equality

$$(-\Delta)^s u(x) = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y).$$

Let us recall that the Fourier transform of the function $(-\Delta)^s u(x)$ is $|\xi|^{2s}\hat{u}(\xi)$. Then, (13) is equivalent to seeing that

$$|\xi|^{2s}\hat{u}(\xi) = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\frac{\partial \hat{U}}{\partial y}(\xi,y). \tag{25}$$

By using the following identities for the Bessel function of third kind (c.f. [9, formula (5.7.9), p. 110)],

$$K_s'(z) = \frac{s}{z}K_s(z) - K_{s+1}(z) \tag{26}$$

and

$$\frac{2s}{z}K_s(z) - K_{s+1}(z) = -K_{s-1}(z) = -K_{1-s}(z), \tag{27}$$

together with (23), we get

$$y^{1-2s}\partial_y \hat{U}(\xi,y) = \frac{|\xi|^{s+1}\hat{u}(\xi)}{2^{s-1}\Gamma(s)}y^{1-s}\left(\frac{2s}{y}K_s(|\xi|y) - K_{s+1}(|\xi|y)\right) = -\frac{|\xi|^{s+1}\hat{u}(\xi)}{2^{s-1}\Gamma(s)}y^{1-s}K_{1-s}(|\xi|y).$$

In view of the behaviour of the Bessel function of third kind K_s, we have that

$$\lim_{y\to 0^+} y^{1-s}K_{1-s}(|\xi|y) = \frac{\Gamma(1-s)|\xi|^{s-1}}{2^s},$$

so we get

$$\lim_{y\to 0^+} y^{1-2s}\partial_y \hat{U}(\xi,y) = -\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}|\xi|^{2s}\hat{u}(\xi).$$

This proves (13), thus finishing the proof. ∎

Alternative proof of (13). By using the following property of the generalised Poisson kernel,

$$\int_{\mathbb{R}^n} P_s(x,y)\,\mathrm{d}x = 1, \qquad y > 0, \tag{28}$$

we can obtain an alternative proof where we do not need to use (26) and (27).

Let $u \in \mathscr{S}(\mathbb{R}^n)$ and consider a solution

$$U(x,y) = (P_s(\,\cdot\,,y) * u)(x)$$

of the extension problem (12). Observe that, by using (28), we can write

$$U(x,y) = \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \int_{\mathbb{R}^n} \frac{(u(z)-u(x))y^{2s}}{(y^2+|x-z|^2)^{(n+2s)/2}}\,\mathrm{d}z + u(x).$$

By differentiating the two sides with respect to y, we obtain

$$y^{1-2s}\partial_y U(x,y) = 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \int_{\mathbb{R}^n} \frac{u(z)-u(x)}{(y^2+|z-x|^2)^{(n+2s)/2}}\,\mathrm{d}z + O(y^2).$$

If we let $y \to 0^+$ and we use the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y) &= 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(z)-u(x)}{(|z-x|^2)^{(n+2s)/2}}\,\mathrm{d}z \\ &= -2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}c_{n,s}^{-1}(-\Delta)^s u(x), \end{aligned} \tag{29}$$

where in the second equality we have used the definition of fractional Laplacian (6).

Finally, recall that

$$c_{n,s} = \frac{s2^{2s}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)}, \tag{30}$$

so, by substituting the expression of (30) in (29) we obtain

$$\lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y) = -2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}c_{n,s}^{-1}(-\Delta)^s u(x) = -\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}(-\Delta)^s u(x),$$

which completes the alternative proof. ∎

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