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Holomorphic functional calculus for sectorial operators

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Abstract: This article is about holomorphic functional calculus on Hilbert spaces. It is an expository paper where we survey the main results in this setting and we present some examples. We define the natural functional calculus for sectorial operators and give an example of a sectorial operator that does not admit a functional calculus.

Resumen: Este artículo trata sobre el cálculo funcional holomorfo en espacios de Hilbert. Es un artículo expositorio donde tratamos los resultados principales y presentamos algunos ejemplos. Definimos el cálculo funcional natural para operadores sectoriales y damos un ejemplo de un operador sectorial que no admite un cálculo funcional.

Keywords: holomorphic functional calculus, sectorial operators, Riesz-Dunford.

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1. Introduction

In this expository paper we develop the main ideas about holomorphic functional calculus for sectorial operators on Hilbert spaces. Of course, we cannot be exhaustive and make a complete presentation of the theory. We present the sketch of some proofs, and the complete demonstration of some results. Our purpose is to show some of the usual procedures and tools in this context. The interested reader can consult, for instance, the references listed at the end.

The functional calculus that we study was formalized in the late 80's and 90's mainly by McIntosh [1, 4, 17, 18]. The motivation was in Kato's square root problem and the operator-method approach to evolution equations of Grisvard and Da Prato. Many examples within the class of operators under consideration can be found among the partial differential operators (elliptic differential operators, Schrödinger operators with singular potentials, Stokes operators...).

The purpose of a functional calculus is to give a meaning to $f(T)$, where f is a complex function defined on a subset of $\mathbb{C}$ and T is an operator defined on a Hilbert space.

For example, in a finite dimensional Hilbert space, we can find exponentials and logarithms of matrices through the theory of linear systems of differential equations. These operations can be regarded as a special case of a functional calculus for operators in a finite dimensional setting.

Another example is the well-known equality

$$\Delta f = \mathcal{F}^{-1}\big(|y|^2\mathcal{F}(f)\big), \tag{1}$$

where Δ represents the Laplace operator, $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$. This equality holds provided that f satisfies certain regularity and decay conditions. From (1) it follows that, if P is a polynomial, we can compute $P(\Delta)$ using the following expression:

$$P(\Delta)f = \mathcal{F}^{-1}(P(|y|^2)\mathcal{F}(f)). \tag{2}$$

Now, if we want to define $m(\Delta)$ for a general complex function m, (2) suggests the following definition:

$$m(\Delta)f = \mathcal{F}^{-1}(m(|y|^2)\mathcal{F}(f)).$$

This is the first step to construct a functional calculus for the Laplace operator. In a second step, we need to specify the Hilbert space H where f belongs to and the class ξ of admissible functions m in this functional calculus. Actually, (1) can be seen as a spectral representation for the operator Δ. This fact allows to extend these ideas and to define a functional calculus for operators on Hilbert spaces by using spectral representations.

The abstract method of defining a functional calculus follows ideas from Haase [8], and it is roughly done in the following way. If T is an operator in a Hilbert space H, we consider a class ξ of functions defined on the spectrum $\sigma(T)$ of T and a mapping $\Phi\colon\ \xi \to B(H)$, where $B(H)$ represents the space of bounded and linear operators on H. Φ is actually a method to assign an operator $f(T)$ defined by $\Phi(f)$. In a first approach, the class ξ is formed by smooth enough functions. Later on, this class of functions will be enlarged by including less regular functions. In this second step, $f(T)$ may be unbounded if T is not in $B(H)$. Since the purpose of functional calculus is to make computations, it is convenient that ξ is an algebra and Φ is an algebra homomorphism. Furthermore, the mapping Φ should be somehow connected to the operator T (note that we write $f(T)$). One of such relations should be that $\Phi((\lambda - z)^{-1}) = R(\lambda, T)$, where $\lambda \in \rho(T)$ and $R(\lambda, T) = (\lambda I - T)^{-1}$ is the resolvent operator. Haase [8] presented an abstract approach to the construction of functional calculus.

This paper is structured as follows. In section 2 we present the holomorphic functional calculus. The definition and the main properties of the sectorial operators on Hilbert spaces are presented in section 3. The holomorphic function algebras, basic for functional calculus, are studied in section 4. Section 5 is focused on the definitions and properties of the holomorphic functional calculus for sectorial operators. Finally, in section 6 we discuss an example: a sectorial operator in a separable Hilbert space that does not have a bounded holomorphic functional calculus.

The theory developed in this paper is concerned with operators in Hilbert spaces and holomorphic functions. The standard definitions and properties about these topics that are used in the sequel can be found in the monographs of Rudin [19, 20]. A detailed study about the holomorphic functional calculus for sectorial operators appears, for instance, in [7, 14, 15, 24], where most of the results in this paper are included.

2. Holomorphic functional calculus

Let H be a Hilbert space. In this section, we assume that $T\colon H \to H$ is a bounded linear operator.

Let f be a holomorphic function defined on $\mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, where R is the radius of convergence of f. Let us also suppose $R > \|T\|$, where $\|T\| = \sup_{z\in H\setminus\{0\}} \frac{\|Tz\|_H}{\|z\|_H}$ is the operator norm of T.

A natural way of defining $f(T)$ is the following one. It is clear that the series defined by

$$f(T) = \sum_{n=0}^{\infty} a_n T^n$$

is convergent in $B(H)$ and $f(T) \in B(H)$. Moreover, if $P_{\|T\|}$ denotes the space of holomorphic functions on $\{z \in \mathbb{C} : |z| < R\}$ for some $R > \|T\|$, the mapping

$$\begin{aligned} \Psi\colon P_{\|T\|} &\longrightarrow B(H) \\ f &\longmapsto f(T) \end{aligned}$$

is an algebra homomorphism. Notice that this way, the definition of $f(T)$ coincides with the natural definition whenever f is a polynomial.

There is also a natural way of defining $f(T)$ when f is a rational function. Let p, q be polynomials such that q does not vanish in $\sigma(T)$. Then, $r = p/q$ is a rational function with no poles in $\sigma(T)$. Write $q(z) = \prod_{j=1}^{n}(\alpha_j - z)$. Since $\alpha_j \notin \sigma(T)$, we have that the resolvent operator $R(\alpha_j, T) = (\alpha_j I - T)^{-1}$ is a bounded operator on H. Therefore, one can define $r(T)$ as

$$r(T) = p(T) \prod_{j=1}^{n} R(\alpha_j, T).$$

Moreover, $r(T) \in B(H)$ and the mapping

$$\begin{aligned} \Psi\colon R_{\sigma(T)} &\longrightarrow B(H) \\ r &\longmapsto r(T) \end{aligned}$$

is and algebra homomorphism. Here $R_{\sigma(T)}$ denotes the spaces of rational functions without poles in $\sigma(T)$.

Our objective is to define a functional calculus extending the two above particular cases (power series with a radius of convergence greater than $\|T\|$ and rational functions without poles in $\sigma(T)$). In order to do this, we consider holomorphic functions in some neighbourhood of the spectrum of $T \in B(H)$.

Let T be a bounded operator in H and $\Omega \subset \mathbb{C}$ an open set containing $\sigma(T)$. For each connected component Ω_i of Ω, we consider a closed contour γ_i in Ω_i around $\sigma(T) \cap \Omega_i$. We let γ be the union of all the γ_i. In particular, γ is a finite collection of smooth closed paths that is contained in $\Omega \setminus \sigma(T)$, see figure 1. Then, we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\mathrm{d}z}{z - \alpha} = \begin{cases} 1, & \alpha \in \sigma(T), \\ 0, & \alpha \notin \Omega. \end{cases}$$

Suppose that f is a holomorphic function in Ω; in short, $f \in \mathcal{H}(\Omega)$. Motivated by the Cauchy integral formula, we define

$$f(T) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(z) R(z, T)\, \mathrm{d}z,$$

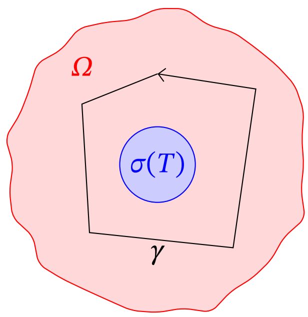

Figure 1: The contour γ around $\sigma(T)$.

where $R(z,T) = (zI - T)^{-1}$ is the resolvent operator. The integral is understood in the $B(H)$-Böchner sense [3]. Since $f \in H(\Omega)$ and γ is contained in $\Omega \setminus \sigma(T)$, the Cauchy integral theorem and Hahn-Banach theorem imply that the integral defining $f(T)$ does not depend on the contour γ satisfying the above conditions. Moreover, $f(T) \in B(H)$.

Theorem 1. *Let $T \in B(H)$. Assume that Ω is an open set in $\mathbb{C}$ containing $\sigma(T)$. Then, the mapping*

$$\begin{aligned} \Phi : \mathcal{H}(\Omega) &\longrightarrow B(H) \\ f &\longmapsto f(T) \end{aligned}$$

satisfies the following properties:

(1) *Φ is an algebra homomorphism.*

(2) *$\Phi(p) = p(T)$, for every polynomial p.*

(3) *If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}(\Omega)$, $f \in \mathcal{H}(\Omega)$ and $f_n \to f$, as $n \to \infty$, uniformly on every compact subset of Ω, then $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$ in $B(H)$.*

Moreover, Φ is the unique mapping from $H(\Omega)$ into $B(H)$ satisfying the properties (1), (2) and (3).

This holomorphic functional calculus is also called the Dunford–Riesz functional calculus. Note that the two previous examples are special cases of the Dunford–Riesz functional calculus. Indeed, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with radius of convergence greater than $\|T\|$, by taking $\Omega = B(0,R)$, we have that $\Phi(f) = \sum_{n=0}^{\infty} a_n T^n$. Also, if $r \in R_{\sigma(T)} \cap \mathcal{H}(\Omega)$ with $\sigma(T) \subset \Omega$, and r is a rational function without poles in $\sigma(T)$, then $\Phi(r) = p(T)\prod_{i=1}^{n}(\alpha_i I - T)^{-1}$. Here, we are assuming $r(z) = \frac{p(z)}{\prod_{i=1}^{n}(\alpha_i - z)}$, where $p(z)$ is a polynomial over $\mathbb{C}$.

The spectral mapping theorem holds for Φ, that is, if Ω is an open set in $\mathbb{C}$ containing $\sigma(T)$ and $f \in H(\Omega)$, then $f(\sigma(T)) = \sigma(f(T))$ [20, Theorem 10.28(b)].

We are going to give an idea on how to extend the holomorphic functional calculus to closed operators. Let T be a closed operator and Ω an open set such that $\sigma(T) \subset \Omega$ and the resolvent set $\rho(T) \cap (\mathbb{C} \setminus \Omega) \neq \varnothing$. We choose $\alpha \in \rho(T) \cap (\mathbb{C} \setminus \Omega)$ and we consider the function

$$R_\alpha(z) = \begin{cases} \frac{1}{z-\alpha} & \text{if } z \in \mathbb{C} \setminus \{\alpha\}, \\ \infty & \text{if } z = \alpha, \\ 0 & \text{if } z = \infty. \end{cases}$$

It is clear that $R_\alpha(\Omega) = W$ is an open set in $\mathbb{C} \setminus \{0\}$. Also, we have that $-R_\alpha(\sigma(T)) = \sigma((\alpha I - T)^{-1})$. Since $\sigma(T) \subset \Omega$, $\sigma((\alpha I - T)^{-1}) \subseteq -W$. We define

$$f(T) := (f \circ (-R_\alpha^{-1})((\alpha I - T)^{-1}),$$

where we use the holomorphic functional calculus defined in theorem 1 on the right hand side.

3. Sectorial operators

In this section we introduce sectorial operators and we discuss their main properties. These operators may not be defined on the whole Hilbert space H but on a dense subspace that we call domain of T (and denote by $D(T)$). Moreover, they may be unbounded. For a (possibly unbounded) linear operator $T\colon D(T) \to H$, we define its spectrum $\sigma(T)$ as the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \mathrm{Id}$ has a bounded inverse.

If $0 < \omega \leq \pi$, by S_ω we denote the open sector, that is symmetric with respect the positive real axis and with opening angle ω, that is,

$$S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\mathrm{Arg}(z)| < \omega\}.$$

Also, we define $S_0 = (0, \infty)$. $\overline{S_\omega}$ denotes the closure of S_ω, for every $0 \leq \omega \leq \pi$.

Definition 2. Let $T\colon D(T) \to H$ be a linear operator. We say that T is a **sectorial operator of angle** $\boldsymbol{\omega} \in [0, \pi)$ (in short, $T \in \mathrm{Sect}(\omega)$) when the following two properties hold:

1) $\sigma(T) \subset \overline{S_w}$,
2) for every $\omega < \alpha < \pi$,

$$M(T, \alpha) := \sup\{\|z R(z, T)\| : z \notin \overline{S_\alpha}\} < \infty.$$

◂

Note that if $T \in \mathrm{Sect}(\omega)$ for some $\omega \in [0, \pi)$, then $(-\infty, 0) \subseteq \rho(T)$ and T is a closed operator. We name sectoriality angle ω_T of the operator T to the number

$$\omega_T = \min\{0 \leq \omega < \pi : T \in \mathrm{Sect}(\omega)\}.$$

We now give a few examples of sectorial operators.

Example 3. If $T \in B(H)$ is self-adjoint and positive, then $T \in \mathrm{Sect}(0)$. ◂

Example 4. Let $0 < \omega < \frac{\pi}{2}$. A family $\{T(z) : z \in S_\omega\} \subseteq B(H)$ is said to be a holomorphic semigroup when

1) $T(z_1)T(z_2) = T(z_1 + z_2)$, $z_1, z_2 \in S_w$,
2) the mapping

$$\begin{aligned} \Psi\colon S_\omega &\longrightarrow B(H) \\ z &\longmapsto T(z) \end{aligned}$$

is holomorphic.

The infinitesimal generator of $\{T(z)\}$ is the operator A defined by

$$Ax = \lim_{t \to 0^+} \frac{T(t) - I}{t} x, \quad x \in D(A),$$

where the domain $D(A)$ of A is formed by all those $x \in H$ such that the limit $\lim_{t\to 0^+} \frac{T(t)x - x}{t}$ exists. Then, the operator $-A$ is in $\mathrm{Sect}(\frac{\pi}{2} - \omega)$. ◂

Example 5. We consider on $\mathbb{C}^2$ the usual inner product defined by

$$(u, v) \cdot (a, b) = u\bar{a} + v\bar{b}, \quad u, v, a, b \in \mathbb{C}.$$

We define the space $\ell^2(\mathbb{C}^2)$ of sequences $z = \{z_n = (u_n\,,\, v_n)\}_{n=1}^\infty$ in $\mathbb{C}^2$ such that

$$\|z\|_2 = \|\{z_n\}_{n=1}^\infty\|_2 := \left(\sum_{n=1}^\infty (|u_n|^2 + |v_n|^2)\right)^{1/2} < \infty.$$

The vector space $\ell^2(\mathbb{C}^2)$ is actually a Hilbert space with the natural operations and the inner product of $\ell^2(\mathbb{C}^2)$ is defined by

$$z \cdot y = \{z_n\}_{n=1}^\infty \cdot \{y_n\}_{n=1}^\infty = \sum_{n=1}^\infty z_n \cdot \bar{y_n}, \quad z, y \in \ell^2(\mathbb{C}^2).$$

We define the operator T on $\ell^2(\mathbb{C}^2)$ as follows: for $z = \{(u_n, v_n)\}_{n=1}^{\infty}$, we define

$$T(z) = \left\{ \begin{pmatrix} 2^{-n} & 1 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n=1}^{\infty}.$$

Therefore, $T \in B(\ell^2(\mathbb{C}^2))$. Note that T is not a sectorial operator for any angle $0 \le \omega < \pi$. Let $\varepsilon < 0$. Let $z = \{(u_n, v_n)\}_n \in \ell^2(\mathbb{C}^2)$. We have that

$$(\varepsilon I - T)(z) = \left\{ \begin{pmatrix} \varepsilon - 2^{-n} & -1 \\ 0 & \varepsilon - 2^{-n} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n=1}^{\infty}.$$

And therefore, the resolvent operator has the following expression:

$$(\varepsilon I - T)^{-1}(z) = \left\{ \begin{pmatrix} (\varepsilon - 2^{-n})^{-1} u_n + (\varepsilon - 2^{-n})^{-2} v_n \\ (\varepsilon - 2^{-n})^{-1} v_n \end{pmatrix} \right\}_{n=1}^{\infty}, \quad z = \{(u_n, v_n)\}_n \in \ell^2(\mathbb{C}^2),$$

and

$$\|(\varepsilon I - T)^{-1}\| \ge \sup_{n \ge 1} (\varepsilon - 2^{-n})^{-2} = \varepsilon^{-2}.$$

Hence, $T \notin \mathrm{Sect}(\omega)$, for any $\omega \in [0, \pi)$. ◂

Example 6. The following differential operators are sectorial:

1) $Tf = -f''$, $f \in D(T) = W^{2,2}(\mathbb{R})$.
2) $Tf = -f''$, $f \in D(T) = \{f \in W^{2,2}(0\,,\,1) : f(0) = f(1) = 0\}$.
3) $Tf = -f''$, $f \in D(T) = \{f \in W^{2,2}(0\,,\,1) : f'(0) = f'(1) = 0\}$.
4) Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2 boundary and

$$Tu = -\Delta u, \ u \in D(T) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Here, by W we denote the usual Sobolev spaces [20, Chapter 8.8]. ◂

Now, we present some of the main properties of sectorial operators.

Proposition 7. *Let H be a Hilbert space and $T \in \mathrm{Sect}(\omega)$, where $0 \le \omega \le \pi$. The following statements hold.*

1) *If T is injective, then $T^{-1} \in \mathrm{Sect}(\omega)$.*
2) *Let $x \in H$. We have that $x \in \overline{D(T)}$ if and only if $\lim_{t\to\infty} t^n(t+T)^{-n}x = x$.*
3) *Let $x \in H$. Then, $x \in \overline{R(T)}$ if and only if $\lim_{t\to 0} T^n(t+T)^{-n}x = x$.*
4) *$\overline{D(T)} = H$ and $H = N(T) \oplus \overline{R(T)}$. T is injective if and only if $R(T)$ is dense in H.*
5) *$T^* \in \mathrm{Sect}(\omega)$ and $\omega_T = \omega_{T^*}$.*

Proof.

1) Suppose that T is injective. We have that $T^{-1} : R(T) \subset H \to D(T) \subset H$. If $\lambda \neq 0$, $\lambda \in \sigma(T)$ if and only if $\frac{1}{\lambda} \in \sigma(T^{-1})$. Moreover $\lambda(\lambda + T^{-1})^{-1} = I - \frac{1}{\lambda}(\frac{1}{\lambda} + T)^{-1}$ provided that $\frac{-1}{\lambda} \in \rho(T)$. Then, $\sigma(T^{-1}) \subset \overline{S_\omega}$ and, for every $\alpha \in (\omega, \pi)$, $M(T^{-1}, \alpha) < \infty$.
2) Since $(t+T)^{-n}x \in D(A)$, for every $t > 0$, we deduce that $x \in \overline{D(A)}$ provided that $x = \lim_{t\to+\infty} t^n(t+T)^{-n}x$. Suppose that $x \in D(T)$. We can write

$$x = t(t+T)^{-1}x + \frac{1}{t}(t(t+T)^{-1})Tx.$$

By iteration of this equality we get

$$x = (t(t+T)^{-1})^n x + \frac{1}{t}\sum_{k=1}^{n}(t(t+T)^{-1})^k Tx.$$

Since $\sup_{t>0}\|t(t+T)^{-1}\| < \infty$, it follows that $\lim_{t\to+\infty}(t(t+T)^{-1})^n x = x$. By using again that $\sup_{t>0}\|t(t+T)^{-1}\| < \infty$, we deduce that $\lim_{t\to+\infty}(t(t+T)^{-1})^n y = y$, for every $y \in \overline{D(T)}$.

3) This property can be proved in a similar way as the previous one.
4) Let $x \in H$. Since $\sup_{n\in\mathbb{N}} \|n(n+T)^{-1}n\| < \infty$, there exists an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and $y \in H$ such that $\phi(n)(\phi(n)+T)^{-1}x \to y$, as $n \to \infty$, in the weak topology of H. That is,

$$\lim_{n\to\infty} \langle \phi(n)(\phi(n)+T)^{-1}x\,,\, z\rangle = \langle y\,,\, z\rangle, \quad z \in H.$$

This implies that $T(\phi(n)+T)^{-1}x \to x-y$, as $n\to\infty$, in the weak topology of H. Now, T is a closed operator. This means that $G(T)$ is a closed subspace of $H\times H$. Hence, $G(T)$ is also weakly closed in $H\times H$. Then, $x=y$. Since the closure of $D(T)$ in the weak topology of H coincides with the closure of $D(T)$ in H, we conclude that $x \in \overline{D(T)}$.

We now prove that $H = N(T) \oplus \overline{R(T)}$. Note that, according to 2), if $x \in N(T)\cap\overline{R(T)}$, then

$$0 = Tx = \lim_{t\to 0^+} (t+T)^{-1}Tx = \lim_{t\to 0^+} T(t+T)^{-1}x = x.$$

Hence, $N(T)\cap\overline{R(T)} = \{0\}$. Let $x \in H$. Since $\sup_{t>0} \|t(t+T)^{-1}x\| < \infty$, there exists a decreasing sequence $(t_n)_{n=1}^{\infty} \subset (0,\infty)$ such that $t_n \to 0$, and $t_n(t_n+T)^{-1}x \to y$, as $n\to\infty$, in the weak topology of H. On the other hand,

$$t_nT(t_n+T)^{-1}x = t_n(x - t_n(t_n+T)^{-1}) \to 0, \quad \text{as } n\to\infty,$$

in H. By recalling that $G(T)$ is a weakly closed subspace of H, it follows that $Ty=0$. Moreover, $T(t_n+T)^{-1}x \to x-y$, in the weak topology of H. Hence, $x-y$ is in the weak closure of $R(T)$ that coincides with the closure of $R(T)$ in H. We conclude that $x \in N(T)+\overline{R(T)}$.
5) This property follows by taking into account that $p(T^*) = \{\bar{\lambda} : \lambda \in p(T)\}$ and that $R(\lambda,T)^* = R(\bar{\lambda}\,,\,T^*)$, for every $\lambda \in p(T)$. ■

Before stating the last properties of sectorial operators, we give some definitions.

Definition 8. A collection of operators $\{T_i\}_{i\in I}$ is said to be **uniformly sectorial of angle** $\omega \in [0,\pi)$ when $T_i \in \mathrm{Sect}(\omega)$, $i \in I$, and, for every $\alpha \in (\omega,\pi)$, $\sup_{i\in I} M(T_i,\alpha) < \infty$. ◂

Definition 9. Suppose that the sequence $\{T_n\}_{n\leq 1}^{\infty}$ is uniformly sectorial of angle ω. We say that $\{T_n\}_{n\in\mathbb{N}}$ is a **sectorial approximation** on S_ω for an operator T when there exists $\lambda \notin \overline{S_\omega}$ such that $\lambda \in \rho(T)$ and $R(\lambda,T_n) \to R(\lambda,T)$, as $n\to\infty$, in $B(H)$. In this case we write $T_n \to T$, (S_ω). ◂

Note that if $\{T_n\}_{n\in\mathbb{N}}$ is a sectorial aproximation on S_ω for T, then $T \in \mathrm{Sect}(\omega)$ and, for every $\lambda \notin \overline{S_\omega}$, $R(\lambda,T_n) \to R(\lambda,T)$, in $B(H)$.

In the next proposition we present some properties concerning sectorial convergence.

Proposition 10. *Suppose that the sequence $\{T_n\}_{n=1}^{\infty}$ uniformly sectorial of angle ω.*

1) *If $T_n \to T$, (S_ω) and T_n, T are injective, then $T_n^{-1} \to T^{-1}$, (S_ω).*
2) *If $T_n \to T(S_\omega)$ and $T \in B(H)$, then there exists $n_0 \in \mathbb{N}$ such that $T_n \in B(H)$ when $n \geq n_0$ and $T_n \to T$, as $n\to\infty$ in $B(H)$.*
3) *If $\{T_n\}_{n=1}^{\infty} \subset B(H)$, $T \in B(H)$ and $T_n \to T$, as $n\to\infty$ in $B(H)$, then $T_n \to T$, (S_w).*

Proof.

1) Follows from proposition 7.1).
2) Assume that $T_n \to T$, (S_ω), and $T \in B(H)$. Then, $-R(-1,T_n) = (I+T_n)^{-1} \to (I+T)^{-1}$ in $B(H)$. The set of invertible operators in $B(H)$ is open in $B(H)$. Also if A_n, $n\in\mathbb{N}$, and A are in $B(H)$, they are invertible and $A_n \to A$, as $n\to\infty$ in $B(H)$, then $A_n^{-1} \to A^{-1}$ as $n\to\infty$ in $B(H)$. Thus 2) is proved.
3) Assume that $\{T_n\}_{n=1}^{\infty} \subset B(H)$, $T \in B(H)$ and $T_n \to T$ as $n\to\infty$ in $B(H)$, $I+T$ is invertible and $(I+T_n)^{-1} \to (I+T)^{-1}$, as $n\to\infty$ in $B(H)$. ■

4. Spaces of holomorphic functions

In this section we present the definitions and the main properties of the spaces of holomorphic functions which will be used to define holomorphic functional calculus in the next section. We recall that $\mathcal{H}(\Omega)$ denotes the space of holomorphic functions on Ω.

Let $\varphi \in (0, \pi]$. We say that a function $f \in \mathcal{H}(S_\varphi)$ is in the Dunford–Riesz class on S_φ, shortly $f \in \mathcal{DR}(S_\varphi)$, when

i) $f \in H^\infty(S_\varphi)$, that is, f is bounded on S_φ;
ii) there exists $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $|z| \to \infty$ with $z \in S_\varphi$;
iii) there exists $\beta > 0$ such that $f(z) = O(|z|^\beta)$, as $|z| \to 0$ with $z \in S_\varphi$.

If there is no possible confusion about the sector, we may write $\mathcal{DR}$ instead of $\mathcal{DR}(S_\varphi)$. Note that, if $f \in \mathcal{DR}(S_\varphi)$, then the function $g(z) = f(1/z)$, $z \in S_\varphi$, is also in $\mathcal{DR}(S_\varphi)$. In short, a holomorphic function is in $\mathcal{DR}$ when it is bounded and tends to zero at the origin and at infinity sufficiently fast. Some examples of functions in $\mathcal{DR}(S_\varphi)$ are the following:

a) $f_1(z) = \frac{z}{1+z^2}$, $z \in S_\varphi$, $0 < \varphi < \frac{\pi}{2}$.
b) $f_2(z) = z\mathrm{e}^{-z}$, $z \in S_\varphi$, $0 < \varphi \leq \frac{\pi}{2}$.
c) $f_3(z) = \sqrt{z}\mathrm{e}^{-\sqrt{z}}$, $z \in S_\varphi$, $0 < \varphi \leq \pi$. Here, if $z = r\mathrm{e}^{\mathrm{i}\theta}$, with $r > 0$ and $\theta \in [-\pi, \pi)$, then $\sqrt{z} = \sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}$.

In the following proposition, equivalent definitions of the functions in $\mathcal{D}R(S_\varphi)$ are stated. The proof of these characterizations is straightforward from the definition.

Proposition 11. *Let $\varphi \in (0, \pi]$. Suppose that $f \in \mathcal{H}(S_\varphi)$. The following properties are equivalent:*

1) $f \in \mathcal{DR}(S_\varphi)$.
2) *There exist $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \min\{|z|^s, |z|^{-s}\}$, $z \in S_\varphi$.*
3) *There exist $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \frac{|z|^s}{1+|z|^{2s}}$, $z \in S_\varphi$.*

We now define another space of functions, namely $\mathcal{DR}_0$. A function $f \in \mathcal{H}(S_\varphi)$ is said to be in $\mathcal{DR}_0(S_\varphi)$ when

1) $f \in H^\infty(S_\varphi)$;
2) there exist $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $|z| \to \infty$ with $z \in S_\varphi$;
3) there exist $r > 0$ and $F \in \mathcal{H}(B(0, r))$ such that $F(z) = f(z)$, $z \in S_\varphi \cap B(0, r)$.

In other words, a function in $\mathcal{DR}_0$ decays at infinity and is holomorphic in a neighborhood of the origin.

Suppose that $f \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$, that is $f = g + h$ with $g \in \mathcal{DR}(S_\varphi)$ and $h \in \mathcal{DR}_0(S_\varphi)$. Since h can be extended as a holomorphic function to a neighborhood of the origin, there exists $c \in \mathbb{C}$ such that $h(z) = c + O(|z|)$, as $z \to 0$ with $z \in S_\varphi$. On the other hand, if $f \in \mathcal{H}(S_\varphi)$ and $c \in \mathbb{C}$, we can write

$$f(z) = \frac{c}{1+z} + \frac{f(z) - c}{1+z} + \frac{z}{1+z} f(z), \quad z \in S_\varphi.$$

By taking these properties in mind, we can establish the following proposition.

Proposition 12. *Let $\varphi \in (0, \pi]$. Assume that $f \in \mathcal{H}(S_\varphi)$. The following assumptions are equivalent:*

1) $f \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$.
2) *$f \in H^\infty$ and it satisfies that*
   a) *there exists $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $|z| \to \infty$ with $z \in S_\varphi$;*
   b) *there exist $\beta > 0$ and $c \in \mathbb{C}$ such that $f(z) = c + O(|z|^\beta)$, as $|z| \to 0$ with $z \in S_\varphi$.*

Note that the function $f(z) = 1$ is not in $\mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$.

We introduce the last space of functions. By $\mathcal{A}(S_\varphi)$ we denote the function space formed by all those $f \in \mathcal{H}(S_\varphi)$ for which there exists $n \in \mathbb{N}$ such that $f(z)(1+z)^{-n} \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$. The functions in $\mathcal{A}(S_\varphi)$ can be characterized as shown in the next proposition.

Proposition 13. *Let $\varphi \in (0, \pi]$. Suppose that $f \in \mathcal{H}(S_\varphi)$. The following properties are equivalent:*

1) *$f \in \mathcal{A}(S_\varphi)$.*
2) *f has the following properties:*
   a) *$f \in \mathcal{H}_c(S_\varphi)$, that is, for every $0 < r < R < \infty$, f is bounded on $S_\varphi \cap \{z \in \mathbb{C} : r \le |z| \le R\}$;*
   b) *There exists $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $z \to \infty$ with $z \in S_\varphi$;*
   c) *There exist $\beta > 0$ and $c \in \mathbb{C}$ such that $f(z) = c + O(|z|^\beta)$, as $z \to 0$ with $z \in S_\varphi$.*
3) *There exist $c \in \mathbb{C}$, $n \in \mathbb{N}$ and $F \in \mathcal{DR}(S_\varphi)$ such that*

$$f(z) = c + (1+z)^n F(z), \quad z \in S_\varphi.$$

If, in addition, f is bounded, 3) holds with $n = 1$.

If $\omega \in [0, \pi)$ and $\mathcal{M}$ represents either $\mathcal{DR}$, $\mathcal{DR}_0$, or $\mathcal{A}$ we define

$$\mathcal{M}[S_\omega] = \bigcup_{\omega < \varphi \le \pi} \mathcal{M}(S_\varphi).$$

5. Holomorphic functional calculus for sectorial operators

In this section we develop a holomorphic functional calculus for sectorial operators and functions in the spaces defined in the previous section. We will be able to give a meaning to the expression $f(T)$ when T is a sectorial operator and f is in the function spaces from the previous section. The basic tool is the Cauchy integral formula. We begin by defining the contours of the integrals.

Let $\varphi \in (0, \pi)$ and $\delta > 0$. We define the path $\Gamma_\varphi := \Gamma_\varphi^+ + \Gamma_\varphi^-$, where

$$\Gamma_\varphi^+(t) = -t\mathrm{e}^{\mathrm{i}\varphi}, \quad t \in (-\infty, 0], \quad \text{and} \quad \Gamma_\varphi^-(t) = t\mathrm{e}^{-\mathrm{i}\varphi}, \quad t \in (0, \infty).$$

Thus, Γ_φ is the boudnary of the sector S_φ oriented in the positive sense. We also consider the path $\Gamma_{\varphi,\delta} = \Gamma_{\varphi,\delta}^+ + \Gamma_{\varphi,\delta}^0 + \Gamma_{\varphi,\delta}^-$, where

$$\Gamma_{\varphi,\delta}^+(t) = -t\mathrm{e}^{\mathrm{i}\varphi},\ t \in (-\infty, -\delta];\ \Gamma_{\varphi,\delta}^0(\theta) = \delta\mathrm{e}^{\mathrm{i}\theta},\ \theta \in (\varphi, 2\pi - \varphi];\ \Gamma_{\varphi,\delta}^-(t) = t\mathrm{e}^{\mathrm{i}\varphi},\ t \in (\delta, \infty).$$

See figure 2. Note that $\Gamma_{\varphi,\delta}$ is the boundary of $S_\varphi \cup B(0,\delta)$ positively oriented. We can think that the paths Γ_φ and $\Gamma_{\varphi,\delta}$ go around the sector S_ω when $0 < \omega < \varphi$.

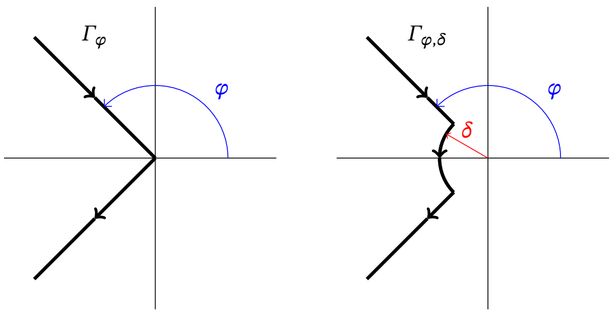

Figure 2: The contours Γ_φ and $\Gamma_{\varphi,\delta}$, for $\varphi = 3\pi/4$.

In the sequel of this section we assume that $T \in \operatorname{Sect}(\omega)$, where $\omega \in (0, \pi]$. Our objective is to define the operator $f(T)$ for every $f \in \mathcal{A}[S_\omega]$. Note that we cannot assure that $\sigma(T) \subset S_\varphi$ when $\varphi \in (\omega, \pi]$.

We first define $f(T)$ when $f \in \mathcal{DR}(S_\varphi)$ and $\varphi \in (\omega, \pi]$.

Proposition 14. *Let $\varphi \in (\omega, \pi]$. Suppose that $f \in \mathcal{DR}(S_\varphi)$. If $\omega < \lambda < \varphi$, the $B(H)$-Bochner integral*

$$\int_{\Gamma_\lambda} f(z)R(z,T)\,\mathrm{d}z$$

converges absolutely (with the $B(H)$ norm). Moreover, if $\omega < \lambda, \lambda' < \varphi$, then

$$\int_{\Gamma_\lambda} f(z)R(z,T)\,\mathrm{d}z = \int_{\Gamma_{\lambda'}} f(z)R(z,T)\,\mathrm{d}z.$$

Proof. The convergence of the integral can be shown using proposition 11.2) and splitting the integral near zero and near infinity. The second statement follows from the Hahn-Banach theorem and the Cauchy integral theorem. ∎

Following proposition 14, we define the following functional calculus. If $\varphi \in (\omega, \pi]$ and $f \in \mathcal{DR}(S_\varphi)$, we define $f(T)$ by

$$f(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z)R(z,T)\,\mathrm{d}z,$$

where $\lambda \in (\omega, \varphi)$. Thus $f(T) \in B(H)$ when $f \in \mathcal{DR}(S_\varphi)$ and $\varphi \in (\omega, \pi]$.

Assume now that $f \in \mathcal{DR}_0(S_\varphi)$. We cannot ensure that the integral in proposition 14 converges if $f(0) \neq 0$. We choose $\delta > 0$ so that f can be holomorphically extended to $B(0,\delta) \setminus S_\varphi$. Proceeding similarly we can define

$$f(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} f(z)R(z,T)\,\mathrm{d}z,$$

where δ, λ are as above, since one can check that the integral is absolutely convergent. Thus, $f(T) \in B(H)$. Note that if $f(0) = 0$, then $f \in \mathcal{DR} \cap \mathcal{DR}_0$ and both definitions for $f(T)$ coincide, by the Cauchy integral theorem. If $f = g + h$ with $g \in \mathcal{DR}$ and $h \in \mathcal{DR}_0$, we define $f(T) = g(T) + h(T)$. This definition does not depend on the choice of g and h. Some properties of the functional calculus we have defined are contained in the following proposition.

Proposition 15. *Let $\varphi \in (\omega, \pi]$.*

1) *Suppose that $x \in N(T)$ and $f = g + h$, where $g \in \mathcal{DR}(S_\varphi)$ and $h \in \mathcal{DR}_0(S_\varphi)$. Then, $f(T)(x) = h(0)x$.*
2) *The mapping $\Psi\colon \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi) \to B(H)$ defined by $\Psi(f) = f(T)$ is a homomorphism of algebras.*
3) *If $f_\mu(z) = \frac{1}{\mu - z}$, where $\mu \notin \overline{S_\varphi}$, then $f_\mu \in \mathcal{DR}_0(S_\varphi)$ and $f_\mu(T) = R(\mu, T)$.*
4) *Suppose that $A \in C(H)$ commutes with $R(\lambda, T)$ for every $\lambda \in \rho(T)$. Then, $Af(T) = f(T)A$ for every $f \in \mathcal{DR} + \mathcal{DR}_0$.*

Proof.

1) Since $x \in N(T)$, we can write, for any $z \in \rho(T)$,

$$zR(z,T)x = zR(z,T)x - R(z,T)Tx = \big(zR(z,T) - TR(z,T)\big)x = (z - T)R(z,T)x = x.$$

Then, by taking $\lambda \in (\omega, \varphi)$ and $\delta > 0$ such that h can be holomorphically extended to $B(0,\delta) \setminus S_\varphi$,

$$\begin{aligned} f(T)x &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} g(z)R(z,T)x\,\mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} h(z)R(z,T)x\,\mathrm{d}z \\ &= \left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \frac{g(z)}{z}\,\mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} \frac{h(z)}{z}\,\mathrm{d}z \right) x. \end{aligned}$$

By using the Cauchy integral theorem, since $g \in \mathcal{DR}$ and $h \in \mathcal{DR}_0$, we conclude that

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \frac{g(z)}{z}\,\mathrm{d}z = 0 \qquad \text{and} \qquad \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} \frac{h(z)}{z}\,\mathrm{d}z = h(0).$$

The first statement is proved.

2) All the properties of an algebra homomorphism are straightforward to prove except for

$$(fF)(T) = f(T) \circ F(T),$$

for every $f, F \in \mathcal{DR} + \mathcal{DR}_0$. The proof is easy but fairly long; it can be done by using some properties of the resolvent operator and the Fubini and Cauchy integral theorems.

3) Let $\mu \notin \overline{S_\varphi}$. It is clear that $f_\mu \in \mathcal{DR}_0(S_\varphi)$. We have that

$$f_\mu(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} \frac{R(z,T)}{\mu - z} \,\mathrm{d}z,$$

where $\lambda \in (\omega, \varphi)$ and $\delta < |\mu|$. The function $\Psi(z) = R(z,T)$ is holomorphic in $\rho(T)$ taking values in $B(H)$. We finish by using the Hahn-Banach and Cauchy integral theorems.

4) Suppose that $f \in \mathcal{DR}(S_\varphi)$. In the general case we could proceed in a similar way. We have that

$$f(T)x = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z) R(z,T) x \,\mathrm{d}z, \qquad x \in H,$$

where $\lambda \in (\omega, \varphi)$. Let $x \in D(A)$. It follows that

$$f(T)Ax = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z) R(z,T) Ax \,\mathrm{d}z.$$

The last $B(H)$-Bochner integral is absolutely convergent in the $H-$norm. The standard properties of the $B(H)$-Bochner integral lead to $f(T)Ax = Af(T)x$. ∎

Finally, we define $f(T)$ for every $f \in \mathcal{A}(S_\varphi)$. Let $f \in \mathcal{A}(S_\varphi)$ with $\varphi \in (\omega, \pi]$. There exists $n \in \mathbb{N}$ such that $f(z)(1+z)^n \in \mathcal{DR} + \mathcal{DR}_0$. Therefore, we can define the operator $f_n(T)$ by

$$f_n(T) = (I+T)^n \left(\frac{f}{p^n}\right)(T),$$

where $p(z) = 1 + z$. We make two small remarks about this definition. Note firstly that, since $(I+T)^{-n} \in B(H)$, $(I+T)^n$ is a closed operator. Then, the operator $f_n(T)$ is closed because $(f/p^n)(T) \in B(H)$. Secondly, one can prove that the definition of f_n does not really depend on n, as long as $f p^{-n} \in \mathcal{DR} + \mathcal{DR}_0$. The proof is not difficult but it is a bit technical, so we omit it. Therefore, we define $f(T) = f_n(T)$ for any admissible n.

Some properties of this functional calculus are summarized in the following proposition. For the sake of conciseness, we omit the proof, which is easy but tedious.

Proposition 16. *Let $\varphi \in (\omega, \pi]$, and $f \in \mathcal{A}(S_\varphi)$.*

i) *If $T \in B(H)$, then $f(T) \in B(H)$.*

ii) *If $S \in B(H)$ and S commutes with T, then S commutes with $f(T)$.*

iii) *Suppose that $g \in \mathcal{A}(S_\varphi)$. We have that*

$$f(T) + g(T) \subset (f+g)(T) \quad \text{and} \quad f(T)g(T) \subset (fg)(T).$$

Furthermore, $D\big((fg)(T)\big) \cap D\big(g(T)\big) = D\big(f(T)g(T)\big)$.

We cannot say that the natural functional calculus commutes with the sum and the product of functions (see proposition 16.iii)). But if $f, g \in \mathcal{A}(S_\varphi)$, we can see that

$$f(T) + g(T) = (f+g)(T),$$
$$f(T)g(T) = (fg)(T),$$

provided that $g(T) \in B(H)$. For instance, if $f(z) = c + (1+z)^n g(z)$, $z \in S_\varphi$, where $c \in \mathbb{C}$, $n \in \mathbb{N}$, and $g \in DR(S_\varphi)$, then $f(T) = c + (1+T)^n g(T)$.

If $0 < \varphi \leq \pi$ and $f \in H(S_\varphi)$, we define the function f^* by

$$f^*(z) = \overline{f(\overline{z})}, \quad z \in S_\varphi.$$

Thus, $f^* \in \mathcal{H}(S_\varphi)$, where f^* is named the conjugate of the function f. It is clear that the function spaces that we have considered in section 4 are invariant with respect to conjugation.

In the following proposition we show how the natural functional calculus acts on adjoints.

Proposition 17. *Let $\varphi \in (\omega, \pi]$ and $f \in \mathcal{A}(S_\varphi)$. Then, $f(T^*) = (f^*(T))^*$.*

Proof. Suppose firstly that $f \in \mathcal{DR}(S_\varphi)$. We have that

$$f^*(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f^*(z) R(z,T)\,\mathrm{d}z \in B(H).$$

Here $\omega < \lambda < \varphi$. For every $x, y \in H$,

$$\begin{aligned}
\langle f^*(T)x, y\rangle &= \Big\langle \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \overline{f(\overline{z})} R(z,T)x\,\mathrm{d}z, y\Big\rangle \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \overline{f(\overline{z})} \langle R(z,T)x, y\rangle\,\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \overline{f(\overline{z})} \langle x, R(\overline{z},T^*)y\rangle\,\mathrm{d}z \\
&= \Big\langle x, \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z) R(z,T^*)y\,\mathrm{d}z\Big\rangle \\
&= \langle x, f(T^*)y\rangle.
\end{aligned}$$

We have taken into account that all the Böchner integrals that appear are absolutely convergent. Then, $(f^*(T))^* = f(T^*)$.

If $f \in \mathcal{DR}_0(S_\varphi)$ we can proceed in a similar way. Assume now that $f \in \mathcal{A}(S_\varphi)$. There exist $n \in \mathbb{N}$, $g \in \mathcal{DR}(S_\varphi)$ and $h \in \mathcal{DR}_0(S_\varphi)$ such that $\frac{f(z)}{(1+z)^n} = g(z) + h(z)$, $z \in S_\varphi$. Then,

$$f(T^*) = (1+T^*)^n F(T^*),$$

where $F = g + h$. Here $(F^*(T))^* = (g^*(T))^* + (h^*(T))^* \in B(H)$. Moreover, we can write

$$\begin{aligned}
(1+T^*)^n &= (-1)^n (1+T^*)^n [R(-1,T))^n]^* [(1+T)^n]^* \\
&= (-1)^n (1+T^*) R(-1,T^*)^n [(1+T)^n]^* = [(1+T)^n]^*.
\end{aligned}$$

We get

$$f(T^*) = [(1+T)^n]^* (F^*(T))^* = (F^*(T)(1+T)^n)^*.$$

According to proposition 15.4), for every $x \in D(T^n) = D((1+T)^n)$, we have

$$F^*(T)(1+T)^n x = (1+T)^n F^*(T)x,$$

or, in other words, $D(T^n) \subset D(f^*(T))$. Since $\overline{D(T)} = \mathcal{H}$, using proposition 7.2), we deduce that $(t+(t+T)^{-1})^n x \to x$, as $t \to \infty$, and $f^*(T)(t+(t+T)^{-1})^n x = (t(t+T)^{-1})^n f^*(T)x \to f^*(T)x$, as $t \to \infty$. Furthermore, $(t+(t+T)^{-1})^n x \in D(T^n)$, for every $t > 0$.

This fact allows us to conclude that the closure $\overline{f^*(T)_{|D(T^n)}} = f^*(T)$ can be written as

$$f(T^*) = (f^*(T)_{|D(T^n)})^* = (\overline{f^*(T)_{|D(T^n)}})^* = (f^*(T))^*. \qquad \blacksquare$$

We can find in the literature extensions of the natural functional calculus. There exists a trade-off between operators and function spaces. When the operator is better we can define a functional calculus for a wider class of functions and vice-versa. We finish this section with the following definition.

Definition 18. Let $\varphi \in (\omega, \pi]$. Suppose that $\mathcal{F}$ is an algebra contained in $\mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$. The natural functional calculus on $\mathcal{F}$ for T is said to be **bounded** when $f(T) \in B(H)$, for every $f \in \mathcal{F}$, and there exists $C > 0$ such that

$$\|f(T)\| \leq C\|f\|_\infty, \qquad \forall f \in \mathcal{F}.$$

6. An operator without a bounded H^∞-calculus

McIntosh and Yagi [18] presented the first example of a sectorial operator without a bounded H^∞-calculus. We are going to give an example that can be found in the work of Le Merdy [11] (see also Haase's thesis [7, p. 49]).

Suppose that H is a separable Hilbert space. We can think for instance $H = \ell^2(\mathbb{N})$. We say that a sequence $\{e_n\}_{n=1}^\infty$ in H is a conditional basis in H when the following two properties hold:

i) For every $x \in H$ there exists a unique sequence $(\mu_n)_{n=1}^\infty \subset \mathbb{C}$ such that $x = \sum_{n=1}^\infty \mu_n e_n$.
ii) There exist a sequence $(\mu_n)_{n=1}^\infty \subset \mathbb{C}$ of complex numbers and a sequence $(\theta_n)_{n=1}^\infty \subset \{-1, 1\}$ of signs such that $\sum_{n=1}^\infty \mu_n e_n$ converges in H, but $\sum_{n=1}^\infty \theta_n \mu_n e_n$ does not converge in H.

According to Lindenstrauss and Tzafriri [16, Proposition 2.b.11], there exists a conditional basis $\{e_n\}_{n=1}^\infty$ of H. We may assume that $\|e_n\| = 1$ for all $n \in \mathbb{N}$. We define the n-th projection operator P_n in the standard way:

$$\begin{aligned} P_n :\ & H \longrightarrow H \\ x = \sum_{k=1}^\infty \mu_k e_k \longmapsto\ & P_n(x) = \sum_{k=1}^n \mu_k e_k. \end{aligned}$$

For every $n \in \mathbb{N}$, $P_n \in B(H)$. Also, one can prove that the sequence $\{P_n\}_{n=1}^\infty$ is bounded in $B(H)$. The constant $M_D = \sup_{n\in\mathbb{N}} \|P_n\|$ is known as the constant basis for $\{e_n\}_{n=1}^\infty$. This constant plays an important role in the theory of basis, but we will only use that M_D is finite.

Let $a = (a_n)_{n=1}^\infty$ be a complex sequence. We define the multiplier operator associated to a as follows: if $x = \sum_{n=1}^\infty \mu_n e_n \in H$ is such that $\sum_{n=1}^\infty a_n \mu_n e_n$ converges in H, then we set

$$T_a x = \sum_{n=1}^\infty a_n \mu_n e_n.$$

These multipliers operators were studied by Venni [23]. Note that the domain

$$D(T_a) = \left\{ \sum_{n=1}^\infty \mu_n e_n \,:\, \sum_{n=1}^\infty a_n \mu_n e_n \in H \right\}$$

is dense in H because the dense set $\left\{ \sum_{n=1}^M \mu_n e_n \in H \,:\, \mu_n \in \mathbb{C}, M \in \mathbb{N} \right\}$ is contained in $D(T_a)$.

Proposition 19. *The multiplier operator T_a is a closed operator.*

Proof. Suppose that $\{x_k\}_{k=1}^\infty \subset D(T_a)$ is such that $x_k \to y$ and $T_a x_k \to z$ as $k \to \infty$, for some $y, z \in H$. Then,

$$\begin{aligned} P_1(x_k) &\xrightarrow{k\to\infty} P_1 y, \\ P_1(T_a x_k) = a_1 P_1 x_k &\xrightarrow{k\to\infty} P_1 z, \end{aligned}$$

and, in general, for every $n \in \mathbb{N}$

$$\begin{aligned} P_{n+1}(x_k) - P_n(x_k) &\xrightarrow{k\to\infty} P_{n+1}(y) - P_n(y), \\ P_{n+1}(T_a x_k) - P_n(T_a x_k) &\xrightarrow{k\to\infty} P_{n+1}(z) - P_n(z). \end{aligned}$$

We deduce that $a_1 P_1(y) = P_1(z)$ and, for every $n \in \mathbb{N}$, $a_{n+1}(P_{n+1}(y) - P_n(y)) = P_{n+1}(z) - P_n(z)$. Hence $y \in D(T_a)$ and $T_a y = z$. ■

Let $(a_n)_{n=1}^{\infty}$ be a sequence of complex numbers. We define

$$\|a\| = \limsup_{n\to\infty} |a_n| + \sum_{n=1}^{\infty} |a_{n+1} - a_n|.$$

This quantity can be infinite. When $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$ we say that $(a_n)_{n=1}^{\infty}$ has bounded variation, and in this case $(a_n)_{n=1}^{\infty}$ is convergent.

Proposition 20. *Let $a = (a_k)$ be a complex sequence and T_a the associated multiplier operator. If $\|a\| < \infty$, then $T_a \in B(H)$ and $\|T_a\| \le M_0\|a\|$.*

Proof. Suppose that $x = \sum_{k=1}^{\infty} \mu_k e_k \in H$ with $(\mu_k)_{k=1}^{\infty} \subset \mathbb{C}$. For $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} a_k\mu_k e_k = \sum_{k=1}^{n} a_k(P_k - P_{k-1})x + a_1P_1x = \sum_{k=1}^{n-1} (a_k - a_{k+1})P_kx + a_nP_nx.$$

Since $\|a\| < \infty$, $\{a_nP_nx\}_{n=1}^{\infty}$ converges and $\{\sum_{k=1}^{\infty}(a_k - a_{k+1})P_kx\}_{k=1}^{\infty}$ is absolutely convergent in norm. Hence, $\sum_{k=1}^{\infty} a_k\mu_k e_k$ converges and

$$\left\|\sum_{k=1}^{\infty} a_k\mu_k e_k\right\| \le M_0\|a\|\|x\|.$$

■

Let $\lambda \in \mathbb{C}$ such that $\lambda \ne a_n$, $n \in \mathbb{N}$. Then, the multiplier operator associated to the sequence $\lambda - a = \{\lambda - a_n\}_{n=1}^{\infty}$ is $T_{\lambda-a} = \lambda I - T_a$ and it is injective. Moreover, $(\lambda I - T_a)^{-1}$ is the multiplier operator associeated to the sequence $\{\frac{1}{\lambda - a_n}\}_{n=1}^{\infty}$.

Proposition 21. *Let $a = (a_n)_{n=1}^{\infty}$ be an increasing sequence of real numbers satisfying that $a_1 > 0$ and $\lim_{n\to\infty} a_n = \infty$. Then, the associated multiplier operator T_a is sectorial of angle 0.*

Proof. Let $\lambda \in \mathbb{C} \setminus [a_1, \infty)$. We define $a(\lambda) = (\frac{1}{\lambda - a_n})_{n=1}^{\infty}$. We can write

$$\|a(\lambda)\| = \sum_{n=1}^{\infty} \left|\frac{1}{\lambda - a_{n+1}} - \frac{1}{\lambda - a_n}\right| = \sum_{n=1}^{\infty} \left|\int_{a_n}^{a_{n+1}} \frac{\mathrm{d}t}{(\lambda - t)^2}\right| \le \int_{a_1}^{\infty} \frac{\mathrm{d}t}{|\lambda - t|^2}.$$

Hence, $\lambda \in \rho(T_a)$. We conclude that $\sigma(T_a) \subset [a_1, \infty)$. Also, if $\lambda = r\mathrm{e}^{\mathrm{i}\theta}$, with $r > 0$ and $\nu \in [-\pi, \pi)$ we get

$$\|\lambda(\lambda I - T_a)^{-1}\| \le M_0|\lambda|\|a(\lambda)\| \le M_0 \int_0^{\infty} \frac{r\,\mathrm{d}t}{r\mathrm{e}^{\mathrm{i}\theta} - t|^2} = M_0 \int_0^{\infty} \frac{\mathrm{d}u}{|\mathrm{e}^{\mathrm{i}\theta} - u|^2}.$$

If $\varphi \in (0, \frac{\pi}{2})$ we have that

$$\int_0^{\infty} \frac{\mathrm{d}u}{|\mathrm{e}^{\mathrm{i}\theta} - u|^2} = \int_0^{\infty} \frac{\mathrm{d}u}{(\cos\theta - u)^2 + \sin^2\theta}$$

is bounded by

$$\int_0^{\infty} \frac{\mathrm{d}u}{u^2 + \sin^2\varphi}, \qquad \text{for } \theta \in [\pi/2, \pi - \varphi] \cup [\varphi - \pi, -\pi/2];$$

$$\int_0^{\infty} \frac{\mathrm{d}u}{(\cos\varphi + u)^2}, \qquad \text{for } \theta \in (\pi - \varphi, \pi) \cup [-\pi, \varphi - \pi];$$

$$\begin{aligned}&\int_0^{2\cos\theta} \frac{\mathrm{d}u}{\sin^2\varphi} + \int_{2\cos\theta}^{\infty} \frac{\mathrm{d}u}{(\cos\theta - u)^2 + \sin^2\varphi}\\ &\quad\le \frac{2\cos\theta}{\sin^2\varphi} + \int_0^{\infty} \frac{\mathrm{d}u}{\frac{1}{4}u^2 + \sin^2\varphi}, \qquad \text{for } \theta \in [\varphi, \pi/2] \cup [-\pi/2, -\varphi].\end{aligned}$$

Hence, $\|\lambda(\lambda I - T_a)^{-1}\| \le C_\varphi$, $\lambda \notin \overline{S_\varphi}$. Thus, we prove that $T_a \in \mathrm{Sect}(0)$. ■

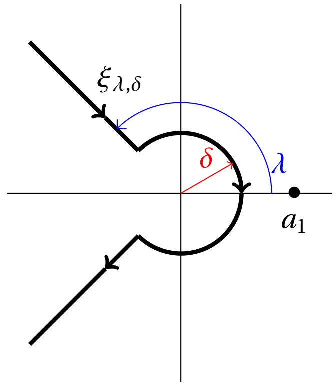

Figure 3: The contour $\xi_{\lambda,\delta}$ and a_1.

Finally, let us show that if $a = (2^n)_{n=1}^{\infty}$, then the operator T_a does not have a bounded functional calculus. Let $f \in \mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$ where $\varphi \in (0, \pi]$. There exists $m \in \mathbb{N}$ such that $f(z)(1+z)^{-m} = g(z) \in \mathcal{DR} + \mathcal{DR}_0$. Since $\sigma(T_a) \subset [a_1, +\infty)$ we can write

$$g(T_a) = \frac{1}{2\pi \mathrm{i}} \int_{\xi_{\lambda,\delta}} g(z) R(z, T_a)\, \mathrm{d}z,$$

where the path $\xi_{\lambda,\delta} = \xi^+_{\lambda,\delta} + \xi^0_{\lambda,\delta} + \xi^-_{\lambda,\delta}$, with $0 < \lambda < \varphi$, $0 < \delta < a_1$ and

$$\xi^+_{\lambda,\delta}(t) = -t\mathrm{e}^{\mathrm{i}\lambda},\ t \in (-\infty, -\delta],\ \xi^-_{\lambda,\delta}(t) = t\mathrm{e}^{-\mathrm{i}\lambda},\ t \in [\delta, \infty),\ \xi^0_{\lambda,\delta}(t) = \delta \mathrm{e}^{-\mathrm{i}\theta},\ \theta \in (-\lambda, +\lambda).$$

See figure 3. By using the Hahn-Banach and Cauchy integral theorems we deduce that

$$g(T_a)(e_n) = \frac{1}{2\pi \mathrm{i}} \int_{\xi_{\lambda,\delta}} g(z) R(z, T_a) e_n\, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\xi_{\lambda,\delta}} \frac{g(z)}{a_n - z}\, \mathrm{d}z\, e_n = g(a_n)\, e_n.$$

Then, if $(\mu_k)_{k=1}^n \subset \mathbb{C}$ with $n \in \mathbb{N}$,

$$f(T_a)\left(\sum_{k=1}^n \mu_k e_k\right) = \sum_{k=1}^n \mu_k (1+T)^m g(a_k)\, e_k = \sum_{k=1}^n \mu_k f(a_k)\, e_k.$$

If the natural functional calculus on $\mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$ for T_a were bounded, then we would have $f(T_a) \in B(H)$ and, hence, the series $\sum_{k=1}^\infty \mu_k f(a_k) e_k$ would converge in H provided that $\sum_{k=1}^\infty \mu_k e_k \in H$.

Now, take the sequence $a_n = 2^n$, $n \in \mathbb{N}$. Since $\{e_n\}$ is a conditional basis, there exist two sequences $(\mu_n)_{n=1}^\infty \subset \mathbb{C}$ and $(\theta_n)_{n=1}^\infty \subset \{-1, 1\}$ such that $\sum_{n=1}^\infty \mu_n e_n$ converges in H and $\sum_{n=1}^\infty \theta_n \mu_n e_n$ does not converge in H. According to Garnett [6, Theorem 1.1, VII.1], there exists $f \in H^\infty(S_{\pi/2}) \cap \mathcal{A}(S_{\pi/2})$ such that $f(2^n) = \theta_n$, $n \in \mathbb{N}$. We conclude that, if T_a is the multiplier operator associated with $\{a_n\}_{n=1}^\infty$, $f(T_a) \notin B(H)$. Hence, the natural functional calculus on $\mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$ for T_a is not bounded.

Multiplier operators like those we have just studied have also been considered to obtain examples related to certain operator theoretical problems [2, 9, 10, 12, 21, 23].

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