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# Geometry of polynomial spaces and polynomial inequalities

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**Abstract:** This paper summarises the contents of the course on geometry of polynomial spaces and polynomial inequalities delivered at the $9^{\text{th}}$ workshop of Functional Analysis organised by the Spanish Functional Analysis Network in Bilbao between the $3^{\text{rd}}$ and the $8^{\text{th}}$ of March, 2019, *in memoriam* of Prof. Bernardo Cascales. We first survey the most relevant results needed to understand polynomials in normed spaces. Then, we provide a few examples of polynomial spaces whose extreme points are fully described, and a couple of applications of the so-called Krein-Milman approach to obtain several sharp polynomial inequalities.

**Resumen:** Este artículo resume el contenido del curso sobre geometría de espacios polinomiales y desigualdades polinomiales que tuvo lugar en la IX Escuela-Taller de Análisis Funcional organizada por la Red de Análisis Funcional y Aplicaciones en Bilbao entre el 3 y el 8 de marzo de 2019, *in memoriam* al profesor Bernardo Cascales. Repasamos los resultados más relevantes para entender los polinomios en espacios normados. Después proporcionamos algunos ejemplos de espacios polinomiales cuyos puntos extremos están completamente descritos y un par de aplicaciones del llamado método de Klein-Milman para obtener desigualdades polinomiales óptimas.

**Keywords:** Bernstein and Markov inequalities, unconditional constants, polarizations constants, polynomial inequalities, homogeneous polynomials, extreme points.

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# 1. Introduction

This paper contains three main blocks. The first one is devoted to introducing polynomials in normed spaces. Although polynomials in a finite number of variables are well known to all undergraduate students, it is not so clear what polynomials in infinitely many variables are. In this block we will present all definitions and basic results required to understand polynomials in an arbitrary normed space, including the polarisation formula and polarisation constants. The second block deals with the geometry of polynomial spaces. The reader is referred to Dineen's book [15] for a modern monograph on polynomials on normed spaces. The characterisation of the extreme points of polynomial spaces is a question that has called the interest of a significant number of researchers in the last decades [1, 5, 10, 11, 18, 22, 24, 28–31, 33, 34]. This problem conveys a tremendous difficulty in most infinite (or even finite) dimensional polynomial spaces of interest, but in very specific cases, a complete and explicit description of the extreme points can be given. We will focus on a number of these particular examples, providing the reader not only with the extreme points of several 3-dimensional polynomial spaces, but also with a formula to calculate the polynomial norm, a parametrisation of the unit sphere and nice pictures of the unit balls of those spaces. Finally, a third section contains the applications of the geometrical results of the second block. In order to understand the applications, a precise introduction to several well-known polynomial inequalities will be provided. It is possible to find a vast diversity of applications in the literature, and therefore we will make a (restrictive) selection consisting of two types of polynomial inequalities, namely, the polynomial Bohnenblust-Hille inequality and the Bernstein-Markov type inequalities.

The arrangement described in the previous paragraph respects the structure of the course on geometry of polynomial spaces and applications delivered during the 9$^{\text{th}}$ workshop of Functional Analysis organized by the Spanish Functional Analysis Network in Bilbao between the 3$^{\text{rd}}$ and the 8$^{\text{th}}$ of March, 2019, *in memoriam* of Prof. Bernardo Cascales.

# 2. Polynomials in normed spaces

In this section we present the essential definitions and results needed to understand polynomials in normed spaces. We begin by recalling a number of basic concepts and definitions related to polynomials in a finite number of variables. In order to handle monomials in $\mathbb{K}^n$, we introduce the following notation. An $n$-dimensional multiindex is an $n$-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N} \cup \{0\}$ for all $i = 1, \dots, n$. If $x = (x_1, \dots, x_n) \in \mathbb{K}^n$, then $|\alpha|$, $\alpha!$ and $x^\alpha$ represent, respectively,

$$\alpha_1 + \dots + \alpha_n, \qquad \alpha_1! \cdots \alpha_n! \qquad \text{and} \qquad x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

With the above notation, a polynomial in $\mathbb{K}^m$ of degree at most $n$ is a linear combination of monomials of the form $x^\alpha$ with $x \in \mathbb{K}^m$ and $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $|\alpha| \leq n$. Hence, a polynomial $P$ in $m$ variables (real or complex) of degree at most $n$ has the form

$$P(x) = \sum_{\substack{\alpha \in (\mathbb{N} \cup \{0\})^m \\ |\alpha| \leq n}} a_\alpha x^\alpha, \qquad \text{for all } x \in \mathbb{K}^m,$$

with $a_\alpha \in \mathbb{K}$. Accordingly, a polynomial $P$ in $m$ variables is homogeneous of degree $n$ if

$$P(x) = \sum_{\substack{\alpha \in (\mathbb{N} \cup \{0\})^m \\ |\alpha| = n}} a_\alpha x^\alpha, \qquad \text{for all } x \in \mathbb{K}^m,$$

with $a_\alpha \in \mathbb{K}$. Our first objective is to recall how to extend the above well-known definition of polynomial and homogeneous polynomial to arbitrary linear spaces.

Let $E$ be a linear space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). From now on, for each $n \in \mathbb{N}$, $\mathcal{L}_a(^nE)$ denotes the space of all $n$-linear forms on $E$. Recall that $L$ is $n$-linear if it is linear in every coordinate. As usual, $E^n := E \times \overset{n)}{\cdots} \times E$. Also, we consider the diagonal mapping

$$\begin{aligned} \Delta_n : \ & E \to E^n \\ & x \mapsto (x, \dots, x). \end{aligned}$$

**Definition 1** (homogeneous polynomials). If $E$ is a linear space over $\mathbb{K}$ and $n \in \mathbb{N}$, we say that $P$ is an **$n$-homogeneous polynomial** if there exists $L \in \mathcal{L}_a(^nE)$ with $P = L \circ \Delta_n$. Equivalently, we write $P = \hat{L}$. The space of all $n$-homogeneous polynomials on $E$ is denoted by $\mathcal{P}_a(^nE)$. We say that $P$ is a polynomial of degree at most $n$ on $E$ if $P = P_n + \dots + P_1 + P_0$, where $P_k \in \mathcal{P}_a(^kE)$ for $k = 1, \dots, n$ and $P_0$ is a constant. ◂

Observe that, if $P \in \mathcal{P}_a(^nE)$, then $P(\lambda x) = \lambda^n P(x)$ for all $x \in E$ and every $\lambda \in \mathbb{K}$. This property is also satisfied by all homogeneous polynomials in a finite number of variables. On the other hand, the $n$-linear form that defines a given $n$-homogeneous polynomial is not uniquely determined. Let us see this with an example. First, notice that for all bilinear forms $L$ on $\mathbb{K}^n$, where $n \in \mathbb{N}$, there exists an $n \times n$ matrix with entries in $\mathbb{K}$ such that $L(z, w) = zAw^{>}$ for all $z, w \in \mathbb{K}^n$. Here $w^{>}$ means, as usual, the transpose of $w$. Hence, all 2-homogeneous polynomials on $\mathbb{K}^n$ are of the form

$$P(z) = L(z, z) = \sum_{i,j=1}^{n} a_{ij} z_i z_j,$$

where $A = (a_{ij})_{i,j=1}^n$. If we consider now $B = \frac{1}{2}(A + A^{>})$, then $zAz^{>} = zBz^{>}$ for every $z \in \mathbb{K}^n$. The latter means that the two bilinear forms determined by the matrices $A$ and $B$ define the polynomial $P$.

The previous example motivates the definition of symmetric multilinear forms.

**Definition 2** (symmetric multilinear forms). Let $E$ be a linear space over $\mathbb{K}$ and let $n \in \mathbb{N}$. An $n$-linear form $L$ is **symmetric** if

$$L(x_1, \dots, x_n) = L\left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right)$$

for all $(x_1, \dots, x_n) \in E^n$ and every permutation $\sigma$ of $\{1, \dots, n\}$. The space of all symmetric $n$-linear forms on $E$ is denoted by $\mathcal{L}_a^s(^nE)$. ◂

*Remark* 3. Let us consider the mapping $s$ from $\mathcal{L}_a(^nE)$ onto $\mathcal{L}_a^s(^nE)$ given by

$$s(L)(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} L\left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right),$$

where $S_n$ is the group of all permutations of $\{1, \dots, n\}$. Then, $s$ is a projection. Furthermore, $L$ and $s(L)$ define the same homogeneous polynomial. Therefore, if $P \in \mathcal{P}_a(^nE)$, it is always possible to choose $L \in \mathcal{L}_a^s(^nE)$ such that $\hat{L} = P$. ◂

Now, using the so-called multinomial formula, it is possible to see that definition 1 does extend the concept of homogeneous polynomial from a finite number of variables to arbitrary linear spaces.

**Proposition 4** (multinomial formula). *Let $E$ be a real or complex linear space, $P \in \mathcal{P}_a(^nE)$, $x_1, \dots, x_k \in E$ and $a_1, \dots, a_k \in \mathbb{K}$. Then,*

$$P\left(\sum_{i=1}^{k} a_i x_i\right) = \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m| = n}} \frac{n!}{m!} a_1^{m_1} \cdots a_k^{m_k} L\left(x_1^{m_1}, \dots, x_k^{m_k}\right),$$

*where $L \in \mathcal{L}_a^s(^nE)$ satisfies $\hat{L} = P$ and*

$$L\left(x_1^{m_1}, \dots, x_k^{m_k}\right) := L\Big(\overbrace{x_1, \dots, x_1}^{m_1}, \dots, \overbrace{x_k, \dots, x_k}^{m_k}\Big).$$

*Proof*. Let $m = (m_1, \dots, m_k) \in \{0, 1, \dots, k\}^k$ with $|m| = n$ and define

$$A_m = \left\{(i_1, \dots, i_k) \in \{0, 1, \dots, k\}^k \text{ such that 1 appears } m_1 \text{ times}, \dots, \ k \text{ appears } m_k \text{ times}\right\}$$

and

$$a_m = \sum_{(i_1, \dots, i_k) \in A_m} L(x_{i_1}, \dots, x_{i_k}).$$

Then, using linearity,

$$P(a_1 x_1 + \dots + a_k x_k) = L((a_1 x_1 + \dots + a_k x_k)^n) = \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m| = n}} a_1^{m_1} \cdots a_k^{m_k} a_m.$$

Observe that, due to the symmetry of $L$, we have

$$a_m = L(x_1^{m_1}, \dots, x_k^{m_k}) \cdot \#(A_m),$$

where $\#(A_m)$ denotes the cardinality of $A_m$. Using elementary combinatorics we arrive at $\#(A_m) = \frac{n!}{m_1! \cdots m_k!}$, which concludes the proof. ■

The importance of the multinomial formula in this context relies on the fact that it can be used to extend the classical definition of polynomials in several variables. Indeed, let $E$ be a real or complex linear space and $F$ a finite dimensional subspace of $E$. Let $\{e_1, \dots, e_k\}$ be a Hammel basis for $F$ and $x = x_1e_1 + \dots + x_ke_k \in F$. Then, using the multinomial formula,

$$\begin{aligned} P(x) &= P(x_1e_1 + \cdots + x_ke_k) \\ &= \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m|=n}} \frac{n!}{m!} x_1^{m_1} \cdots x_k^{m_k} L\left(e_1^{m_1}, \dots, e_k^{m_k}\right) = \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m|=n}} a_m x^m, \end{aligned}$$

which shows that the restriction of an $n$-homogeneous polynomial in the sense of definition 1 to a $k$-dimensional space is an $n$-homogeneous polynomial in $k$ variables.

## 2.1. The polarization formula

If $E$ is a linear space and $P \in \mathcal{P}_a(^nE)$, we have seen in remark 3 that there exists $L \in \mathcal{L}_a^s(^nE)$ such that $P = \hat{L}$. We can go further and prove that this symmetric $n$-linear form that determines $P$ is unique, and we call it polar of $P$. The polarization formula does not only prove the uniqueness of the symmetric $n$-linear form that defines a given $n$-homogeneous polynomial: additionally, it provides an explicit expression of the polar in terms of the polynomial it defines. There are many forms of the polarization formula. We have chosen one taken from Dineen's book [15] that uses Rademacher functions.

**Theorem 5** (polarization formula). *Let $E$ be a real or complex linear space, $P \in \mathcal{P}_a(^nE)$, $L \in \mathcal{L}_a^s(^nE)$ and assume that $\hat{L} = P$. If $(x_1, \dots, x_n) \in E^n$, then*

$$L(x_1, \dots, x_n) = \frac{1}{n!} \int_0^1 r_1(t) \cdots r_n(t) P(r_1(t)x_1 + \dots + r_n(t)x_n)\, \mathrm{d}t, \tag{1}$$

*where $r_j$ is the $j$-th Rademacher function, defined by*

$$r_j(t) = \operatorname{sign}(\sin(2^j \pi t)) \qquad \text{for all } 1 \le j \le n.$$

*Proof.* Let $m = (m_1, \dots, m_n) \in (\mathbb{N} \cup \{0\})^n$. Recall that the Rademacher functions satisfy

$$\int_0^1 r_1^{m_1+1}(t) \cdots r_n^{m_n+1}(t)\, \mathrm{d}t = \int_0^1 r_1^{m_1+1}(t)\, \mathrm{d}t \cdots \int_0^1 r_n^{m_n+1}(t)\, \mathrm{d}t \tag{2}$$

and

$$\int_0^1 r_j^{m_j+1}(t)\, \mathrm{d}t = \begin{cases} 1 & \text{if } m_j + 1 \text{ is even,} \\ 0 & \text{if } m_j + 1 \text{ is odd.} \end{cases} \tag{3}$$

If we assume $|m| = n$, the product $\int_0^1 r_1^{m_1+1}(t)\, \mathrm{d}t \cdots \int_0^1 r_n^{m_n+1}(t)\, \mathrm{d}t$ vanishes unless $m_1 = \dots = m_n = 1$, in which case its value is 1. Now, using the multinomial formula,

$$\begin{aligned}
&\int_0^1 r_1(t)\cdots r_n(t)\,P\left(r_1(t)\,x_1+\ldots+r_n(t)\,x_n\right)\mathrm{d}t\\
&\quad=\int_0^1 r_1(t)\cdots r_n(t)\,L\left(r_1(t)\,x_1+\ldots+r_n(t)\,x_n,\ldots,r_1(t)\,x_1+\ldots+r_n(t)\,x_n\right)\mathrm{d}t\\
&\quad=\int_0^1 r_1(t)\cdots r_n(t)\sum_{\substack{m\in(\mathbb{N}\cup\{0\})^n\\|m|=n}}\frac{n!}{m!}r_1^{m_1}(t)\cdots r_n^{m_n}(t)\,L\left(x_1,\ldots,x_n\right)\mathrm{d}t\\
&\quad=\sum_{\substack{m\in(\mathbb{N}\cup\{0\})^n\\|m|=n}}\frac{n!}{m!}\int_0^1 r_1^{m_1+1}(t)\cdots r_n^{m_n+1}(t)\,\mathrm{d}t L\left(x_1,\ldots,x_n\right)=n!L(x_1,\ldots,x_n).
\end{aligned}$$

■

*Remark* 6. Let $P\in\mathcal{P}_a(^nE)$ and $L\in\mathcal{L}_a^s(^nE)$ be the polar of $P$. It has already been mentioned that there are many forms of the polarization formula. To see this we just need to replace the Rademacher functions by any set of functions satisfying the identities (2) and (3). For instance, we may consider any set of $n$ independent and orthonormal random variables $r_1,\ldots,r_n$ on $[0,1]$ taking values on $\mathbb{K}$ and

$$\Psi=\bar{r}_1\cdots\bar{r}_n\cdot P\left(\sum_{i=1}^n r_i x_i\right).$$

Proceeding as in the proof of theorem 5, the expectancy of $\Psi$ would be given by

$$\mathbb{E}[\Psi]=n!L(x_1,\ldots,x_n),$$

that is,

$$L(x_1,\ldots,x_n)=\frac{1}{n!}\mathbb{E}[\Psi],$$

which is a much more general way to express the polarization formula (1). Thus, if $r_1,\ldots,r_n$ are $n$ independent Bernouilli random variables taking the value $-1$ with probability $1/2$ and $1$ with probability $1/2$, we would have

$$L\left(x_1,\ldots,x_n\right)=\frac{1}{2^n n!}\sum_{\varepsilon_i=\pm1}\varepsilon_1\cdots\varepsilon_n P\left(\sum_{i=1}^n\varepsilon_i x_i\right). \tag{4}$$

This is a very convenient form to put down the polarization formula. ◂

## 2.2. Continuity in polynomial spaces

All polynomials in finitely many variables are continuous. However, this is far from being true when polynomials on an infinite dimensional normed space are considered. Actually, continuity fails to be universal even for linear forms on an infinite dimensional normed space.

Let $(E,\|\cdot\|)$ be a normed space over $\mathbb{K}$. We represent the space of continuous $n$-homogeneous polynomials, the space of continuous $n$-linear forms and the space of continuous symmetric $n$-linear forms, respectively, by $\mathcal{P}(^nE)$, $\mathcal{L}(^nE)$ and $\mathcal{L}^s(^nE)$. Also, for $P\in\mathcal{P}(^nE)$ and $L\in\mathcal{L}^s(^nE)$, we define

$$\begin{aligned}
\|P\|&=\sup\{|P(x)|\,:\,\|x\|\le1\},\\
\|L\|&=\sup\{|L(x_1,\ldots,x_n)|\,:\,\|(x_1,\ldots,x_n)\|\le1\},
\end{aligned}$$

where

$$\|(x_1,\ldots,x_n)\|=\sup\{\|x_i\|\,:\,1\le i\le n\}.$$

These definitions are intended to introduce a norm in $\mathcal{P}(^nE)$ and $\mathcal{L}(^nE)$. However, at this stage we do not even know whether $\|P\|$ or $\|L\|$ are finite. As a matter of fact, the finiteness of $\|P\|$ or $\|L\|$ characterizes the continuity of $P$ or $L$, as we will see later. First we present a fundamental result in the theory of polynomials in normed spaces also known as the polarization inequality. The proof provided below is taken from Dineen's book [15].

**Theorem 7** (polarization inequality). *Let $E$ be a normed space. Then,*

$$\|P\| \le \|L\| \le \frac{n^n}{n!}\|P\|$$

*for every $L \in \mathcal{L}^s({}^nE)$ and $P \in \mathcal{P}({}^nE)$ such that $L$ is the polar of $P$.*

*Proof.* The first inequality is trivial since $P$ is a restriction of $L$. To prove the second inequality, we use the polarization formula (4):

$$\begin{aligned}
\|L\| &= \sup\{|L(x_1,\dots,x_n)| : \|(x_1,\dots,x_n)\| \le 1\}\\
&= \sup\left\{\left|\frac{1}{2^n n!}\sum_{\varepsilon_i=\pm1}\varepsilon_1\cdots\varepsilon_n P\left(\sum_{i=1}^n \varepsilon_i x_i\right)\right| : \|(x_1,\dots,x_n)\| \le 1\right\}\\
&\le \frac{1}{2^n n!}\sum_{\varepsilon_i=\pm1}\sup\left\{\left|P\left(\sum_{i=1}^n \varepsilon_i x_i\right)\right| : \|(x_1,\dots,x_n)\| \le 1\right\}\\
&= \frac{n^n}{2^n n!}\sum_{\varepsilon_i=\pm1}\sup\left\{\left|P\left(\frac{1}{n}\sum_{i=1}^n \varepsilon_i x_i\right)\right| : \|(x_1,\dots,x_n)\| \le 1\right\}\\
&\le \frac{n^n}{n!}\|P\|.
\end{aligned}$$

■

As is well known, boundedness is a characteristic property of continuous linear forms on any normed space. A similar result holds for homogeneous polynomials, as we are about to see. The proof of the following result is inspired in Dineen's book [15].

**Theorem 8.** *Let $E$ be a normed space over $\mathbb{K}$ and $P \in \mathcal{P}_a({}^nE)$. Then, the following are equivalent:*

(i) *$P$ is continuous in $E$.*

(ii) *$P$ is continuous at 0.*

(iii) *$P$ is bounded over $\mathsf{B}_E$, the closed unit ball of $E$.*

*Proof.* That (i) implies (ii) is trivial. We prove now that (ii) implies (iii). By continuity at 0, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|P(x)| < \varepsilon$ whenever $\|x\| < 2\delta$. Therefore, if $x \in \mathsf{B}_E$ we have

$$|P(x)| = \left|P\left(\delta\cdot\frac{x}{\delta}\right)\right| = \frac{1}{\delta^n}|P(\delta x)| < \frac{\varepsilon}{\delta^n}.$$

Thus, $P$ is bounded on $\mathsf{B}_E$.

Finally, we show that (iii) implies (i). Choose an arbitrary $x_0$ in $E$ and take $x \in E$ with $\|x - x_0\| \le 1$. In particular, $\|x\| \le 1 + \|x_0\|$. Then, by Newton's binomial formula and Martin's theorem, we have

$$\begin{aligned}
|P(x) - P(x_0)| &= |P((x - x_0) + x_0) - P(x_0)|\\
&= \left|\sum_{j=0}^{n-1}\binom{n}{j}L\left(x_0^j, (x - x_0)^{n-j}\right)\right|\\
&\le \sum_{j=0}^{n-1}\binom{n}{j}\|L\|\,\|x_0\|^j\,\|x - x_0\|^{n-j}\\
&\le \frac{n^n}{n!}\|P\|\,\|x - x_0\|\sum_{j=0}^{n-1}\binom{n}{j}\|x_0\|^j\\
&\le \frac{n^n}{n!}(1 + \|x_0\|)^n\,\|P\|\,\|x - x_0\|.
\end{aligned}$$

Recall that $\|P\|$ is finite since $P$ is bounded on $\mathsf{B}_E$. Without loss of generality, we may also assume that $\|P\| > 0$. Hence, for an arbitrary $\varepsilon > 0$ we can take

$$\delta = \min\left\{1, \frac{n!}{n^n(1 + \|x_0\|)^n\|P\|}\right\} > 0$$

and we have that $|P(x) - P(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$, that is, $P$ is continuous at $x_0$. ■

*Remark* 9. Let $E$ be a real or complex normed space. Theorems 5 and 8 show two relevant facts:

1. If $P \in \mathcal{P}_a(^nE)$ and $L \in \mathcal{L}_a^s(^nE)$ is its polar, then $P$ is bounded (continuous) if and only if $L$ is bounded (continuous).
2. The spaces $\mathcal{P}(^nE)$ and $\mathcal{L}^s(^nE)$ are topologically isomorphic and $\mathcal{L}^s(^nE) \ni L \mapsto \hat{L} \in \mathcal{P}(^nE)$ is a natural isomorphism whose inverse is provided by the polarization formula. ◂

Using the axiom of choice it is easy to construct non-bounded (and therefore non-continuous) polynomials.

**Example 10.** Let $E$ be any normed space of dimension $\mathfrak{c}$ (here $\mathfrak{c}$ is the continuum, or the cardinality of $\mathbb{R}$). Let $\mathcal{B} = \{e_x : x \in \mathbb{R}\}$ be a Hammel basis of normalized vectors of $E$ indexed in $\mathbb{R}$ and define $L \in \mathcal{L}_a^s(^nE)$ on $\mathcal{B}$ by $L(e_{x_1}, \dots, e_{x_n}) = x_1 \cdots x_n$. On the rest of $E$, $L$ is defined by linearity. Then, the $n$-homogeneous polynomial induced by $L$ is not bounded on $\mathsf{B}_E$, since $\mathcal{B} \subset \mathsf{B}_E$, but

$$\lim_{x\to\infty} P(e_x) = \lim_{x\to\infty} x^n = \infty. \qquad ◂$$

In general, for any normed space $E$, the algebraic size, measured in terms of dimension, of the set of non-bounded $n$-homogeneous polynomials (respectively non-bounded symmetric $n$-linear forms) is maximal. Consider the sets $\mathcal{NBL}^s(^nE)$ and $\mathcal{NBP}(^nE)$ of, respectively, all the non-bounded symmetric $n$-linear forms and all the non-bounded $n$-homogeneous polynomials on $E$. Then, Gámez-Merino, Muñoz-Fernández, Pellegrino, and Seoane-Sepúlveda [16] proved in 2012 the following.

**Theorem 11.** *If $n \in \mathbb{N}$ and $E$ is a normed space of infinite dimension $\lambda$, then the sets $\mathcal{NBL}^s(^nE) \cup \{0\}$ and $\mathcal{NBP}(^nE) \cup \{0\}$ contain a $2^\lambda$-dimensional subspace. We say then that the sets $\mathcal{NBL}^s(^nE)$ and $\mathcal{NBP}(^nE)$ are $2^\lambda$-lineable.*

## 2.3. Polarization constants

If $E$ is a real or complex normed space $P \in \mathcal{P}(^nE)$ and $L \in \mathcal{L}^s(^nE)$ is the polar of $P$, according to Martin's theorem (theorem 7),

$$\|L\| \le \frac{n^n}{n!}\|P\|.$$

The constant $\frac{n^n}{n!}$ cannot be replaced by a smaller constant in general since equality can be attained for the space $E = \ell_1^n$ and the polynomial $\Phi_n(x_1, \dots, x_n) := x_1 \cdots x_n$ and its polar. However, $\frac{n^n}{n!}$ can indeed be replaced by a smaller estimate for specific spaces. This motivates the following definition.

**Definition 12** (polarization constants). If $E$ is a normed space over $\mathbb{K}$, we define the ***n*-th polarization constant** of $E$ as

$$\mathbb{K}(n; E) := \inf\{K > 0 : \|L\| \le K\|P\|,\ \forall P \in \mathcal{P}(^nE) \text{ and } \hat{L} = P\}. \qquad ◂$$

A somewhat more general concept than that of polarization constant arises from the following result by Harris [19].

**Theorem 13.** *Let $E$ be a complex normed space and $n_1, \dots, n_k \in \mathbb{N}$. If $n = n_1 + \cdots + n_k$ and $L \in \mathcal{L}^s(^nE)$, then*

$$\sup\{|L(x_1^{n_1}, \dots, x_k^{n_k})| : \|x_i\| = 1,\ 1 \le i \le k\} \le \frac{n_1! \cdots n_k!\, n^n}{n_1^{n_1} \cdots n_k^{n_k}\, n!}\|\hat{L}\|.$$

A similar result with a different constant can be proved when $E$ is a real normed space. All this serves as a motivation for the following definition.

**Definition 14** (generalized polarization constants). Let $E$ be a normed space over $\mathbb{K}$ and $n, n_1, \dots, n_k \in \mathbb{N}$ with $n = n_1 + \cdots + n_k$. Then, we define the **generalized polarization constant** $\mathbb{K}(n_1, \dots, n_k; E)$ as

$$\mathbb{K}(n_1, \dots, n_k; E) := \inf\{c : |L(x_1^{n_1}, \dots, x_k^{n_k})| \le c\|\hat{L}\|,\ \forall L \in \mathcal{L}^s(^nE),\ \|x_i\| = 1\}. \qquad ◂$$

From theorem 7 we deduce that

$$1 \leq \mathbb{K}(n; E) \leq \frac{n^n}{n!}$$

for any normed space $E$ over $\mathbb{K}$, but the exact value of $\mathbb{K}(n; E)$ for many choices of $E$ has remained as an unresolved problem until today. The calculation of the exact value of the polarization constants of specific spaces seems to be a challenging problem and yet, significant progress has been made. Of particular interest are the works of Sarantopoulos [37] and Kirwan, Sarantopoulos, and Tonge [23], where the spaces satisfying $\mathbb{K}(n; E) = \frac{n^n}{n!}$ are studied. At the other end of the scale, according to an old result, if $E$ is an Euclidean space over $\mathbb{K}$, then $\mathbb{K}(n; E) = 1$ [2, 21, 38] (see Dineen's book [15] for a modern exposition). Furthermore, Benítez and Sarantopoulos [4] proved that $\mathbb{R}(n; E) = 1$ implies that $E$ is a real Euclidean space. However, $\mathbb{C}(n; E) = 1$ does not necessarily imply that $E$ is a complex Euclidean space. The value of $\mathbb{K}(n; \ell_p)$ is known for some choices of $p$ (see for instance [36]), but most of the polarization constants of the classical spaces are still unknown nowadays. For a complete account on polarization constants, we recommend Rodríguez-Vidanes's work [35].

The use of the Krein-Milman approach (which will be described right after theorem 15) in combination with a description of the extreme points of certain polynomial spaces may produce good results in the difficult task of calculating polarization constants. The next section is devoted to the study of the geometry of certain polynomial spaces.

## 3. Geometry of some 3-dimensional polynomial spaces

Let $E$ be a finite dimensional normed space. Recall that $C \subset E$ is a convex body if it is a compact convex set with nonempty interior. A point $e \in C$ is an extreme point of $C$ if it is not an interior point of any segment contained in $C$. We use the notation $\text{ext}(C)$ to represent the set of all the extreme points of $C$. According to the Krein-Milman theorem (or its finite dimensional version proved by Minkowski in 3-dimensional spaces and by Steinitz for any dimension), the set of the extreme points of a convex body $C$ in the finite dimensional normed space $E$ determines $C$. The precise formulation of this result is the following.

**Theorem 15** (Minkowski-Steinitz). *If $E$ is a finite dimensional normed space and $C \subset E$ is a convex body, then*

(i) $\text{ext}(C) \neq \varnothing$.

(ii) $C = \text{co}(\text{ext}(C))$.

*Note that* $\text{co}(A)$ *is the convex hull of the set $A$.*

This result has been used in a large variety of settings to optimize convex functions. In fact, the result that allows the optimization is the following:

> If $C \subset E$ is a convex body in the real finite dimensional normed space $E$ and $f : C \to \mathbb{R}$ is a convex function that attains its maximum in $C$, then there is a point $e \in \text{ext}(C)$ such that $f(e) = \min\{f(x) : x \in C\}$.

We will address to this result as the Krein-Milman approach from now on. A combination of the Krein-Milman approach and an exhaustive description of the extreme points of the unit ball of a polynomial space provides, in many cases, sharp polynomial inequalities. The previous argument motivates the study of the geometry of polynomial spaces. As a matter of fact, many publications have dealt with this question in the past. Konheim and Rivlin [24], as late as in 1966, characterised the extreme points of the space of real polynomials of degree not exceeding $n$, namely $\mathcal{P}_n(\mathbb{R})$, endowed with the norm

$$\|P\| = \sup\{|P(x)| : x \in [-1, 1]\}.$$

Unfortunately, Konheim and Rivlin's results do not provide an explicit representation of the extreme points of $\mathcal{P}_n(\mathbb{R})$. Choi, Kim, and Ki [11], on the one hand, and Choi and Kim [10] on the other, characterised the extreme points of $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_\infty^2)$. Grecu [18] extended those results to $\mathcal{P}(^2\ell_p^2)$ for arbitrary $p \geq 1$.

Aron and Klimek [1] characterised the extreme points of the space of the real quadratic polynomials on $[-1,1]$ and the unit disk in $\mathbb{C}$. Muñoz-Fernández and Seoane-Sepúlveda [33] studied the extreme points of the real trinomials on $[-1,1]$, whereas Neuwirth [34] did the same thing on the unit disk in $\mathbb{C}$. Kim [22] studied polynomials on an hexagon or an octagon. Special attention has also been given to polynomials on non-balanced convex bodies. Thus, Muñoz-Fernández, Révész, and Seoane-Sepúlveda [31] studied the geometry of the space $\mathcal{P}(^2\Delta)$ of the 2-homogeneous polynomials on the simplex $\Delta$ (the triangle of vertices $(0,0)$, $(1,0)$ and $(0,1)$). Milev and Naidenov [28, 29] studied the extreme points of the space of polynomials (homogeneous or not) of degree at most 2 on $\Delta$. Gámez-Merino, Muñoz-Fernández, Sánchez, and Seoane-Sepúlveda [17] studied the geometry of the space $\mathcal{P}(^2\square)$ of the 2-homogeneous polynomials on the unit square $\square = [0,1]^2$. Muñoz-Fernández, Pellegrino, Seoane-Sepúlveda, and Weber [30] characterised the extreme points of the space $\mathcal{P}(^2D(\alpha,\beta))$ of the 2-homogeneous polynomials on the circular sectors $D(\alpha,\beta) = \{re^{i\theta} : r \in [0,1],\ \theta \in [\alpha,\beta]\}$ for $\beta - \alpha = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and $\beta - \alpha \geq \pi$. These results have been generalised in a recent work by Bernal-González, Muñoz-Fernández, Rodríguez-Vidanes, and Seoane-Sepúlveda [5] for an arbitrary length of the interval $[\alpha,\beta]$.

In the rest of this section we will provide a few illustrative examples representing a tiny fraction of the results mentioned above about geometry of polynomial spaces.

## 3.1. The geometry of $\mathcal{P}_2(\mathbb{R})$

Here we consider the space $\mathcal{P}_2(\mathbb{R})$ of the real polynomials $ax^2+bx+c$ of degree not greater than 2, endowed with he norm

$$\|P\|_{\mathcal{P}_2(\mathbb{R})} = \sup\{|P(x)| : |x| \leq 1\}.$$

Elementary calculus shows that

$$\|(a,b,c)\|_{\mathcal{P}_2(\mathbb{R})} = \begin{cases} \left|\frac{b^2}{4a} - c\right| & \text{if } a \neq 0,\ \left|\frac{b}{2a}\right| < 1 \text{ and } \frac{c}{a} + 1 < \frac{1}{2}\left(\left|\frac{b}{2a}\right| - 1\right)^2, \\ |a+c| + |b| & \text{otherwise.} \end{cases}$$

The geometry of this 3-dimensional space was investigated by Aron and Klimek [1] (see also the work of Muñoz-Fernández and Seoane-Sepúlveda [33]). The conclusions extracted from these papers are summarised in the following result. From now on, if $E$ is a normed space, $\mathsf{B}_E$ and $\mathsf{S}_E$ stand for the closed unit ball and the unit sphere of $E$, respectively. Also, $\operatorname{graph}(f)$ stands for the graph of the function $f$.

**Theorem 16.** *Define* $\Gamma(a) = 2(\sqrt{2a} - a)$ *and*

$$U = \{(a,b) \in \mathbb{R}^2 : a < 0 \text{ and } |b| \leq \min\{|a|, \Gamma(|a|)\}\},$$
$$V = \left\{(a,b) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1,1] : |b| \geq |a|\right\},$$
$$W = \{(a,b) \in \mathbb{R}^2 : a > 0 \text{ and } |b| \leq \min\{|a|, \Gamma(|a|)\}\}.$$

*If for every* $(a,b) \in \mathbb{R}^2$ *we define*

$$f_+(a,b) = 1 - a - |b|, \qquad f_-(a,b) = -f_+(-a,b),$$

*and for every* $(a,b) \in \mathbb{R}^2$ *with* $a \neq 0$ *we define*

$$g_+(a,b) = \frac{b^2}{4a} - 1, \qquad g_-(a,b) = -g_+(-a,b),$$

*then*

(a) $\mathsf{S}_{\mathcal{P}_2(\mathbb{R})} = \operatorname{graph}(f_+|_{W\cup V}) \cup \operatorname{graph}(f_-|_{U\cup V}) \cup \operatorname{graph}(g_+|_W) \cup \operatorname{graph}(g_-|_U)$.

(b) $\operatorname{ext}(\mathsf{B}_{\mathcal{P}_2(\mathbb{R})}) = \{\pm(t, \pm\Gamma(t), 1 - t - \Gamma(t)) : t \in [1/2, 2]\} \cup \{\pm(0,0,1)\}$.

We show a picture of $\mathsf{S}_{\mathcal{P}_2(\mathbb{R})}$ in figure 1.

For fixed $m > n$ in $\mathbb{N}$, the geometry of the space $\{ax^m + bx^n + c : a,b,c \in \mathbb{R}\}$ endowed with the sup norm on $[-1,1]$ has been studied by Muñoz-Fernández and Seoane-Sepúlveda [33] for all possible choices of $m,n$. The results depend strongly on whether $m$ and $n$ are even or odd.

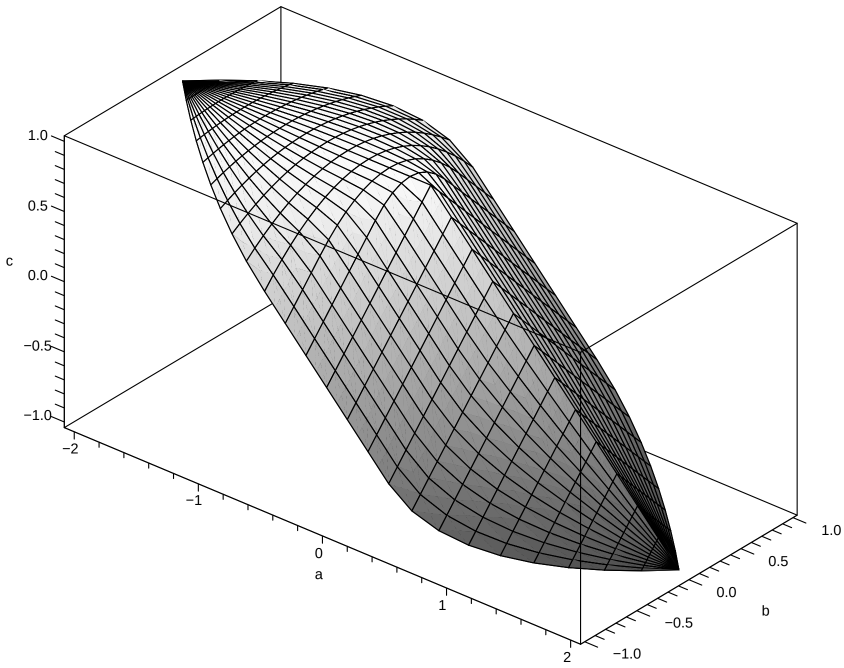


**Figure 1:** Unit sphere of $\mathcal{P}_2(\mathbb{R})$.

## 3.2. The geometry of $\mathcal{P}(^2\Delta)$

Recall first that $\mathcal{P}(^2\Delta)$ is the space of polynomials $P(x,y) = ax^2 + by^2 + cxy$ endowed with the norm defined by

$$\|P\|_\Delta = \sup\{|P(\mathbf{x})| : \mathbf{x} \in \Delta\},$$

where $\Delta$ represents the region enclosed by the triangle in $\mathbb{R}^2$ of vertices $(0,0)$, $(0,1)$ and $(1,0)$ (or simplex, for short). All the results in this section are taken from the work of Muñoz-Fernández, Révész, and Seoane-Sepúlveda [31]. First, it is convenient to have a formula to calculate the norm $\|\cdot\|_\Delta$.

**Theorem 17**. *Let $a, b, c \in \mathbb{R}$ and $P(x,y) = ax^2 + by^2 + cxy$. Then,*

$$\|P\|_\Delta = \begin{cases} \max\left\{|a|, |b|, \left|\frac{c^2-4ab}{4(a-c+b)}\right|\right\} & \text{if } a-c+b \neq 0 \text{ and } 0 < \frac{2b-c}{2(a-c+b)} < 1, \\ \max\{|a|, |b|\} & \text{otherwise.} \end{cases} \tag{5}$$

Now we provide a parametrisation of $\mathsf{S}_{\mathcal{P}(^2\Delta)}$ and describe the geometry of $\mathsf{B}_{\mathcal{P}(^2\Delta)}$. We use the notations $\mathsf{S}_\Delta$ and $\mathsf{B}_\Delta$ for short.

**Theorem 18**. *If we define the mappings*

$$f_+(a,b) = 2 + 2\sqrt{(1-a)(1-b)}$$

*and*

$$f_-(a,b) = -f_+(-a,-b) = -2 - 2\sqrt{(1+a)(1+b)},$$

*for every $(a,b) \in [-1,1]^2$, and the set*

$$F = \{(a,b,c) \in \mathbb{R}^3 : (a,b) \in \partial[-1,1]^2 \text{ and } f_-(a,b) \leq c \leq f_+(a,b)\},$$

*where $\partial[-1,1]^2$ is the boundary of $[-1,1]^2$, then*

(a) $\mathsf{S}_\Delta = \operatorname{graph}(f_+|_{[-1,1]^2}) \cup \operatorname{graph}(f_-|_{[-1,1]^2}) \cup F$.

(b) $\operatorname{ext}(\mathsf{B}_\Delta) = \{\pm(1, t, -2-2\sqrt{2(1+t)}), \pm(t, 1, -2-2\sqrt{2(1+t)}) : t \in [-1,1]\}$.

You can find a picture of $\mathsf{S}_\Delta$ in figure 2.

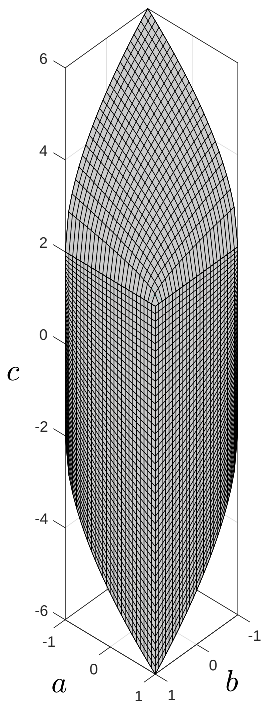


**Figure 2:** Unit sphere of $\mathcal{P}(^2\Delta)$.

### 3.3. The geometry of $\mathcal{P}(^2D(\alpha,\beta))$

First, recall that $\mathcal{P}(^2D(\alpha,\beta))$ is the 3-dimensional space of the polynomials $ax^2+by^2+cxy$ endowed with the norm

$$\|P\|_{D(\alpha,\beta)} := \sup\{|P(\mathbf{x})| : \mathbf{x} \in D(\alpha,\beta)\}.$$

It is a simple exercise to show that the spaces $\mathcal{P}(^2D(\alpha,\alpha+\beta))$ and $\mathcal{P}(^2D(0,\beta))$ are isometric. We write $D(\beta)$ instead of $D(0,\beta)$ for simplicity. Actually, the isometry is given by the matrix

$$\begin{pmatrix} \cos^2\alpha & \sin^2\alpha & \frac{\sin 2\alpha}{2} \\ \sin^2\alpha & \cos^2\alpha & -\frac{\sin 2\alpha}{2} \\ -\sin 2\alpha & \sin 2\alpha & \cos 2\alpha \end{pmatrix}.$$

This isometry allows us to restrict our attention to the study of the geometry of $\mathsf{B}_{D(\beta)}$.

A moment's thought reveals that, if $\beta \ge \pi$, then $\mathsf{B}_{D(\beta)} = \mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$, where $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ stands for the closed unit ball of the space $\mathcal{P}(^2\ell_2^2)$ of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the sup norm over the unit disk. The extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ were described by Choi and Kim [10]. An alternative approach was provided by Muñoz-Fernández, Pellegrino, Seoane-Sepúlveda, and Weber [30].

**Theorem 19.** *Let $\beta \ge \pi$ and define $f(a,b) = 2\sqrt{1+ab-|a+b|}$ on $[-1,1]^2$. Then,*

(a) $\|P\|_{D(\beta)} = \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2+c^2}\right)$*, for all $P \in \mathcal{P}(^2D(\beta))$.*

(b) $\mathsf{S}_{\mathcal{P}(^2D(\beta))} = \mathrm{graph}(f) \cup \mathrm{graph}(-f)$.

(c) $\mathrm{ext}(\mathsf{B}_{\mathcal{P}(^2D(\beta))}) = \{\pm(t,-t,2\sqrt{1-t^2}) : t \in [-1,1]\} \cup \{\pm(1,1,0)\}$.

The reader can find a graph of $\mathsf{S}_{\mathcal{P}(^2\ell_2^2)}$ in figure 3.

Let us give just another example in this section, taken from the work of Muñoz-Fernández et al. [30].

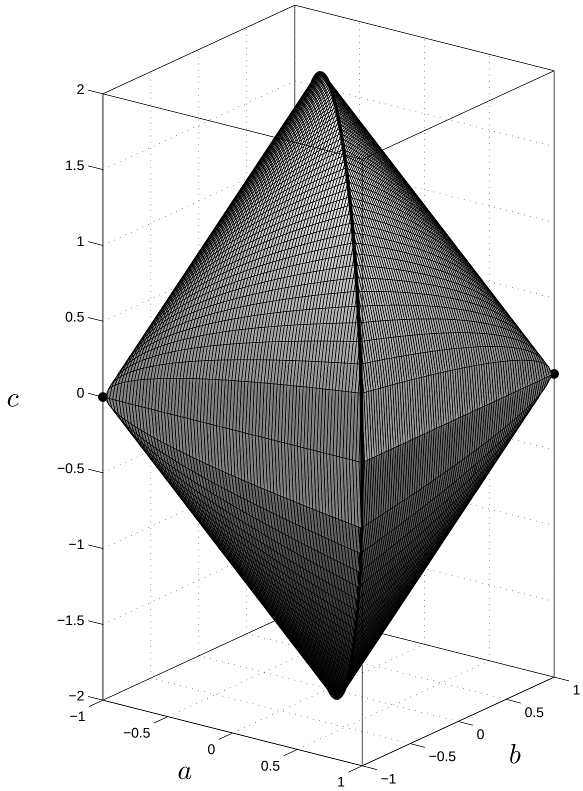


**Figure 3:** $\mathsf{S}_{\mathcal{P}(^2\ell_2^2)}$. The extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ are drawn with a thick line and dots.

**Theorem 20.** *If we define the mappings*

$$G_1(a,b) = 2\sqrt{(1-a)(1-b)}$$

*and*

$$G_2(a,b) = -G_1(-a,-b) = -2\sqrt{(1+a)(1+b)},$$

*for every* $(a,b) \in [-1,1]^2$*, and the set*

$$F = \{(a,b,c) \in \mathbb{R}^3 : (a,b) \in \partial[-1,1]^2 \text{ and } G_2(a,b) \leq c \leq G_1(a,b)\},$$

*where* $\partial[-1,1]^2$ *is the boundary of* $[-1,1]^2$*, then*

(a) $\|P\|_{D(\frac{\pi}{2})} = \max\left\{|a|, |b|, \frac{1}{2}|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}|\right\}$ *for every* $P \in \mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$.

(b) $\mathsf{S}_{D(\frac{\pi}{2})} = \operatorname{graph}(G_1) \cup \operatorname{graph}(G_2) \cup F$.

(c) $\operatorname{ext}(\mathsf{B}_{D(\frac{\pi}{2})}) = \{\pm\left(1, t, -2\sqrt{2(1+t)}\right), \pm\left(t, 1, -2\sqrt{2(1+t)}\right) : t \in [-1,1]\} \cup \{\pm(1,1,0)\}$.

The reader can find a sketch of $\mathsf{S}_{\mathcal{P}(^2D(\frac{\pi}{2}))}$ in figure 4.

## 4. Polynomial inequalities

A number of polynomial inequalities can be tackled using the Krein-Milman approach described right after theorem 15. In this section we will introduce some problems of interest together with a sample of the type of results that can be achieved using the Krein-Milman approach.

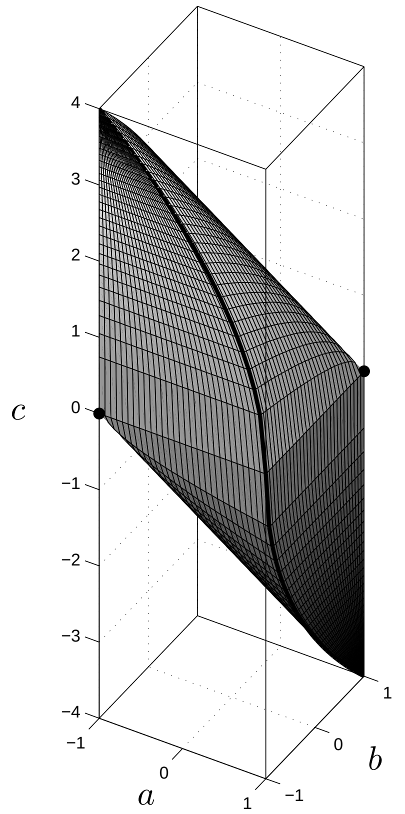


**Figure 4:** $\mathsf{S}_{D(\frac{\pi}{2})}$. The extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ are drawn with a thick line and dots.

## 4.1. The Bohnenblust-Hille inequality and related problems

The $\ell_q$ norm of the coefficients of polynomials in $\mathcal{P}(^m\mathbb{K}^n)$ is given by

$$|P|_q := \begin{cases} \left(\sum_{|\alpha|=m} |a_\alpha|^q\right)^{\frac{1}{q}} & \text{if } 1 \le q < +\infty, \\ \max\{|a_\alpha| \,:\, |\alpha| = m\} & \text{if } q = +\infty, \end{cases}$$

for every $P \in \mathcal{P}(^m\mathbb{K}^n)$ with coefficients $a_\alpha$. Observe that

$$|\cdot|_q \le |\cdot|_s \le d^{\frac{1}{s}-\frac{1}{q}} |\cdot|_q,$$

for $1 \le s \le q$, where $d$ is the dimension of $\mathcal{P}(^m\mathbb{K}^n)$. These norms appear in a number of problems of interest. They can also be used to estimate the difficult-to-calculate polynomial norm of the spaces $\mathcal{P}(^m\ell_p^n)$. Let us denote the norm in $\mathcal{P}(^m\ell_p^n)$ by $\|\cdot\|_p$. Since the norms $|\cdot|_q$ and $\|\cdot\|_p$ are equivalent for all $p, q \ge 1$, there exist $k$ and $K$ depending on $p, q, m, n$ such that

$$k\|P\|_p \le |P|_q \le K\|P\|_p,$$

for all $P \in \mathcal{P}(^m\mathbb{R}^n)$. The optimal values of the constants $k$ and $K$ can be calculated in many situations using the Krein-Milman approach. Indeed, as for the constant $K$, the target function to which the Krein-Milman approach could be applied is

$$B_{\|\cdot\|_p} \ni P \mapsto |P|_q.$$

Here is where the geometry of $B_{\|\cdot\|_p}$ can be used to optimise the target functions in the case when we have a description of its extreme points.

The equivalence constants which we have just introduced are closely related to the famous polynomial Bohnenblust-Hille constants. Let us call $K_{m,n,q,p}$ the best (smallest) value of $K$ in (4.1). The $m$-th polynomial Bohnenblust-Hille constant is nothing but an upper bound for $K_{m,n,\frac{2m}{m+1},\infty}$. The reason why the specific choice $q = \frac{2m}{m+1}$ and $p = \infty$ is of interest rests on the fact that, if $q \geq \frac{2m}{m+1}$, then there exists a constant $D_{m,q} > 0$ depending only on $m$ and $q$ such that

$$|P|_q \leq D_{m,q} \|P\|_\infty, \tag{6}$$

for all $P \in \mathcal{P}(^m\mathbb{K}^n)$ and every $n \in \mathbb{N}$. Moreover, any constant fitting in (6) for $q < \frac{2m}{m+1}$ depends necessarily on $n$. This result was proved by Bohnenblust and Hille [7] in 1931. Observe that any plausible choice for $D_{m,q}$ in (6) must satisfy $D_{m,q} \geq \sup\{K_{m,n,q,\infty} \,:\, n \in \mathbb{N}\}$. The best (in the sense of smallest) possible choice for $D_{m,q}$ in (6) when $q = \frac{2m}{m+1}$ is called the polynomial Bohnenblust-Hille constant. It is interesting to notice that there exists a considerable difference between the polynomial Bohnenblust-Hille constants for real and complex polynomials. For this reason, the polynomial Bohnenblust-Hille constants are usually denoted by $D_{\mathbb{K},m}$.

Moreover, if we keep $n \in \mathbb{N}$ fixed, the best (smallest) $D_m(n) > 0$ in

$$|P|_{\frac{2m}{m+1}} \leq D_m(n) \|P\|_\infty,$$

for all $P \in \mathcal{P}(^m\mathbb{K}^n)$, is denoted by $D_{\mathbb{K},m}(n)$. Observe that $D_{\mathbb{K},m}(n) = K_{m,n,\frac{2m}{m+1},\infty}$. The calculation of the Bohnenblust-Hille constants $D_{\mathbb{K},m}$ and $D_{\mathbb{K},m}(n)$ has motivated a large amount of papers, but their exact values are still unknown except for very restricted choices of $m$ and $n$. The best lower and upper estimates on $D_{\mathbb{K},m}$ and $D_{\mathbb{K},m}(n)$ known nowadays can be found in the literature [3, 8, 9, 12–14, 20, 25].

We present below a simple application of the Krein-Milman approach to calculate the value of $D_{\mathbb{R}}(2)$ based on the following result by Choi, Kim, and Ki [11].

**Theorem 21.** *The set* $\operatorname{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))})$ *of extreme points of the unit ball of* $\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))$ *is given by*

$$\operatorname{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))}) = \{\pm x^2,\ \pm y^2,\ \pm(tx^2 - ty^2 \pm 2\sqrt{t(1-t)}xy) \,:\, t \in [1/2, 1]\}.$$

**Theorem 22.** *Let $f$ be the real-valued function given by*

$$f(t) = \left[2t^{\frac{4}{3}} + (2\sqrt{t(1-t)})^{\frac{4}{3}}\right]^{\frac{3}{4}}.$$

*We have that* $D_{\mathbb{R},2}(2) = f(t_0) \approx 1.837\,373$*, where*

$$t_0 = \frac{1}{36}\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right) \approx 0.867\,835.$$

*Moreover, the following normalized polynomials are extreme for this problem:*

$$P_2(x, y) = \pm\left(t_0x^2 - t_0y^2 \pm 2\sqrt{t_0(1-t_0)}xy\right).$$

*Proof.* We just have to notice that, due to the convexity of the $\ell_p$-norms and theorem 21, we have

$$D_{\mathbb{R},2}(2) = \sup\{|\mathbf{a}|_{\frac{4}{3}} \,:\, \mathbf{a} \in \mathsf{B}_{\mathcal{P}(^2\ell_\infty^2\mathbb{R})}\} = \sup\{|\mathbf{a}|_{\frac{4}{3}} \,:\, \mathbf{a} \in \operatorname{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2\mathbb{R})})\} = \sup_{t\in[1/2,1]} f(t).$$

The function $f$ is maximized using elementary calculus. The help of computer packages of symbolic calculus such as Matlab may be helpful to prove that $f$ attains its maximum in $[1/2, 2]$ at $t = t_0$, thus concluding the proof. ∎

## 4.2. Bernstein and Markov type inequalities

Bernstein and Markov inequalities are estimates on the growth of the derivatives of polynomials. The famous Russian chemist D. Mendeleev (the author of the periodic table of elements) was among the pioneers that studied these types of estimates. In particular, he was interested in the following problem:

> If $p(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and we define $\|p\|_{[\alpha,\beta]} := \max\{|p(x)| : x \in [\alpha, \beta]\}$, then what is the smallest possible constant $M_2(\alpha, \beta) > 0$ so that $|p'(x)| \le M_2(\alpha, \beta)\|p\|_{[\alpha,\beta]}$ for every $x \in [\alpha, \beta]$ and every quadratic polynomial $p$?

Considering an appropriate change of variable, namely $x \to [\alpha + \beta + (\beta - \alpha)x]/2$, it can be seen that $M_2(\alpha, \beta) = 2/(\beta - \alpha)M_2(-1, 1)$ and, hence, we can restrict ourselves to (quadratic) polynomials on the standard interval $[-1, 1]$. Mendeleev gave his own solution to the problem proving that $M_2(-1, 1) = 4$. Mendeleev's result was generalised by A. A. Markov in 1889 for polynomials of arbitrary degree [26]. What A. A. Markov proved was that

$$\|P'\|_{[-1,1]} \le n^2\|P\|_{[-1,1]}, \tag{7}$$

with equality for the $n$-th Chebyshev polynomial of the first kind, defined, for $x \in [-1, 1]$, by $T_n(x) = \cos(n \arccos x)$. V. A. Markov [27] (brother of A. A. Markov) provided in 1892 a sharp estimate on the norm of the $k$-th derivative of a polynomial of arbitrary degree. A. A. Markov's inequality (7) can be improved in the inner points of $[-1, 1]$. Let $M_n(x)$ be the optimal constant in

$$|P'(x)| \le M\|P\|_{[-1,1]}, \qquad \text{for all } P \in \mathcal{P}_n(\mathbb{R}).$$

According to a classical result due to Bernstein [6], we have $M_n(x) \le \frac{n}{\sqrt{1-x^2}}$ in $(-1, 1)$. Both the uniform Markov type estimates on the norm of the derivative and the pointwise estimates due to Bernstein have been generalised in many different ways in the last century. One of the most popular generalisations is due to Harris in 2010 [19]. He proved that the old A. A. Markov constant $n^2$ is valid for polynomials on any real Banach space, that is, if $P$ is a polynomial of arbitrary degree $n$ on a real Banach space $E$, then $\|DP(x)\| \le n^2\|P\|$. Obviously, the constant $n^2$ is optimal in the general case too. Many Bernstein and Markov type estimates can be obtained by applying the Krein-Milman approach. We present here a worked out example where the Markov and Bernstein optimal estimates are obtained for the space of trinomials $\mathcal{P}_{m,n} = \{ax^m + bx^n + c\}$ endowed with the norm

$$\|ax^m + bx^n + c\|_{m,n} = \sup\{|ax^m + bx^n + c| : x \in [-1, 1]\}.$$

Observe that the polynomial $ax^m + bx^n + c$ in $\mathcal{P}_{m,n}$ can be identified with $(a, b, c)$ in $\mathbb{R}^3$. The geometry of the space $\mathcal{P}_{m,n}$ was studied by Muñoz-Fernández and Seoane-Sepúlveda [33], and the optimal value of the Markov constant $M_{m,n}$ and the Bernstein function $M_{m,n}(x)$ where calculated by Muñoz-Fernández, Sarantopoulos, and Seoane-Sepúlveda [32] when $m$ is odd and $n$ is even. We reproduce here the complete reasoning, based on the Krein-Milman approach and the following characterisation of the extreme points of $\mathsf{B}_{m,n}$ (unit ball of $\mathcal{P}_{m,n}$) when $m$ is odd and $n$ is even.

**Theorem 23**. *If $m, n \in \mathbb{N}$ are such that $m$ is odd, $n$ is even and $m > n$, the extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{m,n})$ are*

$$\{\pm(0, 2, -1), \pm(1, 1, -1), \pm(1, -1, 1), \pm(0, 0, 1)\}.$$

**Theorem 24**. *Let $m, n \in \mathbb{N}$ be such that $m$ is odd, $n$ is even and $m > n$. Then,*

$$M_{m,n}(x) = \begin{cases} 2n|x|^{n-1} & \text{if } 0 \le |x| \le \left(\frac{n}{m}\right)^{\frac{1}{m-n}}, \\ mx^{m-1} + n|x|^{n-1} & \text{if } \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \le |x| \le 1. \end{cases} \tag{8}$$

*Proof*. If $x \in [-1, 1]$, by definition we have that

$$M_{m,n}(x) = \sup_{p \in \mathsf{B}_{m,n}} |p'(x)|.$$

It suffices to work just with the extreme polynomials of $M_{m,n}$, which are given in theorem 23. Notice that the contribution of $\pm 1$ to $M_{m,n}(x)$ is irrelevant. Hence, it suffices to consider the polynomials

$$p_1(x) = \pm(2x^n - 1), \quad p_2(x) = \pm(x^m + x^n - 1) \quad \text{and} \quad p_3(x) = \pm(x^m - x^n + 1).$$

Therefore,

$$\begin{aligned} M_{m,n}(x) &= \max\{|p_1'(x)|, |p_2'(x)|, |p_3'(x)|\} \\ &= \max\{2n|x|^{n-1}, |mx^{m-1} + nx^{n-1}|, |mx^{m-1} - nx^{n-1}|\} \\ &= \max\{2n|x|^{n-1}, mx^{m-1} + n|x|^{n-1}\} \\ &= |x|^{n-1} \max\{2n, m|x|^{m-n} + n\}, \end{aligned}$$

and since $2n \le m|x|^{m-n} + n$ and $\left(\frac{n}{m}\right)^{\frac{1}{m-n}} \le |x|$ are equivalent, the result follows immediately. ■

**Corollary 25.** *If $m, n \in \mathbb{N}$ are such that $m$ is odd, $n$ is even and $m > n$, then*

$$M_{m,n} = M_{m,n}(\pm 1) = m + n,$$

*and equality is attained for the polynomials $p(x) = \pm(x^m + x^n - 1)$ and $p(x) = \pm(x^m - x^n + 1)$.*

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