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Axial algebras of Monster type $(2\eta, \eta)$ for orthogonal groups over $\mathbb{F}_2$

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Abstract: Axial algebras are commutative non-associative algebras generated by special elements called axes satisfying a prescribed fusion law. They were introduced by Hall, Rehren and Shpectorov. This class has applications in physics, group theory and also elsewhere in mathematics. Here, we introduce the axial algebras of Monster type $(2\eta, \eta)$ and give an overview of the flip construction for such algebras. We also apply it to the non-degenerate orthogonal groups $O^{\varepsilon}(2k, 2)$. We describe the classes of involutions (flips) of $O^{\varepsilon}(2k, 2)$ and for each flip we investigate the corresponding flip subalgebra. In this way, we build a new rich family of examples of algebras of Monster type $(2\eta, \eta)$.

Resumen: Las álgebras axiales son álgebras conmutativas no asociativas generadas por elementos especiales, llamados ejes, que satisfacen una ley de fusión prescrita. Fueron introducidas por Hall, Rehren y Shpectorov. Tienen aplicaciones en física, teoría de grupos y también en otras áreas de las matemáticas. Aquí introducimos las álgebras axiales de tipo Monster $(2\eta, \eta)$ y damos una visión general de la construcción por involución para tales álgebras. También la aplicamos a los grupos ortogonales no degenerados $O^{\varepsilon}(2k, 2)$. Describimos las clases de involuciones (*flips*) de $O^{\varepsilon}(2k, 2)$ y para cada *flip* investigamos la correspondiente subálgebra. De este modo, construimos una nueva e interesante familia de ejemplos de álgebras de tipo Monster $(2\eta, \eta)$.

Keywords: Axial algebras, Monster type, orthogonal groups.

MSC2010: 17D99.

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1. Introduction

In 2009, Alexander Ivanov [5] turned the key properties used by Masahiko Miyamoto and Shinya Sakuma in their calculations of vertex operator algebras generated by two Ising vectors into the axioms of a new class of algebras called Majorana algebras. A Majorana algebra is a commutative non-associative algebra A over the field of real numbers generated by special idempotents with the fusion law $\mathcal{M}\left(\frac{1}{4}, \frac{1}{32}\right)$ and satisfying some additional properties. In 2015, Jon Hall, Felix Rehren and Sergey Shpectorov [3, 4] refined and generalised the axioms of Majorana algebras and these new axioms became the axioms of axial algebras.

In this text, we start by providing the background on axial algebras. Then we introduce the notion of 3-transposition groups, from which we derive the Matsuo algebras. After that, we turn to the flips of Matsuo algebras and we explain how a flip leads to a flip subalgebra that is an algebra of Monster type $(2\eta, \eta)$. In the case of orthogonal groups over the field with two elements, we obtain two types of flips and further split them into conjugacy classes. For each class we determine the dimension of the flip subalgebra. Our approach is similar to that of Vijay Joshi [6] who completed the case of symplectic groups over $\mathbb{F}_2$.

2. Background

2.1. Axial algebras

Let $\mathbb{F}$ be a field. All algebras in this text are over $\mathbb{F}$ and they are non-associative, that is, not necessarily associative. For a set $\mathcal{F}$, the set of all subsets of $\mathcal{F}$ is denoted by $2^{\mathcal{F}}$.

Definition 1. Let $\mathcal{F}$ be a finite subset of $\mathbb{F}$ and $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ be a symmetric binary operation. The pair $(\mathcal{F}, *)$ is a *fusion law* over $\mathbb{F}$. ◂

Examples of fusion laws can be seen in Tables 1 and 2.

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

Table 1: Fusion law $\mathcal{J}(\eta)$

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 2: Fusion law $\mathcal{M}(\alpha, \beta)$

Definition 2. Let A be a commutative algebra. For $a \in A$, the adjoint endomorphism $\mathrm{ad}_a : A \to A$ is defined by $b \mapsto ab$ for all $b \in A$. ◂

For $\lambda \in \mathbb{F}$, let $A_\lambda(a) = \{b \in A : ab = \lambda b\}$ be the λ-eigenspace of ad_a.

Definition 3. Let $\mathcal{F}$ be a *fusion law* over $\mathbb{F}$. Then, $a \in A$ is an $\mathcal{F}$-axis if

(i) a is an idempotent, i.e., $a^2 = a$;

(ii) ad_a is semisimple and every eigenvalue of ad_a is in $\mathcal{F}$, i.e., $A = A_{\mathcal{F}}(a) = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$;

(iii) $A_\lambda A_\mu \subseteq \bigoplus_{\nu \in \lambda * \mu} A_\nu(a)$ for all $\lambda, \mu \in \mathcal{F}$, where $\lambda * \mu$ is the product in $\mathcal{F}$, hence a subset of $\mathcal{F}$. ◂

Definition 4. Let A be a commutative algebra. We call A an *$\mathcal{F}$-axial algebra* if it is generated by a set of $\mathcal{F}$-axes. ◂

Definition 5. An $\mathcal{F}$-axis a is *primitive* if $A_1(a) = \langle a \rangle$, that is, $A_1(a)$ is 1-dimensional. An $\mathcal{F}$-axial algebra is *primitive* if it is generated by a set of primitive $\mathcal{F}$-axes. ◂

Jordan algebras are examples of axial algebras.

Definition 6 ([7]). A *Jordan algebra* is a commutative algebra A satisfying the following condition:

(J) *(Jordan Identity)* $x^2(yx) = (x^2y)x$ for all $x, y \in A$. ◂

Every idempotent in a Jordan algebra satisfies the Peirce decomposition that amounts to the fusion law $\mathcal{J}(\frac{1}{2})$.

Definition 7. An axial algebra of *Jordan type* η is a primitive axial algebra generated by a set of axes satisfying the fusion law $\mathcal{J}(\eta)$. ◂

Definition 8. An axial algebra of *Monster type* (α, β) is a primitive axial algebra generated by a set of axes satisfying the fusion law $\mathcal{M}(\alpha, \beta)$. ◂

The Griess algebra is a 196, 844-dimensional algebra over the field of real numbers. This algebra is an axial algebra of Monster type $\left(\frac{1}{4}, \frac{1}{32}\right)$.

2.2. 3-Transposition groups

Definition 9 ([1]). Suppose that G is a finite group and C is a normal subset of involutions (elements of order 2) of G. If C generates G and for all $c, d \in C$, $o(cd)$ is at most 3, then the pair (G, C) is a 3*-transposition group*. ◂

Let V be a vector space over $\mathbb{F}_2$, $q : V \to \mathbb{F}_2$ a non-degenerate quadratic form and $(\cdot, \cdot)$ the associated symplectic form. Let $G = O^{\varepsilon}(2k, 2)$ be the orthogonal group associated with V and q, where $\dim V = n = 2k$ and q is of type $\varepsilon \in \{+, -\}$. Take $w \in V$. The map $r_w : u \mapsto u + (u, w)w$ is called a *transvection* and it lies in G if $q(w) = 1$. Take $C = \{r_w : w \in V, q(w) = 1\}$ to be the class of transvections. Then, (G, C) is a 3-transposition group.

2.3. Matsuo algebras

Definition 10. Let (G, C) be a 3-transposition group and $\mathbb{F}$ be a field with $\operatorname{char} \mathbb{F} \neq 2$. Take $\eta \in \mathbb{F}$, $\eta \neq 0, 1$. Let $A = M_\eta(G, C)$ be the algebra with the basis C and the product $\circ$ defined by

$$c \circ d = \begin{cases} c & \text{if } c = d, \\ 0 & \text{if } o(cd) = 2, \\ \frac{\eta}{2}(c + d - e) & \text{if } o(cd) = 3, \end{cases}$$

where $c, d \in C$ and $e = c^d$. ◂

This algebra A is the *Matsuo algebra* corresponding to (G, C) and it is of Jordan type η.

2.4. Flip subalgebras

Consider a Matsuo algebra $A = M_\eta(G, C)$.

Definition 11. A *flip* is an involutive automorphism of A. ◂

Involutive automorphisms of G preserving C act on A, and hence they are flips.

Definition 12. Let $a, b \in C$ be such that $ab = 0$, i.e., a and b are orthogonal. Then, $a + b$ is called a *double axis*. ◂

Definition 13. Let σ be a flip of A. The *flip subalgebra* is generated by all single and double axes contained in the fixed subalgebra A_σ. ◂

Theorem 14 ([2]). *Every flip subalgebra is a primitive axial algebra of Monster type* $(2\eta, \eta)$.

3. Flip subalgebras in the orthogonal case

The following theorems give us the information about the flip subalgebras for all possible flips in the orthogonal case.

Theorem 15 ([8]). *Let U be a maximal totally isotropic subspace of V with a basis $\{u_1, \dots, u_k\}$, where each u_i is non-singular. Let $\sigma = \tau_i = r_{u_1} r_{u_2} \cdots r_{u_i}$ for all $1 \leq i \leq k$. If i is odd, then the dimension of the flip subalgebra is $2^{n-3} + 2^{n-i-2}$. If i is even, then the dimension is $2^{n-3} + 2^{n-i-2} - \delta 2^{k-2}$, where $\delta = 1$ for plus type and $\delta = -1$ for minus type.*

Theorem 16 ([8]). *Let U be a maximal totally singular subspace of V with a basis $\{u_1, \dots, u_{k-\beta}\}$, where $\beta = 0$ for plus type and $\beta = 1$ for minus type. Let $\sigma = \sigma_s = \sigma_{U_1} \sigma_{U_2} \cdots \sigma_{U_s}$, $1 \leq s \leq \left\lfloor \frac{k-\beta}{2} \right\rfloor$, where $U_j = \langle u_{2j-1}, u_{2j} \rangle$, for all $1 \leq j \leq \left\lfloor \frac{k-\beta}{2} \right\rfloor$, and $\sigma_{U_j} = \prod_{0 \neq u \in U_j} r_u$. Then, the dimension of the flip subalgebra is $2^{n-2} + 2^{n-2s-2} - \delta 2^{k-1}$.*

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