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Local Bollobás type properties and diagonal operators

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Abstract: We study properties related to the density of norm-attaining operators. To do so, we introduce the set $\mathcal{A}_{\|\cdot\|}(X,Y)$ of norm-one norm-attaining operators from X to Y such that, given some $\varepsilon > 0$, there exists $\eta(\varepsilon,T)$ such that, if $\|T(x)\| > 1-\eta$, then there is x_0 with $\|x_0 - x\| < \varepsilon$ and T attains its norm at x_0. These are operators such that, whenever they almost attain their norm at a point, they do attain it at a nearby point. The analogous set $\mathcal{A}_{\text{nu}}$ for the numerical radius is also introduced and studied. We give examples of operators that belong to these sets and, in particular, we give a characterisation of what diagonal operators belong to these sets for the classical Banach sequence spaces. The contents are from [5].

Resumen: Estudiamos propiedades relacionadas con la densidad de operadores que alcanzan su norma. Para ello, introducimos el conjunto $\mathcal{A}_{\|\cdot\|}(X,Y)$ de operadores de norma uno de X en Y tales que, dado un $\varepsilon > 0$, existe $\eta(\varepsilon,T)$ de forma que, si $\|T(x)\| > 1-\eta$, entonces existe un x_0 con $\|x_0 - x\| < \varepsilon$ y T alcanza su norma en x_0. Estos son operadores tales que, si casi alcanzan su norma en un punto, la alcanzan en un punto cercano. El conjunto análogo $\mathcal{A}_{\text{nu}}$ para el radio numérico también es introducido y estudiado. Vamos a dar ejemplos de operadores que pertenecen a estas clases y, en particular, daremos una caracterización de qué operadores diagonales pertenecen a estos conjuntos para los espacios de Banach de sucesiones clásicos. Los contenidos forman parte de [5].

Keywords: Bishop-Phelps-Bollobás, norm attaining operators, diagonal operators.

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1. Introduction

In this contribution, we will summarize some of the results from the recent work [5] that were presented in a talk in the 3rd BYMAT Conference in 2020.

1.1. Notation and terminology

Let X and Y be Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by B_X, S_X and X^* the closed unit ball, the unit sphere and the topological dual of X, respectively. $\mathcal{L}(X, Y)$ represents the space of bounded and linear operators from X to Y, and we shall write $\mathcal{L}(X) = \mathcal{L}(X, X)$. We will use the notation c_0, ℓ_1, ℓ_∞ and ℓ_p $(1 < p < \infty)$ for the classical Banach sequence spaces, and the notation $x = (x(1), x(2), x(3), \dots)$ will be used for any $x \in X$ where X is a Banach sequence space. If $x \in X$ and $x^* \in X^*$, the dual action may be written as $\langle x^*, x\rangle$ or as $x^*(x)$ indistinctly.

We say that an operator $T\colon X \to Y$ *attains its norm* (or is *norm attaining*) if there exists some $x \in S_X$ such that $T(x) = \|T\| = \sup_{x\in B_X} |T(x)|$ (that is, when the supremum is actually attained), and the set of norm attaining operators is denoted by $\mathrm{NA}(X, Y)$. The set of states of X is defined as $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. The *numerical radius* of an operator $T\colon X \to X$ is defined as $\nu(T) := \sup_{(x,x^*)\in\Pi(X)} |x^*(x)|$. It is easy to see that the numerical radius is a seminorm which satisfies $0 \le \nu(T) \le \|T\|$ for every operator $T\colon X \to X$. We say that an operator $T\colon X \to X$ *attains its numerical radius* (or is *numerical radius attaining*) if there exist some $(x, x^*) \in \Pi(X)$ such that $|x^*(T(x))| = \nu(T)$, and the set of such operators is denoted by $\mathrm{NRA}(X)$.

If X and Y are Banach sequence spaces, an operator $T\colon X \to Y$ is said to be *diagonal* if it is defined as $T(x) = (\alpha_1 x(1), \alpha_2 x(2), \alpha_3 x(3), \dots)$ for all $x \in X$, for some bounded sequence of scalars $\{\alpha_k\}_{k=1}^{+\infty} \in \mathbb{K}$.

1.2. Brief historical background

In 1961, Bishop and Phelps [3] proved that, for any Banach space X, the set $\mathrm{NA}(X, \mathbb{K})$ is always dense in X^*. Bollobás [4] gave a numerical refinement of that result in 1970, stating that you can always approximate a functional $x^* \in S_{X^*}$ and a point $x \in S_X$ at which it almost attains its norm by a pair $(y, y^*) \in \Pi(X)$. It is natural to wonder if any of these results also hold in the case of operators instead of functionals, however Lindenstrauss [8] proved that there exist spaces X and Y such that $\mathrm{NA}(X, Y)$ is not dense in $\mathcal{L}(X, Y)$.

In order to study quantitatively when a pair of spaces satisfies that $\mathrm{NA}(X, Y)$ is dense in $\mathcal{L}(X, Y)$, Acosta, Aron, García and Maestre [2] introduced in 2008 the Bishop-Phelps-Bollobás property, abbreviated as BPBp (see [2, Definition 1.1]). Roughly speaking, a pair of Banach spaces (X, Y) has the BPBp if, whenever we have an operator $T \in S_{\mathcal{L}(X,Y)}$ and a point $x \in S_X$ at which it almost attains its norm, we can always approximate them by an operator $S \in S_{\mathcal{L}(X,Y)}$ and a point $y \in S_X$ at which it attains its norm. We refer the interested reader to the survey [1] and references therein for more information and background on the BPBp.

Motivated by that work, Guirao and Kozhushkina [7] introduced in 2013 the *Bishop-Phelps-Bollobás property for numerical radius* (BPBp-nu for short), which is a natural adaptation of the BPBp to the case of numerical radius instead of norms. These properties, as well as several variations, have been profusely studied in the recent years. One of those properties is particularly relevant to this work, the $\mathbf{L}_{o,o}$, which is a local version of the BPBp where $S = T$ and η depends on the previously fixed T (see [6, Definition 2.1]).

1.3. Introducing the problem

Most of the works studying BPBp-like properties focus in finding what spaces can, or can not, satisfy properties of that kind. In the very recent work [5], the authors tackled the study from a different point of view: finding what *operators* can satisfy properties of that kind. We are going to present here briefly some of the findings from that work. We start by introducing two necessary concepts for this study, which are classes of operators such that whenever they almost attain their norm (or numerical radius) at some point (or state), they do attain it at a nearby point (or state). Note that this concept is closely related to the $\mathbf{L}_{o,o}$.

Definition 1. Let X, Y be Banach spaces.

(i) $\mathcal{A}_{\|\cdot\|}(X, Y)$ stands for the set of all norm-attaining operators $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon, T)$, there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

(ii) $\mathcal{A}_{\text{nu}}(X)$ stands for the set of all numerical radius attaining operators $T \in \mathcal{L}(X, X)$ with $\nu(T) = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $(x, x^*) \in \Pi(X)$ satisfies $|x^*(T(x))| > 1 - \eta(\varepsilon, T)$, there is $(x_0, x_0^*) \in \Pi(X)$ such that $|x_0^*(T(x_0))| = 1$, $\|x_0 - x\| < \varepsilon$, and $\|x_0^* - x^*\| < \varepsilon$. ◂

In Section 2 we will list some of the results from [5], focusing mainly in those where the involved operators are diagonal operators.

2. Results

2.1. First results

In order to determine what operators satisfy certain properties, it is natural to begin wondering what happens with operators between finite dimensional spaces and functionals. [5, Theorem 2.1] claims that, if X is a finite dimensional Banach space and Y is any Banach space, then every operator from $S_{\mathcal{L}(X,Y)}$ is in $\mathcal{A}_{\|\cdot\|}(X, Y)$, and every operator $T \in \mathcal{L}(X)$ with $\nu(T) = 1$ is in $\mathcal{A}_{\text{nu}}(X)$. As for functionals that may or may not belong to $\mathcal{A}_{\|\cdot\|}$, [5, Theorems 2.1 and 2.2] study the cases when $X = c_0$, ℓ_1, ℓ_∞ and the case where X is uniformly convex.

What can be said about other operators? Is there any relation between the sets $\mathcal{A}_{\|\cdot\|}(X, X)$ and $\mathcal{A}_{\text{nu}}(X)$ in general? The following examples should make it clear that this is, in fact, not trivial, even in the particular case of Hilbert spaces.

Example 2. Consider the following operators $T\colon \ell_2 \to \ell_2$:

(i) If $T(x) := x$, for all $x \in \ell_2$, then $T \in \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ and $T \in \mathcal{A}_{\text{nu}}(\ell_2)$.

(ii) If $T(x) := (0, x(1), x(2), x(3), x(4), \dots)$, for all $x \in \ell_2$, then $T \in \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ but $T \notin \mathcal{A}_{\text{nu}}(\ell_2)$.

(iii) If $T(x) := (2x(2), -2x(1), x(3), 0, 0, \dots)$, for all $x \in \ell_2$, then $T \notin \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ but $T \in \mathcal{A}_{\text{nu}}(\ell_2)$.

(iv) If $T(x) := \left(x(1), \frac{1}{2}x(2), \frac{2}{3}x(3), \frac{3}{4}x(4), \frac{4}{5}x(5), \dots\right)$, for all $x \in \ell_2$, then $T \notin \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ and $T \notin \mathcal{A}_{\text{nu}}(\ell_2)$, even though T attains both its norm ant its numerical radius, and $\|T\| = \nu(T) = 1$. ◂

We refer the reader to [5] for a collection of results and examples in the matter involving, for example, compact operators, adjoint operators, canonical projections, Hilbert spaces, and direct sums of spaces.

2.2. Diagonal operators

Let us examine items (i) and (iv) from Example 2. In both cases, T is a diagonal operator satisfying $\|T\| = \nu(T) = 1$ and $T \in \text{NA}(\ell_2, \ell_2) \cap \text{NRA}(\ell_2)$; however, their situations are completely different regarding the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$. In [5], a characterisation is made to determine what are the diagonal operators that belong to the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$ when the involved spaces are the classical Banach sequence spaces c_0 and ℓ_p $(1 \leq p \leq +\infty)$. We will summarize here some of the intuitions behind as well as the main results.

Example 3. Consider the diagonal operators $T\colon c_0 \to c_0$ defined as $T(x) = (\alpha_1 x(1), \alpha_2 x(2), \dots)$ for all $x \in c_0$, all of which satisfying $\|T\| = \nu(T) = 1$:

(i) If $\alpha_n := \frac{n}{n+1}$, then T can not be in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$, since T does not attain its norm or numerical radius. So at least one of the α_n needs to have absolute value 1 to be in our sets.

(ii) If $\alpha_1 = 1$ and $\alpha_n = 1 - \frac{1}{n}$ for $n > 1$, then T is also not in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$, since the only points $x \in S_{c_0}$ where it attains its norm are of the form $s \cdot e_1$ with $|s| = 1$, but the sequence $\{|T(e_{n+1})|\}_{n=1}^{+\infty}$ is strictly increasing and converges to 1, and similar with the numerical radius. So to be in our sets, not only one of the α_n needs to have $|\alpha_n| = 1$, but also, those that are not 1 have to be far from 1 (that is, 1 can not be an accumulation point of $\{|\alpha_n|\}_{n=1}^{+\infty}$).

(iii) Finally, if $\alpha_1 = 1$ and $\alpha_n = \frac{1}{n}$ for $n > 1$, then T is in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cap \mathcal{A}_{\text{nu}}(c_0)$, since the only points where the norm is almost attained are close to some point $x \in S_{c_0}$ with $|x(1)| = 1$ (similar for $\mathcal{A}_{\text{nu}}$). ◂

The intuitions presented above addresses us to necessary and sufficient conditions for a diagonal operator to be in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$, and they can be generalized to other spaces and to the complex case. We summarize the main results on the matter from [5, Theorems 2.13, 2.15 and 2.17, and Corollary 2.16].

Theorem 4. *Let (X, Y) be (c_0, c_0), (ℓ_p, ℓ_p) $(1 \leq p \leq +\infty)$ or (ℓ_p, c_0) $(1 \leq p < +\infty)$. Let $T\colon X \to Y$ be the norm one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ if and only if both of these conditions are satisfied:*

(i) *There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.*

(ii) *If $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, then either $J = \mathbb{N}$ or $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$.*

Theorem 5. *Given $1 \leq p < +\infty$, let $T\colon c_0 \to \ell_p$ be the norm one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ if and only if there is some $N \in \mathbb{N}$ such that $\alpha_n = 0$ for all $n > N$.*

Theorem 6. *Let $X = c_0$ or ℓ_p, $1 \leq p < \infty$. Let $T\colon X \to X$ be the numerical radius one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{nu}(X)$ if and only if the following two conditions hold:*

(i) *There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.*

(ii) *If $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, then the cardinality of the set $\{\alpha_n : n \in J\}$ is finite and $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$ when $J \neq \mathbb{N}$.*

In particular, if $\{\alpha_n\}_{n=1}^{+\infty} \subset \mathbb{R}$, $T \in \mathcal{A}_{nu}(X)$ if and only if $T \in \mathcal{A}_{\|\cdot\|}(X, X)$.

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