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On circles enclosing many points

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Abstract: We prove that in every set of n red and n blue points in the plane there are a red and a blue point such that every circle having them in its boundary encloses at least $n(1-1/\sqrt{2})-o(n)$ other points of the set. This is a bichromatic version of a problem introduced by Neumann-Lara and Urrutia. In addition, we show that every set S of n points contains two points such that every circle passing through them encloses at most $\lfloor \frac{2n-1}{3} \rfloor$ other points of S. The results are proved using properties of order-k Voronoi diagrams, in the spirit of the work of Edelsbrunner, Hasan, Seidel and Shen on this problem.

Resumen: Demostramos que en cualquier conjunto de n puntos rojos y n puntos azules en el plano existen un punto rojo y un punto azul tales que cualquier circunferencia que pase por ellos contiene en su interior al menos $n(1-1/\sqrt{2})-o(n)$ puntos del conjunto. Esta es una versión bicromática de un problema propuesto por Neumann-Lara y Urrutia. También probamos que todo conjunto S de n puntos en el plano contiene dos puntos tales que cualquier circunferencia que pase por ellos contiene como mucho $\lfloor \frac{2n-1}{3} \rfloor$ otros puntos de S. Las demostraciones usan propiedades de los diagramas de Voronoi de orden k, al estilo del trabajo de Edelsbrunner, Hasan, Seidel y Shen en este problema.

Keywords: point set, circle containment, Voronoi diagram.

MSC2010: 52C99.

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1. Introduction

Let $\ell(n)$ be the largest number such that every set S of n points in general position in the plane has the following property: There exist $p, q \in S$ such that every circle passing through p and q contains at least $\ell(n)$ other points of S. Neumann-Lara and Urrutia [7] introduced this problem and obtained the bound $\ell(n) \geq \lceil \frac{n-2}{60} \rceil$. This bound was not tight and hence it was improved in a series of papers [1, 4, 5]. The best known bound up-to-date was obtained by Edelsbrunner *et al.* [3], who proved that $\ell(n) \geq n(\frac{1}{2} - \frac{1}{\sqrt{12}}) + O(1) \approx \frac{n}{4.7}$. Later, Ramos and Viaña [9] obtained an independent proof of this lower bound and further proved the following result:

Theorem 1 ([9]). *Every set S of n points in general position in the plane contains two points such that each circle passing through them encloses at least k and at most $n-k-2$ points of S, for $k = (\frac{1}{2} - \frac{1}{\sqrt{12}})n - o(n)$.*

We present an alternative proof of Theorem 1 making use of properties of order-k Voronoi diagrams. The techniques that we use in our proof allow us to obtain two new results: An upper bound condition, Theorem 2, and a bichromatic result, Theorem 3, stated below. The chromatic problem was introduced by Prodromou [8] with d dimensions and $\lfloor \frac{d+3}{2} \rfloor$ colors. In the particular case $d = 2$, it is proved that every set of n red points and m blue points contains a red point and a blue point such that every circle passing through them encloses $\frac{n+m}{36}$ other points of the set. Our result improves this bound.

This is an extended abstract of manuscript [2].

2. Circles and Voronoi diagrams

An order-k Voronoi diagram of a point set S is a subdivision of the plane into regions such that all the points in the same region have the same k closest points of S. The borders between regions are segments of the perpendicular bisectors between pairs of points in S. This is a key concept in our proof of Theorem 1 because the segments of the order-k Voronoi diagram of S are precisely the centers of the circles through two points of S that enclose exactly $k-1$ other points of S [6]. We say that a segment of the perpendicular bisector b_{pq} of p and q has weight k if all the circles through p and q with center in such segment enclose k other points of S. Thus, the segments of the order-k Voronoi diagram have weight $k-1$, see Figure 1.

3. New results

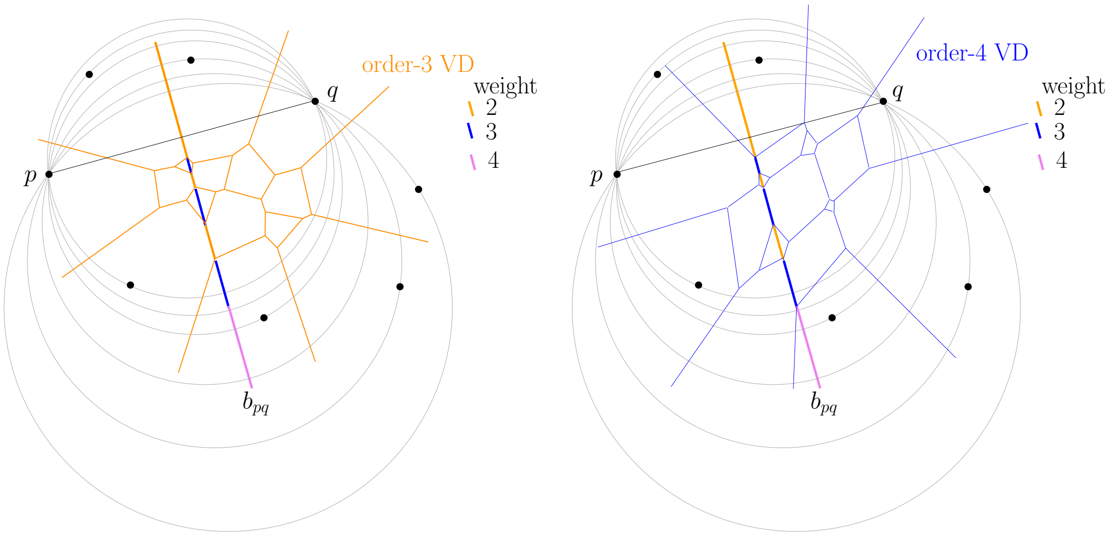
Following the ideas in the previous section, we study an upper bound version of the circle containment problem. Let $u(n)$ be the smallest number such that every set S of n points in general position in the plane has the following property: There exist $p, q \in S$ such that every circle passing through p and q contains at most $u(n)$ other points of S. In Theorem 2 we prove that $u(n) \leq \lfloor \frac{2n-1}{3} \rfloor$.

Theorem 2. *Let S be a set of $n \geq 3$ points in general position in the plane. Then, S contains two points such that every circle passing through them encloses at most $\lfloor \frac{2n-1}{3} \rfloor$ points of S.*

Adapting the proof of Theorem 1 to only consider circles passing through a red point and a blue point, we obtain the following result.

Theorem 3. *Every set S of n red points and $m = \lfloor cn \rfloor$, for $c \in (0, 1]$, blue points in general position in the plane contains a red point p and a blue point q such that any circle passing through them encloses at least $\frac{n+m-\sqrt{n^2+m^2}}{2} - o(n+m)$ points of S.*

For $n = m$, Theorem 3 gives the bound $n(1 - \frac{1}{\sqrt{2}}) - o(n) \approx \frac{n}{3.4}$.



(a) (b)

Figure 1: Relation between the order-k Voronoi diagram and the circle containment problem. (a) The segments of weight 2 are edges of the order-3 Voronoi diagram; (b) The segments of weight 3 are edges of the order-4 Voronoi diagram.

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