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Hyperreflexivity of the space of module homomorphisms between non-commutative L^p-spaces

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Abstract: Let $\mathcal{M}$ be a von Neumann algebra, and let $0 < p, q \leq \infty$. Then, the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all right $\mathcal{M}$-module homomorphisms from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ is a reflexive subspace of the space of all continuous linear maps from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$. Further, the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive in each of the following cases:

(i) $1 \leq q < p \leq \infty$;
(ii) $1 \leq p, q \leq \infty$ and $\mathcal{M}$ is injective, in which case the hyperreflexivity constant is at most 8.

Resumen: Sea $\mathcal{M}$ un álgebra de von Neumann y sean $0 < p, q \leq \infty$. El espacio $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ de los homomorfismos de $\mathcal{M}$-módulos a derechas de $L^p(\mathcal{M})$ en $L^q(\mathcal{M})$ es un subespacio reflexivo del espacio de aplicaciones lineales y continuas de $L^p(\mathcal{M})$ en $L^q(\mathcal{M})$. Además, el espacio $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ es hiperreflexivo en los siguientes casos:

(i) $1 \leq q < p \leq \infty$;
(ii) $1 \leq p, q \leq \infty$ y $\mathcal{M}$ es inyectiva, en cuyo caso la constante de hiperreflexividad es no mayor que 8.

Keywords: non-commutative L^p-spaces, injective von Neumann algebras, reflexive subspaces, hyperreflexive subspaces, module homomorphisms.

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1. Introduction

Let $\mathcal{X}, \mathcal{Y}$ be quasi-Banach spaces, and let $\mathcal{S}$ be a closed linear subspace of $B(\mathcal{X}, \mathcal{Y})$. In accordance with Loginov and Šul'man [8], $\mathcal{S}$ is called reflexive if

$$\mathcal{S} = \{T \in B(\mathcal{X}, \mathcal{Y}) : T(x) \in \overline{\{S(x) : S \in \mathcal{S}\}}\ \forall x \in \mathcal{X}\}.$$

Following Larson [6, 7], in the case where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $\mathcal{S}$ is called hyperreflexive if there exists C such that

$$\operatorname{dist}(T, \mathcal{S}) \le C \sup_{\|x\|\le 1} \inf\{\|T(x) - S(x)\| : S \in \mathcal{S}\},$$

for all $T \in B(\mathcal{X}, \mathcal{Y})$, and the optimal constant is called the hyperreflexivity constant of $\mathcal{S}$.

If $\mathcal{H}$ is a Hilbert space and $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$, then the double commutant theorem shows that $\mathcal{A}$ is a reflexive subspace of $B(\mathcal{H})$. Christensen [2–4] showed that many von Neumann algebras are hyperreflexive, but the general case is still open.

The non-commutative L^p-spaces that we consider throughout are those introduced by Haagerup (see [5, 9, 10]). Let $\mathcal{M}$ be a von Neumann algebra. For each $0 < p \le \infty$, the space $L^p(\mathcal{M})$ is a contractive Banach $\mathcal{M}$-bimodule or a contractive p-Banach $\mathcal{M}$-bimodule according to $1 \le p$ or $p < 1$, and we will focus on the right $\mathcal{M}$-module structure of $L^p(\mathcal{M})$.

Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be quasi-Banach right $\mathcal{A}$-modules. An operator $T \in B(\mathcal{X}, \mathcal{Y})$ is a right $\mathcal{A}$-module homomorphism if

$$T(xa) = T(x)a \quad (x \in \mathcal{X},\ a \in \mathcal{A}).$$

We write $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ for the space of right $\mathcal{A}$-module homomorphisms from $\mathcal{X}$ to $\mathcal{Y}$.

For $T \in B(\mathcal{X}, \mathcal{Y})$ and $a \in \mathcal{A}$, define linear maps $aT,\ Ta : \mathcal{X} \to \mathcal{Y}$ by

$$(aT)(x) = T(xa), \quad (Ta)(x) = T(x)a \quad (x \in \mathcal{X}).$$

Let $\operatorname{ad}(T) : \mathcal{A} \to B(\mathcal{X}, \mathcal{Y})$ denote the inner derivation implemented by T, so that

$$\operatorname{ad}(T)(a) = aT - Ta \quad (a \in \mathcal{A}).$$

It is clear that T is a right $\mathcal{A}$-module homomorphism if and only if $\operatorname{ad}(T) = 0$, and, in the case where $\mathcal{X}$ and $\mathcal{Y}$ are Banach $\mathcal{A}$-modules, the constant $\|\operatorname{ad}(T)\|$ is intended to estimate the distance from T to the space $\operatorname{Hom}_{\mathcal{A}(\mathcal{X},\mathcal{Y})}$.

In [1], we study the reflexivity and hyperreflexivity of the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$, where $\mathcal{M}$ is a von Neumann algebra and $0 < p, q \le \infty$.

2. Bilinear maps and orthogonality

Our research is based on the analysis of bilinear maps that satisfy a certain orthogonality property.

Proposition. *Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{Z}$ be a topological vector space, and let $\varphi : \mathcal{M}\times\mathcal{M} \to \mathcal{Z}$ be a continuous bilinear map.*

(i) *Suppose that*

$$e \in \operatorname{Proj}(\mathcal{M}) \implies \varphi(e, e^\perp) = 0.$$

Then,

$$\varphi(a, 1_{\mathcal{M}}) - \varphi(1_{\mathcal{M}}, a) = 0 \quad (a \in \mathcal{M}).$$

(ii) *Suppose that $\mathcal{Z}$ is a normed space and let the constant $\varepsilon \ge 0$ be such that*

$$e \in \operatorname{Proj}(\mathcal{M}) \implies \|\varphi(e, e^\perp)\| \le \varepsilon.$$

Then,

$$\|\varphi(a, 1_{\mathcal{M}}) - \varphi(1_{\mathcal{M}}, a)\| \le 8\varepsilon\|a\| \quad (a \in \mathcal{M}).$$

This result remains true even if $\mathcal{M}$ is just a unital C^*-algebra of real rank zero (see [1, Theorem 1.2]).

Proposition. *Let $\mathcal{M}$ be a von Neumann algebra and let $0 < p, q \leq \infty$. Let $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$.*

(i) *If*

$$e \in \operatorname{Proj}(\mathcal{M}) \implies e^{\perp} T e = 0,$$

then $T \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$.

(ii) *If $p, q \geq 1$, then*

$$\|\operatorname{ad}(T)\| \leq 8 \sup_{\|x\| \leq 1} \inf \{\|T(x) - \Phi(x)\| : \Phi \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\}.$$

This proposition is a direct consequence of the previous one, and it can be shown in much more general situations (see [1, Theorems 2.2, 2.3 and 2.4]).

3. Reflexivity and hyperreflexivity

The first part of the second proposition leads us to this result.

Theorem (Alaminos, Godoy and Villena [1, Corollary 2.11]). *Let $\mathcal{M}$ be a von Neumann algebra and let $0 < p, q \leq \infty$. Then, the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is reflexive.*

In [1], we show a slightly stronger result: if $T : L^p(\mathcal{M}) \to L^q(\mathcal{M})$ is a linear map such that

$$T(x) \in \overline{\{\Phi(x) : \Phi \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\}},$$

then $T \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$. Note that the continuity of T is not required.

The hyperreflexivity of these spaces is a substantially more complex problem. In fact, we need additional hypotheses to solve it and the general case is still open.

Theorem (Alaminos, Godoy and Villena [1, Theorems 3.7, 3.8 and 3.9]). *Let $\mathcal{M}$ be a von Neumann algebra and let $1 \leq p, q \leq \infty$.*

(i) *If $p = \infty$ or $q = 1$, then $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive and the hyperreflexivity constant is less or equal than 8.*

(ii) *If $\mathcal{M}$ is injective and $p, q \geq 1$, then $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive and the hyperreflexivity constant is less or equal than 8.*

(iii) *If $q < p$, then $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive and the hyperreflexivity constant is less or equal than a constant $C_{p,q}$ that does not depend on $\mathcal{M}$.*

The proof of the first and second parts of the theorem consists of showing that, given $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$, there is a homomorphism Φ such that

$$\|T - \Phi\| \leq \|\operatorname{ad}(T)\|,$$

and then the second part of the second proposition concludes the demonstration.

The third part is shown by assuming towards a contradiction that for each $n \in \mathbb{N}$ there is a von Neumann algebra $\mathcal{M}_n$ and an operator $T_n \in B(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))$ such that

$$\operatorname{dist}(T_n, \operatorname{Hom}_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) > n\|\operatorname{ad}(T_n)\| \quad (n \in \mathbb{N}).$$

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