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An introduction to knot homology theories

Jorge Becerra
University of Groningen
j.becerra@rug.nl

Abstract: We give a brief exposition of how the Alexander polynomial $\Delta_K(t)$ and the Jones polynomial $J_K(t)$, two classical and powerful knot invariants, can be upgraded to (bi)graded groups $\widehat{HFK}(K)$ and $Kh(K)$ such that the polynomials can be recovered by taking some (graded) Euler characteristic. These groups are called *knot Floer homology* and *Khovanov homology*, respectively, as they share common features with the homology groups of spaces. These groups retain more information about the knots than the polynomials in the sense that they not only encode but also strengthen their properties.

Resumen: Damos una breve exposición de cómo el polinomio de Alexander $\Delta_K(t)$ y el polinomio de Jones $J_K(t)$, dos invariantes de nudos clásicos pero robustos, generalizan a grupos (bi)graduados $\widehat{HFK}(K)$ y $Kh(K)$, de modo que los polinomios se pueden recuperar tomando cierta característica de Euler (graduada). Estos grupos se llaman *homología de nudos de Floer* y *homología de Khovanov*, respectivamente, ya que comparten características similares a las de la homología de espacios. Estos grupos retienen más información sobre los nudos que los polinomios en el sentido de que no solo heredan sus propiedades sino que las hacen más fuertes.

Keywords: knots, Khovanov homology, knot Floer homology.

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1. Categorification

We start off by briefly discussing the idea of categorification. Let us borrow from Lurie [3] the term *category number* as a loose measure of the amount of abstraction involved in a mathematical idea, construction, theorem, etc. The most concrete kind of mathematics, such as numbers, or polynomials (arrays of numbers) belong to category number zero. One level up, in category number one, we find mathematical structures such as sets, groups, topological spaces, etc. These objects have certain structure and maps between them which preserve the structure. Category number two refers to classes of mathematical structures, that is, categories, where not only do we have arrows between the structures but also arrows between the categories. And the ladder continues all the way up.

The problem of *categorification* consists of taking an object, statement, construction, etc. which happens in some category number, and lifting it to another such taking place at a higher level, being able to recover the original object, statement, construction, etc. in a rather simple way (*decategorification*).

Example 1. The category of finite sets is a categorification of the natural numbers: every $n \in \mathbb{N}$ is lifted to the finite set S_n of n elements. Decategorification consists of taking cardinality, $\#S_n = n$. ◂

Example 2. The category of finite dimensional chain complexes over a field k is a categorification of the integers. Decategorification sends a chain complex C_* to its Euler characteristic $\chi(C_*) \coloneqq \sum_i (-1)^i \dim_k C_i$. ◂

Example 3. Singular homology categorifies the Euler characteristic of finite-dimensional CW-complexes (and hence the (non-)orientable genus of closed surfaces):

$$\chi(X) = \chi(H_*(X;k)) = \sum_i (-1)^i \dim_k H_i(X,k).$$

The homology groups of a CW-complex carry much more information about it than its Euler characteristic. The success of homology relies on the following properties:

- $H\colon \mathsf{Top} \to \mathsf{grVect}_k$ is a functor.
- $H(X)$ only depends on the homotopy type of X, and the homology of the one-point space is one copy of k concentrated in degree 0.
- There is an isomorphism $H(X \times Y;k) \cong H(X;k) \otimes_k (H(Y;k)$ for spaces X, Y (Künneth formula).
- There are computational tools: Mayer-Vietoris, long exact sequences, etc. ◂

We will try to mimic the previous example 3 for knot polynomial invariants in the next section.

2. Knots and Khovanov homology

Knot theory has proved to be an important subject with many connections with category theory, physics, quantum algebra, manifold theory, biology, etc. We recall that a knot K is a smooth[1] embedding $S^1 \to S^3$. A classical problem in knot theory consists of distinguishing knots up to isotopy. Roughly speaking, two knots are isotopic when one can be deformed in the three-dimensional space into the other without cutting the rope and pasting the endpoints later. A similar discussion follows for links (multiple component knots).

There are two classical link polynomial invariants that we present here:

- The *Alexander polynomial* $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$ of a link L captures topological information about the link embedding, more precisely about the complement of the link in S^3. It is completely determined by the condition $\Delta_{\text{unknot}} = 1$ and the *skein relation*

$$\Delta_{L_+} - \Delta_{L_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta_{L_0},$$

[1] If you do not like derivatives you can safely replace “smooth” by “piecewise linear”.

where L_+, L_- and L_0 are links that are identical except in a small ball where they look like , and , respectively.

- The *Jones polynomial* $J_L(q) \in \mathbb{Z}[q, q^{-1}]$ of a link L has a combinatorial nature, but it can be interpreted as a certain path integral in terms of Chern-Simons theory [7]. It is determined by the condition[2] $J_{\text{unknot}} = q + q^{-1}$ and the skein relation

$$q^2 J_{L_+} - q^2 J_{L_-} = (q - q^{-1}) J_{L_0},$$

where L_+, L_- and L_0 are as before.

Just like with the Euler characteristic of CW-complexes, the Alexander and Jones polynomials have category number zero, and we would like to lift them to some "homology-like" theories, with similar features to the ones singular homology has.

Let Link be the category whose objects are isotopy classes of oriented links in S^3 and whose arrows $L \to L'$ are orientation-preserving homeomorphism classes of bordisms from L to L', that is, compact oriented surfaces $\Sigma \subseteq S^3 \times I$ such that $\partial\Sigma = -L \amalg L'$. We also let $\mathsf{bigrVect}_{\mathbb{Z}/2}$ be the category of bigraded $\mathbb{Z}/2$-vector spaces.

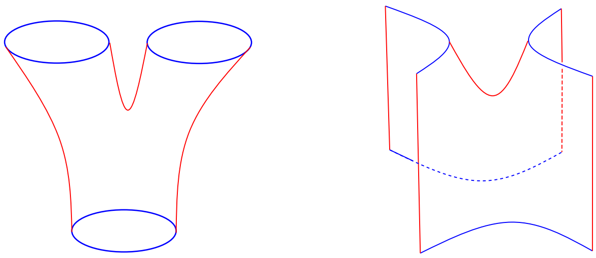

Figure 1: Some (local) pictures of link bordisms.

Theorem 4 (Khovanov [2]). *There exists a functor*

$$Kh\colon \mathsf{Link} \to \mathsf{bigrVect}_{\mathbb{Z}/2}$$

satisfying

(i) *If $\Sigma\colon L \to L'$ is an isotopy, then $Kh(\Sigma)\colon Kh(L) \xrightarrow{\cong} Kh(L')$ is an isomorphism.*

(ii) $Kh(\text{unknot}) = \mathbb{Z}/2_{(0,1)} \oplus \mathbb{Z}/2_{(0,-1)}$

(iii) $Kh(L_1 \amalg L_2) \cong Kh(L_1) \otimes_{\mathbb{Z}/2} Kh(L_2)$.

(iv) *If L is a link, denote by L_0 and L_∞ links identical to L except around one crossing of the form where they have been modified as and , respectively. Then there is an exact triangle*

$$\begin{array}{ccc} Kh(L_0) & \longrightarrow & Kh(L) \\ & \nwarrow \quad \swarrow & \\ & Kh(L_\infty) & \end{array}$$

(v) *The Jones polynomial is the graded Euler characteristic of Kh:*

$$J_L(q) = \chi_{gr}(Kh(L)) = \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Z}/2} Kh^{i,j}(L).$$

[2] Sometimes it is normalised so that $J_{\text{unknot}} = 1$, but for the purpose of the exposition we do not do that.

Bar-Natan's work [1] has been very influential. Another excellent exposition is [6].

Example 5. The Khovanov homology of the right-handed trefoil $\overline{3}_1$ is $Kh^{i,j}(\overline{3}_1) = \mathbb{Z}/2$ for $(i,j) = (0,1)$, $(0,3)$, $(2,5)$, $(3,9)$ and trivial otherwise. Therefore,

$$\chi_{gr}(Kh(\overline{3}_1)) = \sum_{i,j}(-1)^i q^j \dim_{\mathbb{Z}/2} Kh^{i,j}(K) = q + q^3 + q^5 - q^9 = J_{\overline{3}_1}(q)$$

as expected. ◂

Remark 6. Khovanov homology is strictly stronger than the Jones polynomial: there is a pair of knots called 5_1 and 10_{132} with the same Jones polynomial but with non-isomorphic Khovanov homology. ◂

3. Knot Floer homology

Discovered independently by Ozsváth and Szabó [4] and Rasmussen [5], the knot Floer homology of a knot $K \subset S^3$ is a bigraded $\mathbb{Z}/2$-vector space

$$\widehat{HFK}(K) = \bigoplus_{m,s\in\mathbb{Z}} \widehat{HFK}_m(K,s)$$

which only depends on the isotopy type of K.

One of the major achievements of knot Floer homology is that it categorifies the Alexander polynomial:

Theorem 7 (Oszváth-Szabó, Rasmussen). *For any knot K we have*

$$\Delta_K(t) = \chi_{gr}(\widehat{HFK}(K)) = \sum_{m,s}(-1)^m t^s \dim_{\mathbb{Z}/2} \widehat{HFK}_m(K,s).$$

Example 8. Knot Floer homology is strictly stronger than the Alexander polynomial: there is a pair of celebrated knots, called the *Conway knot* $11n_{34}$ and the *Kinoshita-Terasaka knot* $11n_{42}$, with the same Alexander polynomial (equal to one) but with non-isomorphic knot Floer homology. ◂

Remark 9. Knot Floer homology strengthens some well-known properties of the Alexander polynomial. For instance, Alexander gives a lower bound for the knot genus[3], $g(K) \geq \frac{1}{2}\deg \Delta_K(t)$; whereas knot Floer detects the knot genus [4]: $g(K) = \max\{s \in \mathbb{Z} : \widehat{HFK}(K,s) \neq 0\}$. ◂

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[3] This is the smallest genus among all one-boundary component closed surfaces in S^3 whose boundary is the given knot.