3rd *BYMAT Conference* (2020)
Invited article

Characters of finite groups

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Abstract: Groups are the mathematical objects formally describing our idea of symmetry. They appear naturally acting on vector spaces as groups of invertible matrices. Group representation theory is the branch of mathematics that studies such actions. More specifically, character theory studies the trace maps associated to those actions. A fundamental question in the field is to understand how much information about a finite group G and its local subgroups can be extracted from the knowledge of the character theory of G. In this note, I will report on recent advances in this topic.

Resumen: La teoría de grupos es la rama de las matemáticas que describe y estudia las simetrías. Los grupos aparecen de forma natural actuando sobre espacios vectoriales como matrices invertibles. La teoría de representaciones se encarga de estudiar dichas acciones como homomorfismos entre grupos y grupos de matrices. En particular, la teoría de caracteres estudia las trazas asociadas a tales homorfismos. Un problema central en el área es descubrir qué información acerca de la estructura de un grupo G y sus subgrupos locales puede leerse en su tabla de caracteres. El propósito de esta monografía es exponer recientes contribuciones a este problema en el marco de las conjeturas globales-locales.

Keywords: finite groups, character tables, Sylow subgroups, global-local conjectures.

MSC2010: 20C15, 20C20.

Acknowledgements: This works is partially supported by Spanish Ministerio de Ciencia e Innovación MTM2017-82690-P and PID2019-103854GB-I00, FEDER funds and the ICMAT Severo Ochoa project SEV-2015-0554.

I would like to thank Eugenio Giannelli, Gabriel Navarro and Mandi Schaeffer Fry for their useful comments on this note.

Reference: Vallejo Rodríguez, Carolina. "Characters of finite groups". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 1-6. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p1.

1. Character theory: a brief historical remark

The theory of groups is the abstract mathematical framework to describe the intuitive human concept of symmetry. It has been said by the mathematician and sci-fi writer E. T. Bell that "wherever groups disclose themselves or can be introduced, simplicity crystallizes out of comparative chaos", a sentence that highlights the usefulness of group theory in other branches of mathematics. In his Erlangen program, F. Klein proposed to study the geometry of a space through its group of symmetries. Outside mathematics, groups play a crucial role in several other disciplines like quantum physics, chemistry and cryptography.

The birth of group theory goes back to the work on polynomial equations of J. L. Lagrange and E. Galois, where groups appeared as permutations of the roots of a polynomial. In his investigations, Galois already introduced key concepts in group theory like those of normal subgroup, simple group and solvable group. However, we owe the first abstract definition of group to Cayley by the end of the 19th century.

Character theory and, more generally, representation theory study how groups act on a vector space. Properties of representations and characters of a group are intimately connected to the algebraic structure of the group itself. This mutual influence is successfully used to study one in terms of the other. Indeed, the study of finite groups bloomed in the early 20th century thanks to the springtime of representation and character theory. The pioneering work of W. Burnside, F. G. Frobenius and I. Schur laid the foundations of this area of mathematics. Burnside proved in 1904 that finite groups whose order is divisible by at most two primes are solvable [3]. This was the first main application of character theory to group theory, and we care to remark that a proof of Burnside's p^aq^b theorem not involving character theoretical arguments was not found until 1972 by D. Goldschmidt [6] and H. Bender [1].

In 1963, W. Feit and J. Thompson proved that groups of odd order are solvable [5]. Their (225 pages long) proof requires a mixture of deep group and character theoretical arguments. For his contributions to this success, Thompson was awarded a Fields Medal in 1970 and an Abel Prize in 2008. Moreover, the solvability of groups of odd order lies at the heart of one of the greatest achievements of mathematics in the last two centuries: the classification of finite simple groups [4].

2. Character tables and Sylow subgroups

In 1963, R. Brauer published an inspiring survey article on the representations of finite groups [2]. In its introduction, Brauer writes that "A tremendous effort has been made by mathematicians for more than a century to clear up the chaos in group theory. Still, we cannot answer some of the simplest questions." Far from being critical of the theory of finite groups, Brauer claims to be fascinated by its mysteries. In that landmark survey, he set up a list of 42 problems that still guides the research on representation theory. Among the most significant problems contained in his article we find the so-called Brauer's $k(B)$-conjecture (Problem 20) and Brauer's height zero conjecture (Problem 23). These conjectures remain open today, although a great deal of work has been devoted to them (see [9, 11, 17], for instance).

In this note, we focus our attention on Brauer's Problem 12. For a finite group G, we denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G (those characters afforded by G-actions on vector spaces without G-invariant subspaces). If $\chi \in \mathrm{Irr}(G)$, then $\chi: G \to \mathbb{C}$ is a function constant on G-conjugacy classes. It is well known, and follows from the Wedderburn decomposition of the algebra $\mathbb{C}G$, that $|\mathrm{Irr}(G)| = k$ is the number of G-conjugacy classes of G. Hence, we can arrange the values of the irreducible characters of G in a $(k \times k)$ matrix $X(G)$ known as the character table of G. The value $\chi(1)$ is the degree of χ, and coincides with the dimension of a G-vector space affording χ. It is customary to arrange $X(G)$ so that its first column is the column corresponding to irreducible character degrees. Also, the first row of $X(G)$ usually contains the values of the principal character $1_G: G \to \mathbb{C}$, coming from the trivial action of G on $\mathbb{C}$. For example, the character table of S_3, the symmetric group on 3 symbols, is

$$X(\mathsf{S}_3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

Character tables are invertible matrices whose rows and columns satisfy amazing numerical relations (see [7, Chapter 2] for details on the Schur orthogonality relations).

Problem (Brauer's Problem 12). Given the character table $X(G)$ of a group G and a prime p dividing the order $|G|$ of G, how much information about the structure of the Sylow p-subgroup P of G can be obtained? In particular, can it be decided whether or not P is abelian? ◂

We recall that a Sylow p-subgroup P of G is a p-subgroup of G of order $|G|_p$, the largest p-power dividing $|G|$. The set $\mathrm{Syl}_p(G)$ of Sylow p-subgroups of G is non-empty, its elements form a G-conjugacy class and they *dominate* the p-subgroups of G. Sylow theory is a cornerstone of group theory. We care to mention that the character table of a group does not determine the isomorphism class of its Sylow subgroups. For instance, the dihedral group D_8 of order 8 and the quaternion group Q_8 have the same character table (after possibly rearranging rows and columns).

The Sylow subgroups of S_3 are abelian for every prime p. How can this information be extracted from $X(\mathsf{S}_3)$? For $p = 3$, we notice that every irreducible character of S_3 has degree coprime to p. The Itô-Michler theorem guarantees that such a condition is equivalent to S_3 having a normal and abelian Sylow 3-subgroup. For $p = 2$, we observe that not every irreducible character degree of S_3 is odd. Nevertheless, the degree of every irreducible character belonging to the *principal* 2*-block* of S_3 is odd (the characters in the principal 2-block of S_3 are the ones corresponding to the first and second row of $X(\mathsf{S}_3)$, but we do not wish to get into technical details at this point of the exposition). This condition is equivalent to S_3 having an abelian Sylow 2-subgroup, by the principal block case of Brauer's height zero conjecture (a key case that has been recently proven [12]).

The character table $X(G)$ of a group G determines the order $|G|$ of the group by the well-known formula

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

In particular, we do not need to appeal to the Itô-Michler theorem nor the principal block case of Brauer's height zero conjecture to deduce that the Sylow subgroups of S_3 are abelian (but we thought it could be a good way of informally introducing their statements). Once we know that $X(G)$ easily determines $|P|$, the order of a Sylow p-subgroup P of G, it makes sense to wonder whether $X(G)$ determines the order $|\mathbf{N}_G(P)|$ of the normalizer of P (see [15, Question 7]). At the time of this writing, that is an open question. A positive answer is known in the special case where $\mathbf{N}_G(P) = P$. In other words, we can tell whether $|\mathbf{N}_G(P)| = |G|_p$ after an easy inspection of $X(G)$. This follows from the main results of [19] and [24] for p odd and $p = 2$, respectively.

What more can be said about the structure of $\mathbf{N}_G(P)$ in $X(G)$? Apart from determining if $\mathbf{N}_G(P)$ is a p-group, we can also determine if $\mathbf{N}_G(P)$ is p*-nilpotent*, that is, if $\mathbf{N}_G(P) = P \times X$. This fact follows from [20] and [21] for p odd and from [22] and [26] for $p = 2$.

Let us go back to the structure of P. We have mentioned above that, by the main result of [12], one can determine if P is abelian by looking at the character degrees of irreducible characters in the principal p-block of $X(G)$. The set of irreducible characters of G belonging to the principal p-block is

$$\{\chi \in \mathrm{Irr}(G) \mid \sum_{x \in G^0} \chi(x) \neq 0\},$$

where $G^0 \subseteq G$ consists of those elements of order not divisible by p. Without getting into further technical details, we care to remark that, by Higman's theorem [7, Theorem 8.21], the set of irreducible characters of G belonging to the principal p-block can be determined after an easy inspection of $X(G)$.

It is also possible to determine whether P is cyclic by looking only at $X(G)$ (an elementary proof can be found in [15, Theorem 8]). A huge step further is to consider whether $X(G)$ determines if P is 2-generated. For example, if a group G has a cyclic (1-generated) Sylow 2-subgroup, then G has a normal 2-complement. In particular, such a group is solvable by Feit-Thompson's odd order theorem [5]. In opposition, there are many nonsolvable groups possessing a 2-generated Sylow 2-subgroup. Actually, the number of isomorphism classes of 2-groups of order 2^n that are 2-generated grows exponentially with n. Despite the greater degree of difficulty, we have recently shown in [16] that $X(G)$ determines if a Sylow 2-subgroup is 2-generated. What happens for odd primes, we do not know. We expect that $X(G)$ determines if P is 2-generated if $p = 3$, but for larger primes the situation might be very different.

3. Global-local conjectures

The purpose of this last section is to briefly describe a deep system of interconnected conjectures underlying most of the results mentioned in the context of Brauer's Problem 12 above. This system consists of the so-called global-local conjectures. The common philosophy behind all these conjectures is that certain essential information on the character theory of a finite group G is encoded in its local subgroups (we refer the reader to [10] for a detailed account on the global-local principle in representation theory). By local subgroups of G we mean its nontrivial p-subgroups and their normalizers, where p is any fixed prime. The most important local subgroups are the Sylow subgroups and their normalizers.

One of the most paradigmatic global-local conjectures is the McKay conjecture.

Conjecture (McKay, 1971). *Let G be a finite group, let p be a prime and $P \in \mathrm{Syl}_p(G)$. The number of* $\mathrm{Irr}(G)$ *of degree not divisible by p equals the number of* $\mathrm{Irr}(\mathbf{N}_G(P))$ *of degree not divisible by p.*

We write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) \text{ is not divisible by } p\}$. As a global-local statement, the McKay conjecture is telling us that global invariant $|\mathrm{Irr}_{p'}(G)|$ is a local invariant, in the sense that it can be computed as $|\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$ in the local subgroup $\mathbf{N}_G(P)$.

In Section 2, we said that wether a group G has a self-normalizing Sylow p-subgroup P, that is, $\mathbf{N}_G(P) = P$, can be read off from $X(G)$. What does the McKay conjecture predict in such a situation? Assume that $\mathbf{N}_G(P) = P$; then, the McKay conjecture asserts that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(P)|$. As the degrees of irreducible characters divide the order of the group, we have that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Lin}(P)|$, where $\mathrm{Lin}(P) = \mathrm{Hom}(P, \mathbb{C}^\times) \cong P/P'$. Hence, if $\mathbf{N}_G(P) = P$, then the McKay conjecture predicts that $|\mathrm{Irr}_{p'}(G)| = |P : P'|$. Unfortunately, this property does not characterize groups with a self-normalizing Sylow p-subgroup (as shown by S_3 for $p = 3$). More than that, we do not know if $X(G)$ determines $|P : P'|$ in the case where $P' > 1$ (not even if we restrict ourselves to the realm of p-solvable groups, as explained in [15]).

Nevertheless, we have mentioned that global-local conjectures underlie most of the results contained in Section 2. The key is held by the so-called Galois version of the McKay conjecture proposed by Navarro [14], also known as the McKay-Navarro conjecture.

The values of $\chi \in \mathrm{Irr}(G)$ are sums of roots of unity and lie in $\mathbb{Q}(\mathrm{e}^{2\pi \mathrm{i}/|G|})$. Given $\sigma \in \mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\mathrm{e}^{2\pi \mathrm{i}/|G|})/\mathbb{Q})$, the function $\chi^\sigma = \sigma(\chi)$ is an irreducible character of G. Hence, $\mathcal{G}$ acts on $\mathrm{Irr}(G)$ (so on the rows of $X(G)$). As the McKay conjecture predicts the existence of a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$, and the Galois group $\mathcal{G}$ acts on both sets, it is natural to wonder if such a bijection can be expected to commute with the action of $\mathcal{G}$ (that is, if such bijection can be expected to be $\mathcal{G}$-equivariant). The general linear group $\mathrm{GL}(2,3)$ provides a negative answer to that question for $p = 3$, as $\mathrm{GL}(2,3)$ has less rational-valued irreducible characters of degree not divisible by 3 than the dihedral group D_{12}, the normalizer of a Sylow 3-subgroup.

Let $\mathcal{H}_p \leq \mathcal{G}$ be the subgroup consisting of those Galois automorphisms $\sigma \in \mathcal{G}$ for which there exists a fixed integer f such that $\sigma(\xi) = \xi^{p^f}$ for every root of unity $\xi \in \mathbb{Q}(\mathrm{e}^{2\pi \mathrm{i}/|G|})$ of order not divisible by p.

Conjecture (McKay-Navarro, 2004). *Let G be a finite group, let p be a prime and $P \in \mathrm{Syl}_p(G)$. There exists an $\mathcal{H}_p$-equivariant bijection* $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$.

In [14, Theorem 5.2 and Theorem 5.3], Navarro proves that the McKay-Navarro conjecture implies the main results of [19] and [24]. We say those results are *proven consequences* of the conjecture. The interest of proving consequences of global-local conjectures is twofold: they help us understand new connections between global and local invariants of a group and, at the same time, they provide new evidence for the validity of these elusive conjectures. In a similar way, the McKay-Navarro conjecture is behind the main results of [26], [21] and [16].

In general, the method used to prove such consequences of the McKay-Navarro conjecture is based on a reduction to simple groups of the statement and then on an exhaustive study of the character theory of the finite nonabelian simple groups (and related groups, as the decorated groups). The first part of the method is what we call proving a *reduction theorem.* The origin of the term goes back to the McKay conjecture.

The McKay conjecture was formulated in 1971 (originally just for simple groups and the prime $p = 2$), after evidence found on symmetric groups and the known sporadic groups. It immediately attracted the interest of the community for the simplicity of its formulation, and celebrated group-theorists started verifying it for different families of groups such as solvable groups and general linear groups. The different verifications used *ad hoc* methods, specific to each family. Despite overwhelming evidence, for several decades no general strategy for proving this easy-to-state conjecture was envisaged. It was not until 2007 that a method was proposed. I. M. Isaacs, G. Malle and G. Navarro [8] showed that in order to prove the McKay conjecture for all finite groups, it was enough to verify the so-called inductive McKay condition only for all finite simple groups. Such a result is what we call a reduction theorem. We care to remark that it is not enough that the conjecture is satisfied for all finite simple groups. The inductive McKay condition is much stronger than the conjecture itself and its verification constitutes a true challenge for simple-group theorists. (We omit a description of the inductive condition here due to its highly technical nature, instead we refer the interested reader to [27].) However, the above strategy has proven to be successful. In 2016, Malle and B. Späth [13] verified the inductive McKay condition for all simple groups at the prime $p = 2$. This has lead to one of the highlights in representation theory of the 21st century: the McKay conjecture holds for the prime 2.

Inspired by the success of the *reduction approach* to the McKay conjecture, we have recently obtained a reduction theorem for the McKay-Navarro conjecture [18]. Namely, we have shown that, in order to prove the McKay-Navarro conjecture in full generality, it is enough to verify the *inductive McKay-Navarro condition* for all finite simple groups. We have already mentioned that verifying the inductive McKay condition is a challenge for simple-groups theorists. It is no surprise that verifying the inductive McKay-Navarro condition constitutes a bigger challenge, as it requires a vast knowledge of the character values of decorated simple groups and the interplay between Galois action and the action of group automorphisms on characters. Some examples of finite simple groups satisfying the inductive McKay-Navarro condition have appeared so far [23, 25], and we are aware that exciting new results will appear soon.

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