3rd *BYMAT Conference* (2020)

Universal spectral covers and the Hitchin map

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Abstract: We study spectral data for pairs (E, φ), where $E \to X$ is a vector bundle over a smooth projective variety and $\varphi : E \to E \otimes V$ is an endomorphism "twisted" by another vector bundle $V \to X$, satisfying a *commuting condition* $\varphi \wedge \varphi = 0$. When $V = \Omega^1_X$ these pairs are known as *Higgs bundles*, which are intimately related to linear representations of the fundamental group of X.

Studying spectral data for this kind of objects consists on describing the fibres of a certain *Hitchin map*. In order to do this, we review the construction of the *universal spectral cover* and the *spectral correspondence* given in a recent paper by Chen and Ngô [2].

Resumen: Realizamos un estudio de los datos espectrales de pares (E, φ), donde $E \to X$ es un fibrado vectorial sobre una variedad proyectiva lisa y $\varphi : E \to E \otimes V$ es un endomorfismo "torcido" por otro fibrado vectorial $V \to X$, satisfaciendo una *condición de conmutación* $\varphi \wedge \varphi = 0$. Cuando $V = \Omega^1_X$ estos pares se conocen como *fibrados de Higgs*, que están íntimamente relacionados con las representaciones lineales del grupo fundamental de X.

Estudiar los datos espectrales para este tipo de objetos consiste en describir las fibras de una cierta *aplicación de Hitchin*. Para hacer esto, repasamos la construcción de la *cubierta espectral universal* y la *correspondencia espectral* dadas en un artículo reciente de Chen y Ngô [2].

Keywords: spectral data, Higgs bundles, Hitchin morphism.

MSC2010: 14D20, 14H60, 14J60.

Acknowledgements: The author is grateful to his thesis advisor Oscar García-Prada for proposing this problem and providing guidance through its study. The author also wants to thank M. S. Narasimhan for pointing to the results of Chen-Ngô and for fruitful discussions about them.

Collaboration with professor Narasimhan has been posible thanks to the support of the Indian Institute of Science and also of the International Centre for Theoretical Sciences during a visit for participating in the program *Moduli of bundles and related structures* (Code: ICTS/mbrs2020/02).

The author's research is funded by the Contract CT63/19-CT64/19 of Universidad Complutense de Madrid.

Reference: Gallego, Guillermo. "Universal spectral covers and the Hitchin map". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 107-110. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p107.

1. Introduction

Let k be an algebraically closed field, X a smooth projective variety over k and $V \to X$ a rank r vector bundle over X. We are interested in studying pairs (E, φ), with $E \to X$ a rank n vector bundle and $\varphi : E \to E \otimes V$ an "endomorphism twisted by V", satisfying a *commuting condition*: $\varphi \wedge \varphi = 0$ as a morphism $E \to E \otimes \wedge^2 V$. Locally, this means that φ can be written as $(\varphi_1, \dots, \varphi_r)$, with $[\varphi_i, \varphi_j] = 0$.

A particularly interesting situation occurs when $V = \Omega^1_X$ is the cotangent bundle of X. In that case we say that such an (E, φ) is a *Higgs bundle*. These objects were introduced by Hitchin in 1987 [5] in the case that X is a Riemann surface, and later generalized by Simpson [6] to Kähler manifolds of higher dimension.

When $k = \mathbb{C}$ is the field of complex numbers, a theorem of Corlette [3] and Simpson [6] identifies the moduli space of *stable* Higgs bundles with the *character variety* parametrizing irreducible representations of the fundamental group $\pi_1(X)$ on $\mathrm{GL}_n(\mathbb{C})$.

We denote by $\mathcal{M}_{n,V}$ the moduli stack of such pairs (E, φ), where E has rank n. The Hitchin morphism is defined as the map

$$h_{n,V} : \mathcal{M}_{n,V} \to \bigoplus_{i=1}^{n} H^0(X, S^i V) : \ (E, \varphi) \mapsto (\sigma_1(\varphi), \dots, \sigma_n(\varphi)),$$

where the $\sigma_i(\varphi)$ are the coefficients of the "characteristic polynomial" of φ,

$$p_\varphi(T) = \det(T - \varphi) = T^n + \textstyle\sum_{i=1}^n \sigma_i(\varphi) T^n.$$

We are interested in *studying the fibres of this morphism.*

When $\dim X = 1$ and the twisting bundle is a line bundle $V = L$, if L^n is base point free, then for a generic $b \in \bigoplus_{i=1}^n (X, L^i)$ there exists a smooth *spectral curve* Y_b, with a finite morphism $Y_b \to X$ such that the fibre of the Hitchin map over L is in correspondence with the Picard group of Y_b. This was shown by Hitchin in 1987 [4] in the case of Higgs bundles ($L = \Omega^1_X$) and later generalized by Beauville, Narasimhan and Ramanan [1] for a general line bundle L.

In particular, what Hitchin proved in [4] is that the Hitchin map endows the moduli space of stable Higgs bundles on a Riemann surface of genus $g \geq 2$ with the structure of an *algebraically integrable system.*

We seek similar results to that of Beauville-Narasimhan-Ramanan for general values of $\dim X$ and $\mathrm{rk}\, V$.

A recent paper by Chen and Ngô [2] has shed some light into this problem, by (among other things) giving an interpretation of the Hitchin map in terms of "universal spectral data" and a proving a spectral correspondence for Higgs bundles over projective surfaces.

In this document we review some of the main ideas of Chen and Ngô's interpretation of the Hitchin map and the spectral correspondence.

2. Universal spectral data

Let E be a k-vector space of dimension n. Consider $\varphi_1, \dots, \varphi_r$ a family of r endomorphisms of E that pairwise commute, that is $[\varphi_i, \varphi_j] = 0$. This family generates a k-algebra $A = k[\varphi_1, \dots, \varphi_r]$. The evaluation morphism $k[X_1, \dots, X_r] \to A \subset \mathrm{End}\, E$ endows E with the structure of a $k[X_1, \dots, X_r]$-module. Geometrically, this can be interpreted as a sheaf $\mathcal{F}$ over the r-dimensional affine space $\mathbb{A}^r_k$ such that $p_* \mathcal{F} = E$, where $p : \mathbb{A}^r_k \to \mathrm{Spec}(k)$ is the natural morphism.

In the particular case where $r = 1$, so $A = k[\varphi]$ for some endomorphism $\varphi : E \to E$, since $k[T]$ is a PID, A can be written as $A = k[T]/(m_\varphi)$, where (m_φ) is the ideal generated by a polynomial m_φ which is by definition the *minimal polynomial* of φ. The roots of m_φ are precisely the eigenvalues $x_1, \dots, x_s \in k$ of φ. If $E = \bigoplus_{i=1}^s E_i$ is the spectral decomposition of E associated to φ and $n_i = \dim E_i$, we can consider the characteristic polynomial $p_\varphi(T) = (T - x_1)^{n_1} \cdots (T - x_s)^{n_s}$. The Cayley-Hamilton theorem asserts that $p_\varphi(\varphi) = 0$, and thus the minimal polynomial m_φ divides p_φ. Geometrically, this implies that the sheaf $\mathcal{F}$ is supported in a closed subscheme of $\mathrm{Spec}(k[T]/(p_\varphi))$.

In the general case of $r > 1$, the ring A is of the form $k[X_1, \dots, X_r]/I$, for I some ideal that in principle can be generated by several polynomials. However, we can take the primary decomposition of I, $I = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_s^{\alpha_s}$. This induces a decomposition of E, $E = \bigoplus_{i=1}^s E_i$. If we denote $n_i = \dim E_i$, we can consider the ideal $J = \mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_s^{n_s}$. Using the Nakayama lemma, one can prove a generalized version of the Cayley-Hamilton theorem asserting that $J \subseteq I$. Geometrically, this means that the sheaf $\mathcal{F}$ is supported in a closed subscheme of $\mathrm{Spec}(k[X_1, \dots, X_r]/J)$.

Therefore, we can associate to the algebra $A = k[\varphi_1, \dots, \varphi_r]$ a formal combination of points of $\mathbb{A}_k^r$ $\mathrm{sd}(A) = \sum_{i=1}^s n_i[x_i]$, with $\sum_{i=1}^s n_i = n$, where x_i is the point of $\mathbb{A}_k^r$ associated to the maximal ideal $\mathfrak{m}_i$. This formal combination $\mathrm{sd}(A)$ is known as the *spectral datum* of A.

The set parametrizing formal combinations of points of $\mathbb{A}_k^r$ of length n is the symmetric product $S^n\mathbb{A}_k^r = (\mathbb{A}^r \times \overset{(n)}{\dots} \times \mathbb{A}^r)/\mathfrak{S}^n$. A classical theorem of Weyl shows that there exists a closed embedding

$$\iota_{n,r} : S^n\mathbb{A}_k^r \to \mathbb{A}(k^r \oplus S^2k^r \oplus \cdots \oplus S^nk^r) : \sum_{i=1}^n [x_i] \mapsto (\sigma_1, \dots, \sigma_n),$$

where S^ik^r is the i-th symmetric product vector space of k^r and $\mathbb{A}$ denotes the functor sending a vector space over k to the associated affine space (regarded as a scheme), that is $\mathbb{A}(V) = \mathrm{Spec}(k[V])$. The elements σ_i are defined as

$$\sigma_1 = x_1 + \dots + x_n, \quad \sigma_2 = x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n, \quad \dots \quad \sigma_n = x_1 \cdots x_n.$$

In particular, for $r = 1$ the map $\iota_{n,r}$ is an isomorphism by the Fundamental Theorem on Symmetric Polynomials.

We can now consider an "universal characteristic polynomial" by defining the map

$$\chi_{n,r} : \mathbb{A}_k^r \times S^n\mathbb{A}_k^r \to \mathbb{A}(S^nk^r) : (x, \sum_{i=1}^n [x_i]) \mapsto (x - x_1) \cdots (x - x_n).$$

Indeed, note that $\chi_{n,r}\left(x, \sum_{i=1}^n [x_i]\right) = x^n - \sigma_1 x^{n-1} + \dots + (-1)^n \sigma_n$. We define the *Cayley scheme* as the 0-fibre of this map,

$$\mathrm{Cayley}_n(\mathbb{A}_k^r) = \chi_{n,r}^{-1}(0).$$

The natural projection $p_{n,r} : \mathrm{Cayley}_n(\mathbb{A}_k^r) \to S^n\mathbb{A}_k^r$ is called the *universal spectral cover*.

The following theorem sums up the main properties of this map:

Theorem 1. *The universal spectral cover $p_{n,r}$ is finite of degree n and generically étale over the set $(S^n\mathbb{A}_k^r)'$ consisting of multiplicity-free formal combinations. For $a = \sum_{i=1}^s n_i[x_i]$, we have*

$$p_{n,r}^{-1}(a) = \mathrm{Spec}\left(\frac{k[X_1, \dots, X_r]}{\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_s^{n_s}}\right).$$

Thus, $p_{n,r}$ is flat if and only if $r = 1$ or $n = 1$. Finally, if $A = k[\varphi_1, \dots, \varphi_r]$, the associated sheaf $\mathcal{F}$ is supported in a closed subscheme of $p_{n,r}^{-1}(\mathrm{sd}(A))$.

The last part of this theorem can be interpreted as a "universal version" of the Cayley-Hamilton theorem.

3. The Hitchin map

We come back now to the situation of the introduction, where X is a smooth projective variety over k and $V \to X$ is a rank r vector bundle over X. We can consider now the scheme $S^n(V/X)$, which is the result of twisting the space $S^n\mathbb{A}_k^r$ by the GL_r-torsor attached to V. In other words, $S^n(V/X)$ is a fibre bundle over X with fibre $S^n\mathbb{A}(V_x)$ over $x \in X$. We denote by $\mathcal{B}_{n,V}$ the set of sections of $S^n(V/X)$. The embedding $S^n\mathbb{A}_k^r \hookrightarrow \mathbb{A}(k^r \oplus S^2k^r \oplus \dots \oplus S^nk^r)$ induces an embedding $\iota_{n,V} : \mathcal{B}_{n,V} \hookrightarrow \bigoplus_{i=1}^n H^0(X, S^iV)$.

It is clear now that this gives a fatorization of the Hitchin morphism as $h_{n,V} = \iota_{n,V} \circ \mathrm{sd}$. Therefore, the problem of studying the fibres of $h_{n,V}$ is reduced to the problem of studying the fibres of the *spectral data map* $\mathrm{sd} : \mathcal{M}_{n,V} \to \mathcal{B}_{n,V}$.

Consider now a section $b \in \mathcal{B}_{n,V}$, which is a map $b : X \to S^n(V/X)$. If we twist the universal spectral cover by the GL_r-torsor attached to V, we get a map $\mathrm{Cayley}_n(V/X) \to S^n(V/X)$. Taking the fibered product of b and this map we get a degree n finite morphism $\pi : Y_b \to X$ factorizing by an embedding $Y_b \hookrightarrow V$. This is the *spectral cover*. Moreover, if b is generically multiplicity free, then π is generically étale. However, π is not a flat morphism in general.

Suppose now that (E, φ) is a pair with spectral data b, then the twisted endomorphism $\varphi : E \to E \otimes V$ can be seen as a morphism $V^\vee \to \operatorname{End} E$, which induces a morphism $S^\bullet V^\vee \to \operatorname{End} E$. Since $S^\bullet V^\vee = p_* \mathcal{O}_X$, the previous morphism defines a coherent sheaf $\mathcal{F}$ on V such that $p_* \mathcal{F} = E$. The universal version of the Cayley-Hamilton theorem implies that the support of $\mathcal{F}$ is contained precisely in the spectral cover Y_b. Therefore, we have a correspondence between those pairs (E, φ) with spectral data b and coherent sheaves $\mathcal{L}$ on Y_b with $\pi_* \mathcal{L} = E$. Moreover, we have the following lemma from commutative algebra:

Lemma 2. *Let A be a regular ring and B a finite A-algebra. Any B-module M is maximal Cohen-Macaulay over B if and only if it is locally free as an A-module.*

This implies the following correspondence:

Theorem 3. *The functor π_* gives an equivalence of categories between pairs (E, φ) with spectral data b and maximal Cohen-Macaulay sheaves on Y_b of generic rank* 1.

The main problem with the above result is that, since the map π is not flat in general, the category of maximal Cohen-Macaulay sheaves on Y_b might be empty. A way to solve this problem is by "modifying" π in order to obtain a flat morphism $\tilde{\pi} : \tilde{Y}_b \to X$. For example, if $\dim X = 1$, since any coherent sheaf on a curve can be decomposed as a direct sum of a locally free and a torsion sheaf, we can obtain a *flat spectral cover* just by removing the torsion of the structure sheaf. When $\dim X = 2$, a construction by Chen and Ngô [2] yields a *Cohen-Macaulayfication* of the spectral curve.

Moreover, note that if the spectral cover is an integral scheme, the corresponding sheaves over it are torsion-free and, if it is smooth, then they are locally free. In the case where the twisting bundle has rank 1 and its n-th power is base point free, Beauville, Narasimhan and Ramanan found that these good conditions on the spectral cover are in fact satisfied for a generic b. It would be interesting then to find similar conditions for the cases of general values for $\dim X$ and $\operatorname{rk} V$.

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