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Best approximations and greedy algorithms

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Abstract: We have obtained the exact order estimates for approximations by greedy algorithms of the classes $L^{\psi}_{\beta,p}$ of periodic functios in the space L_q for some relations between parameters p and q.

Resumen: Se han obtenido las estimaciones de orden exacto para las aproximaciones por algoritmos *greedy* de las clases $L^{\psi}_{\beta,p}$ de funciones periódicas en el espacio L_q, para algunas relaciones entre los parámetros p y q.

Keywords: greedy approximation, greedy algorithms, best approximations.

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1. Introduction

Let L_q be a space of functions f which are 2π-periodic and summable to a power q, $1 \le q < \infty$ (resp., essentially bounded for $q = \infty$), on the segment $[-\pi, \pi]$. The norm in this space is defined as follows:

$$\|f\|_{L_q} = \|f\|_q = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(x)|^q \,\mathrm{d}x\right)^{1/q}, & 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{x\in[-\pi,\pi]} |f(x)|, & q = \infty. \end{cases}$$

For a function $f \in L_1$, we consider its Fourier series

$$\sum_{k\in\mathbb{Z}} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx},$$

where $\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$ are the Fourier coefficients of the function f. In what follows, we always assume that the function $f \in L_1$ satisfies the condition

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0.$$

Further, let $\psi \neq 0$ be an arbitrary function of natural argument and let β be an arbitrary fixed real number. If a series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\hat{f}(k)}{\psi(|k|)}\mathrm{e}^{\mathrm{i}(kx+\beta\frac{\pi}{2}\operatorname{sign} k)}$$

is the Fourier series of a summable function, then, following Stepanets [3], we can introduce the (ψ, β)-derivative of the function f and denote it by f_β^ψ. By L_β^ψ we denote the set of functions f satisfying this condition. In what follows, we assume that the function f belongs to the class $L_{\beta,p}^\psi$ if $f \in L_{\beta,p}^\psi$ and

$$f_\beta^\psi \in U_p = \{\varphi : \varphi \in L_p, \|\varphi\|_p \le 1\}, \quad 1 \le p \le \infty.$$

If $\psi(|k|) = |k|^{-r}$, $r > 0$, and $k \in \mathbb{Z} \setminus \{0\}$, then the (ψ, β)-derivative of the function f coincides with its (r, β)-derivative (denoted by f_β^r) in the Weyl–Nagy sense.

We give the definition of the greedy approximation under investigation. Let $\{\hat{f}(k(l))\}_{l=1}^\infty$ be the Fourier coefficients $\{\hat{f}(k)\}_{k\in\mathbb{Z}}$ of the function $f \in L_1$, arranged in non-increasing order of their absolute value, i.e.,

$$|\hat{f}(k(1))| \ge |\hat{f}(k(2))| \ge \dots$$

Denote for $f \in L_q$

$$G_m(f, x) = \sum_{l=1}^{m} \hat{f}(k(l))\mathrm{e}^{\mathrm{i}k(l)x}$$

and, if $F \subset L_q$ is a certain function class, then we set

$$G_m(F)_q \coloneqq \sup_{f\in F} \|f(\cdot) - G_m(f, \cdot)\|_q. \tag{1}$$

At present, there are many works devoted to the investigation of quantity (1) for important classes of functions. For details and the corresponding references, see, e.g., [7].

By B we denote the set of functions ψ satisfying the following conditions:

(i) ψ is positive and nonincreasing;

(ii) there exists a constant $C > 0$ such that $\frac{\psi(\tau)}{\psi(2\tau)} \le C$, $\tau \in \mathbb{N}$.

Thus, the functions $1/\tau^r$, $r > 0$; $\ln^\gamma(\tau + 1)/\tau^r$, $\gamma \in \mathbb{R}$, $r > 0$, $\tau \in \mathbb{N}$, and some other functions belong to the set B.

For the quantities A and B, the notation $A \asymp B$ means that there exist positive constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$. If $B \le C_2 A$ ($B \ge C_1 A$), then we can write $B \ll A$ ($B \gg A$). All C_i, $i = 1, 2, \dots$, encountered in our paper may depend only on the parameters appearing in the definitions of the class and metric in which we determine the error of approximation.

2. Main results

The following assertion is true:

Theorem. *Let $1 < p < q \le 2$, $\psi \in B$, $\beta \in \mathbb{R}$ and let, in addition, there exist $\varepsilon > 0$ such that the sequence $\psi(t)t^{\frac{1}{p}-\frac{1}{q}+\varepsilon}$, $t \in \mathbb{N}$, does not increase. Then, the following order estimate is true:*

$$G_m(L^{\psi}_{\beta,p})_q \asymp \psi(m)m^{\frac{1}{p}-\frac{1}{2}}.$$

Proof. The upper bounds follow from the estimate for the approximation of functions from the classes $L^{\psi}_{\beta,p}$ by their Fourier sums [3]:

$$\mathcal{E}_m(L^{\psi}_{\beta,p})_2 = \sup_{f\in L^{\psi}_{\beta,p}} \left\| f(x) - \sum_{k=-m}^{m} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx} \right\|_2 \asymp \psi(m)m^{\frac{1}{p}-\frac{1}{2}}.$$

We now determine the lower bounds. We will use the Rudin-Shapiro polinomials $\mathcal{R}_l(x)$:

$$\mathcal{R}_l(x) = \sum_{j=2^{l-1}}^{2^l-1} \varepsilon_j \mathrm{e}^{\mathrm{i}jx}, \quad \varepsilon_j = \pm 1,\ x \in \mathbb{R},$$

satisfying the order estimate (see, e.g., [1]) $\|\mathcal{R}_l\|_\infty \ll 2^{l/2}$.

We also need the well-known de la Vallee-Poussin kernels

$$V_m(x) = \frac{1}{m}\sum_{l=m}^{2m-1} D_l(x), \quad x \in \mathbb{R},\ m \in \mathbb{N},$$

where $D_l(x) = \sum_{|k|\le l} \mathrm{e}^{\mathrm{i}kx}$ is the Dirichlet kernel.

Further, for $\varepsilon = \pm 1$ we set $\Lambda_{\pm 1} := \{k : \widehat{\mathcal{R}}_l(k) = \pm 1\}$, and let $\varepsilon = \pm 1$ be such that $|\Lambda_\varepsilon| > |\Lambda_{-\varepsilon}|$. Then, for given m, we take $l \in \mathbb{N}$ from the relation $2^{l-2} \le m < 2^{l-1}$, take a small positive parameter δ and consider a function

$$f(x) = C_3\psi(2^l)2^{l(\frac{1}{p}-1)} f_1(x), \quad C_3 > 0,$$

where $f_1(x) = V_m(x) + \varepsilon\delta\mathcal{R}_m(x)$ and $0 < \delta \le m^{\frac{1}{2}-\frac{1}{p}}$.

We now show that, for a certain choice of the constant $C_3 > 0$, the function f belongs to the class $L^{\psi}_{\beta,p}$. To this end, it suffices to verify that $\|f^{\psi}_{\beta}\|_p \ll 1$.

For this purpose, we use the estimate [2] $\|t^{\psi}_{\beta}\|_p \ll \psi^{-1}(n)\|t\|_p$ (for any polynomial $t \in T_n$, $1 < p < \infty$), and the well-known relation (see, e.g., [4]) $\|V_{2^l}\|_p \asymp 2^{l(1-\frac{1}{p})}$, $1 \le p \le \infty$.

Hence, we can write

$$\begin{aligned}\|f^{\psi}_{\beta}\|_p &\ll \psi^{-1}(m)\|f\|_p \le \psi^{-1}(m)\psi(2^l)2^{l(\frac{1}{p}-1)}(\|V_m\|_p + \delta\|\mathcal{R}_m\|_p)\\ &\le \psi^{-1}(m)\psi(2^l)2^{l(\frac{1}{p}-1)}(\|V_m\|_p + \delta\|\mathcal{R}_m\|_\infty)\\ &\ll \psi^{-1}(m)\psi(2^l)2^{l(\frac{1}{p}-1)}(2^{l(1-\frac{1}{p})} + 2^{l(\frac{1}{2}-\frac{1}{p})}2^{\frac{l}{2}}) \ll 1.\end{aligned}$$

This implies that, for a proper choice of the constant $C_3 > 0$, function $f \in L^{\psi}_{\beta,p}$.

By using the estimate (see, e.g., [5, p. 581]) that, for $1 \le q \le 2$ and $1 < p \le 2$,

$$\|f_1 - G_m(f_1)\|_q \gg m^{\frac{1}{2}},$$

we obtain

$$\sup_{f\in L^{\psi}_{\beta,p}} \|f - G_m(f)\|_q \gg \psi(2^l)2^{l(\frac{1}{p}-1)}\|f_1 - G_m(f_1)\|_q \gg \psi(m)m^{\frac{1}{p}-1}m^{\frac{1}{2}} = \psi(m)m^{\frac{1}{p}-\frac{1}{2}}.$$

The required lower bound is established, which proves the theorem. ■

Remark. The assertion of the theorem for a special case of the classes $W^r_{p,\beta}$ was established by Temlyakov [6]. ◂

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