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Yano's extrapolation theorem

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Abstract: Yano's extrapolation theory provides a tool to obtain estimates on L^1 spaces starting from information on estimates for L^p spaces for every $1 < p < p_0$, for some $p_0 > 1$. This document provides an introduction to this theory by sketching the proof of Yano's extrapolation theorem [3]. The main tool developed in the proof of this theorem is the technique known as *layer cake*, which is nowadays used in many other proofs in Fourier analysis.

Resumen: La teoría de extrapolación de Yano da una herramienta para obtener estimaciones en espacios L^1 partiendo de información sobre estimaciones para espacios L^p para todo $1 < p < p_0$, para algún $p_0 > 1$. Este documento da una introducción a esta teoría esbozando la demostración del teorema de extrapolación de Yano [3]. La herramienta principal desarrollada en la prueba de este teorema es la técnica conocida como *layer cake*, que se usa a día de hoy en muchas demostraciones en análisis de Fourier.

Keywords: extrapolation theory, Yano's theorem, endpoint estimates, layer cake method.

MSC2010: 42B99, 46E30.

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1. Introduction

If you are familiar with functional analysis, or more specially with the theory of bounding operators, you have probably heard about operator interpolation techniques, where we use operator bounding at the “ends” of a family of spaces, to get bounds on the rest of the spaces of the family.

However, when it comes to extrapolation techniques, the intention is precisely the opposite. That is, use what we know about the bounds of a certain operator in the “interior” of a family of spaces to obtain bounds of this operator at the endpoints of the family. Beyond concrete points, what characterizes this type of theorems is the use of a specific type of techniques. In this sense, two great schools stand out: Yano’s and Rubio de Francia’s.

In these pages, we intend to present in a simple way the extrapolation technique of Yano, by explaining the theorem that he published in 1951 [3]. The various applications of this method to the study of the bounding of several operators is a unique knowledge paradigm in the field of Fourier analysis.

2. Contextualization of the problem

Definition 1 ($L \log L$ space). For a measurable function f defined in (a,b), we will say that $f \in L^{*k}[a,b]$ if

$$\int_a^b |f(x)| \log^k\left(1+f^2(x)\right) \mathrm{d}x, \qquad k>0.$$ ◂

Note that this is not a norm, even though it allows us to characterize the functions in that space.

Definition 2. As usual, for two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, for an operator

$$T\colon X \to Y,$$

we are going to define the norm of T as

$$\|T\| = \sup_{\|f\|_X \le 1} \frac{\|Tf\|_Y}{\|f\|_X}.$$ ◂

In Fourier analysis, we are often concerned with operators T which transform a measurable function f defined in $[0,2\pi]$ into another measurable function also defined in $[0,2\pi]$ such that

(i) for every $p>1$ we have
$$\|T\|_{L^p} \le A_p,$$

(ii) for every $f \in L^{*k}[0,2\pi]$ we have
$$\|Tf\|_{L^1[0,2\pi]} \le A_k \int_0^{2\pi} |f(x)| \log^k\left(1+f^2(x)\right) \mathrm{d}x + B_k,$$

where A_p, A_k and B_k are constants depending only on p, k and k, respectively.

Usually, given an operator T, it is checked that T satisfies each one of the above conditions separately. But, what if we could deduce that T satisfied the last condition based on T satisfying the first one? This is what Yano’s theorem allows us to do.

Theorem 3. *Let T be a sublinear operator which transforms every integrable function to a measurable function, both being defined in a finite interval (a,b) such that*

(i) $|Tf| = |T(-f)|$,

(ii) *the inequality*

$$\|T\|_{L^p[a,b]} \leq \frac{A}{(p-1)^k},$$

holds for $1 < p \leq 2$*, for some* $k > 0$ *and the constant* A *depending only on the length of the interval* (a,b).

Then, we have that for every $f \in L^{*k}[a,b]$

$$\|Tf\|_{L^1[a,b]} \leq A_k \int_0^{2\pi} |f(x)| \log^k \left(1 + f^2(x)\right) \mathrm{d}x + B_k,$$

where A_k *and* B_k *are constants depending only on* k *and the lenght of the interval* $b - a$.

For the proof, we only need to check the theorem in the case where $f \geq 1$ because for any other function, we can decompose it into the difference of two functions greater than one and apply the condition (i) in order to obtain the desired result. Indeed,

$$f = \left(f\chi_{f\geq 0} + 1\right) - \left(1 - f\chi_{f<0}\right) = g_1 - g_2,$$

where $g_1, g_2 \geq 1$.

So, given an arbitrary function $f \geq 1$, we decompose it in the following way:

$$f = \sum_{n\geq 0} 2^n f_n, \qquad \text{where} \quad f_n = 2^{-n} f \chi_{\{2^n \leq f < 2^{n+1}\}}.$$

The sublinearity of the operator T allow us to work with these special functions f_n which have the particularity that $1 \leq f_n(x) < 2$ for every $n \geq 0$ and any $x \in [a,b]$. Moreover, the definition of these functions makes it possible for us to return to the initial function f if desired.

In fact, from the above decomposition and applying the Hölder inequality, it is easy to see that for any sequence $\{p_n\}$ of exponents such that $1 < p_n \leq 2$ it is satisfied that

$$\|Tf\|_{L^1[a,b]} \leq C \sum_n 2^n \|Tf_n\|_{L^{p_n}[a,b]} \leq C \sum_n \frac{2^n}{(p_n - 1)^k} \|f_n\|_{L^{p_n}[a,b]},$$

with the constant C only depending on the length of the interval and on the constant A which appears on the second hypothesis of the theorem.

At this point, it only remains to choose the exponent p_n, not fixed yet, to conclude the desired theorem. For instance, we choose

$$p_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 + \dfrac{1}{n} & \text{if } n \neq 0. \end{cases}$$

For the end of the theorem we only need to use Young's inequality in the right way and the fact that

$$2^n f_n(x) n^k \leq C|f(x)| \log^k \left(1 + f^2(x)\right),$$

for every $x \in [a,b]$.

This way of treating a function, by decomposing it into simpler and control-bounded functions, is known as the *layer cake* method. This technique has been used in many other proofs in order to obtain extrapolation theorems, but also in other areas of Fourier analysis. See, for example [1] or the proof of Lemma 1.4.20 in [2, page 56], where the partition made is a little bit different since it considers the sets

$$A_n = \{x \in X : f^*(2^{n+1}) < |f(x)| \leq f^*(2^n)\},$$

where f^* denotes the rearrangement invariant of f.

References

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