3rd *BYMAT Conference* (2020)

Singular and fractal properties of functions associated with three-symbolic system of real numbers coding

✉ Iryna Zamrii
State University of Telecommunications
irinafraktal@gmail.com

Abstract: We consider continuous singular and piecewise singular functions defined in terms of given polybasic three-symbolic representation of real numbers, which depend on three parameters and are a generalization of classic ternary representations of real numbers. Their local and global properties (structural, variational, extreme, differential, integral and fractal) are studied. We investigate a family of continuous functions that store a central digit in a given polybasic three-symbolic representation of real numbers, that depends on three parameters and is a generalization of classic ternary representation of real numbers. It is proved that the set of such functions is continuous. A special role in this family has unique strictly decreasing function, called the inversor of digits. We also thoroughly study the properties of several model representatives of countable subclasses of functions with one and two infinite levels, respectively. They are piecewise singular. We found equivalent definitions for them.

Resumen: En esta contribución, se consideran funciones continuas singulares y singulares a trozos, definidas en términos de una determinada representación polibásica trisimbólica de los números reales, que depende de tres parámetros y es una generalización de la representación ternaria clásica de los números reales. Se estudian sus propiedades locales y globales (estructurales, variacionales, extremas, diferenciales, integrales y fractales). Se investiga una familia de funciones continuas que almacenan un dígito central en una determinada representación polibásica trisimbólica de los números reales, que depende de tres parámetros y es una generalización de la representación ternaria clásica de los números reales. Se demuestra que el conjunto formado por dichas funciones es continuo. Una función especial en esta familia tiene una única función estrictamente decreciente, llamada inversor de los dígitos. También se estudian a fondo las propiedades de varias representaciones de subclases contables de funciones con uno y dos niveles infinitos respectivament. Estas son singulares a trozos y encontramos una definición equivalente para ellas.

Keywords: singular function, Q_3-representation digits of real number, self-affine set, Hausdorff-Besicovitch dimension, the set of level of function.

MSC2010: 26A45, 28A80, 39B22, 28A78, 11T71.

Reference: Zamrii, Iryna. "Singular and fractal properties of functions associated with three-symbolic system of real numbers coding". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 139-142. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p139.

1. Introduction

Continuous functions often differ fundamentally in their local properties. We are not talking about smooth functions, but about functions with a complex local structure, which have "features" in each arbitrarily small interval. This "category" of functions includes tortuous and singular functions. The first ones do not have intervals of monotonicity, but in each arbitrarily small segment they have the largest and the smallest value. The theorems of Banach-Mazurkiewicz and Zamfirescu show that families of such functions are "numerous" and, therefore, deserve attention. Recently, interest in such functions has been growing, and many works have been devoted to them, in particular [1–4]. There is a common problem for them: the problem of having an effective "apparatus" of tasks and research. Following the principle "from simple to complex", functions that have relatively simple local properties have been studied so far, namely: they have the properties of self-similarity and self-affinity. For this class of functions, we are looking for alternative ways of setting and studying, seeing some potential in the theory of functional equations.

The well-known Lebesgue theorem states that every monotonic function f can be decomposed into a linear combination $\alpha_1 f_d + \alpha_2 f_{ac} + \alpha_3 f_s$ ($\alpha_i > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$) of three monotonic functions: discrete f_d, absolutely continuous f_{ac} and singular f_s. Moreover, in 1981 T. Zamfirescu proved that the "majority" of continuous monotonic functions are singular, since the latter in the metric space of all continuous monotonic functions with supremum-metric form a set of Be of the second category. For more than 100 years of development, the theory of singular functions has been enriched mainly due to individual theories (individual functions or finitely parametric families of functions have been studied), but the general theory is still poorly developed, it contains little, but it is small. At the same time, the study of singular functions has recently been intensified due to their connection with the theory of fractals.

On the basis of the general interest in singular functions, a natural interest in nonmonotonic singular functions and nontrivial mixtures of singular and absolutely continuous functions arises. Examples of nonmonotonic singular Kantor-type functions (functions whose constancy intervals form a set of full measure) are easily constructed. The first examples of nowhere monotonic singular functions were constructed in the 1950s by Indian mathematicians (Shukla U. K., Gard K. M.). Simple examples of singular monotonic functions nowhere appear in the works of M. V. Pratsiovytyi and A. N. Agadjanov. There are only a few works dedicated to such functions. Mixtures of singular and absolutely continuous functions have not yet been the subject of serious study. The logical question is: in which "relatively simple" classes are such functions dominant? And where do they "appear" naturally?

There are a number of problems associated with singular functions, one of which is the problem of effective ways to set and research them. Recently, various systems of representation of real numbers with both finite and infinite alphabets have been used for this purpose, one of which is Q-representation of numbers, first introduced in 1986 by M. V. Pratsiovytyi [3]. It was used to study singular distribution functions. We use Q-representation of numbers to study nonmonotonic piecewise singular functions. We are interested, in particular, in the fractal aspect of the study.

2. Object of research

The investigated functions are defined in terms of Q_3-representations of real numbers from the segment $[0, 1]$, namely: $x = \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left[\beta_{\alpha_k(x)} \prod_{j=1}^{k-1} q_{\alpha_j(x)}\right] \equiv \Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, where (q_0, q_1, q_2) is a predetermined set of positive numbers such that $q_0 + q_1 + q_2 = 1$, $\beta_0 = 0$, $\beta_1 = q_0$, $\beta_2 = q_0 + q_1$, $\alpha_n(x) \in \{0, 1, 2\}$.

The main object of research is a continuous function f satisfying the conditions

$$f(\Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{Q_3}_{\gamma_1\gamma_2\ldots\gamma_n\ldots} \quad \text{and} \quad \gamma_n = \begin{cases} \gamma_n(y) = \gamma_n(\alpha_1, \alpha_2, \ldots, \alpha_n), \\ \gamma_n = 1 \Leftrightarrow \alpha_n = 1. \end{cases} \tag{1}$$

That is, if $\gamma_n = 1$ if and only if $\gamma_n = 1$, then the function f stores the digit 1 (without magnification). The set of all such functions is denoted by P_c.

3. Results

Theorem 1. *The set of P_c continuous functions on $[0;1]$, which store the number 1 in the Q_3-representation of numbers, is continuum.*

Proof. To prove this fact, it suffices to show that the function f, which for a predetermined $y_0 \in C \equiv C[Q_3, \{0,2\}] = \{x : x = \Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},\ \alpha_1 \in \{0,2\}\}$ gives $f(1) = y_0$. From this, the equivalence of subset P_c and continuum set C is obvious, which is equivalent to continuum P_c. ∎

A trivial example of a function that satisfies this definition is the function $f(x) = x$. Another example of such a function is the inversor, which is a continuous function I with inhomogeneous differential properties [4], which is denoted by the equality $I(\Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{Q_3}_{[2-\alpha_1][2-\alpha_2]\ldots[2-\alpha_n]\ldots}$. The function I can be defined as the only solution to the system of functional equations $f(\beta_i + q_i x) = \beta_{[2-i]} + q_{[2-i]} f(x)$, $i = 0, 1, 2$, in the class of continuous functions.

Theorem 2 ([4]). *The inversor I has the following properties:*

(i) *it is a correctly defined continuous monotone function and singular if $q_0 \neq q_2$;*

(ii) *its graph $\Gamma_I = \{(x, I(x)) : x \in [0,1]\}$ is a self-affine set, namely: $\Gamma_I = \bigcup_{i=0}^{2} \phi_i(\Gamma_I) \equiv \phi(\Gamma_I)$, where ϕ_i is an affine transformation such that* $\phi_i : \begin{cases} x' = q_i x + \beta_i, \\ y' = -q_{[2-i]} y + \beta_{[3-i]}, \quad i \in A_3; \end{cases}$

(iii) *there is equality:* $\displaystyle\int_0^1 I(x)\,\mathrm{d}x = \frac{2q_0q_1 + q_0^2}{1 - 2q_0q_1 - q_1^2}.$

Using the definition of the inversor, you can effectively specify functions with some symmetries of the graph, which belong to the set P_c. For example, a function f that is the only solution in the set P_c of the functional equation $f(x) = f(I(x))$, $x \in [0,1]$ under condition $f(\Delta^{Q_3}_{(0)}) = \Delta^{Q_3}_{(02)}$.

Consider another example, the function $g(x)$.

Define the digits γ_n of the function $g(x)$ as follows. Let $\{0,2\} \ni i$ be a fixed parameter. We put

(i) $\gamma_1 = \alpha_1$; moreover, if $\alpha_1 = 1$, then $\gamma_{1+r} = \alpha_{1+r}$, $r \in N$, $g(\Delta^{Q_3}_{(0)}) = \Delta^{Q_3}_{(02)}$ and $g(\Delta^{Q_3}_{(2)}) = \Delta^{Q_3}_{(20)}$;

(ii) if $\alpha_1 = \ldots = \alpha_m = i$ and $\alpha_{m+1} = 2 - i$, then $\gamma_{m+1+r} = \begin{cases} \alpha_{m+1+r}, & m \text{ is odd number}, \\ 2 - \alpha_{m+1+r}, & m \text{ is even number}; \end{cases}$

(iii) if $\alpha_1 = \ldots = \alpha_m = i$, $\alpha_{m+1} = \ldots = \alpha_{m+k} = 1$ and $\alpha_{m+k+1} = 2 - i$, then $\gamma_{m+k+1+r} = \begin{cases} \alpha_{m+k+1+r}, & m = 2t+1, \\ 2 - \alpha_{m+k+1+r}, & m = 2t. \end{cases}$

If $\alpha_{m+k+1} = i$, then $\gamma_{m+k+1+r} = \begin{cases} 2 - \alpha_{m+k+1+r}, & m = 2t+1, \\ \alpha_{m+k+1+r}, & m = 2t, \end{cases}$ where $m, k, r, t \in N$.

It is obvious that the function $g(x)$ is correctly defined and satisfies (1), i.e., it belongs to P_c.

Theorem 3. *The function $g(x)$ thus defined is the only solution of the functional equation with the conditions*

$$\begin{cases} g(I(x)) = I(g(x)), \quad x \in [0,1], \\ g(\Delta^{Q_3}_{(0)}) = \Delta^{Q_3}_{(02)}, \\ g(\Delta^{Q_3}_{(2)}) = \Delta^{Q_3}_{(20)}. \end{cases}$$

Proof. From the definition g for $i, j \in \{0,2\}$ of the function, the relations follow:

$$g(\Delta^{Q_3}_{1\alpha_1\ldots\alpha_n\ldots}) = \Delta^{Q_3}_{1\alpha_1\ldots\alpha_n\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m+1}[2-i]\alpha_{2m+3}\ldots\alpha_{2m+3+n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots i}_{2m+1}[2-i]\alpha_{2m+3}\ldots\alpha_{2m+3+n}\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m}[2-i]\alpha_{2m+2}\ldots\alpha_{2m+2+n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots i[2-i]}_{2m}i[2-\alpha_{2m+2}]\ldots[2-\alpha_{2m+2+n}]\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m+1}\underbrace{1\ldots 1}_{k}j\alpha_{r_1}\ldots\alpha_{r_n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots i}_{2m+1}\underbrace{1\ldots 1}_{k}[2-i]\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_1}\right]\ldots\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_n}\right]\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m}\underbrace{1\ldots 1}_{k}j\alpha_{r_1}\ldots\alpha_{r_n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots[2-i]}_{2m}\underbrace{1\ldots 1}_{k}i\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_1}\right]\ldots\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_n}\right]\ldots}.$$

It is easy to see that for all the above relations, equality $g(I(x)) = I(g(x))$ holds. Therefore, equality $g(I(x)) = I(g(x))$ holds for the function g.

We show that g is the only solution. Suppose that in the set P_c there exists a function ψ different from g which satisfies the conditions of the theorem. That is, there are points $x_0 = \Delta^{Q_3}_{c_1c_2\ldots c_n\ldots}$ that $y_0 = \psi(x_0) \neq g(x_0) = y_1$. Then for each such number x_0 there exists the smallest natural k such that $h = \alpha_k(y_0) \neq \alpha_k(y_1) = l$, $h \neq 1 \neq l$, $h, l \in \{0, 2\}$ and $\alpha_j(y_0) = \alpha_j(y_1)$ for $j < k$. We choose among them x_0 for which k is the smallest. If there are more than one x_0, then we take the one for which the number h is the smallest.

From the fact that $\alpha_j(y_0) = \alpha_j(y_1)$ for $j < k$ it follows that y_0 and y_1 belong some segment $\Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}}$.

Let $h < l$. From the above it follows that $h = 0$, $l = 2$. Then, there exists a point $x_1 < x_0$, i.e., $x_1 \in \Delta^{Q_3}_{c_1c_2\ldots c_{k-1}[c_k-1]}$. Given the previous considerations, consider different Q_3-rational values: $x_0 \equiv \Delta^{Q_3}_{c_1c_2\ldots c_{k-1}c_k(0)} = \Delta^{Q_3}_{c_1c_2\ldots c_{k-1}[c_k-1](2)} \equiv x_1$. The function ψ at the points x_0 and x_1 takes values

$$y_0 = \psi(x_0) = \Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}h\tau_1\tau_2\ldots\tau_n\ldots} = \Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}0\tau_1\tau_2\ldots\tau_n\ldots}, \quad y_0^* = \psi(x_1) = \Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}ds_1s_2\ldots s_n\ldots},$$

where $d \in A_3$, $\tau_i, s_i \in \{0, 2\}$, $i \in N$. If the conditions $\tau_i = 2$, $s_i = 0$ and $d = 1$ are satisfied for all i, then the function ψ coincides with g, which contradicts the assumption. If the sequences (τ_n) and (s_n) are arbitrarily different from the previous case or $d \neq 1$, then the function ψ is discontinuous. Therefore, $\psi \notin P_c$ which contradicts the assumption.

If $h > l$, i.e., $h = 2$ and $l = 0$, then the numbers y_0 and y_0^* are equal if and only if the conditions $\tau_i = 0$, $s_i = 2$ and $d = 1$ are satisfied for all i. In this case ψ coincides with g, which contradicts the assumption. For the remaining values of d, (τ_n), (s_n) we obtain a discontinuous function, i.e., $\psi \notin P_c$.

Therefore, only function g satisfies the conditions of the theorem. The theorem is proved. ■

Theorem 4. *The function g has the following properties: for $q_0 = q_2$, it is a piecewise linear, and for $q_0 \neq q_2$ it is a mixture of singular and piecewise linear; the graph of the function g is "symmetrically similar" with respect to the point $(\Delta^{Q_3}_{(1)}; \Delta^{Q_3}_{(1)})$; and it has two infinite levels $y_0 = \Delta^{Q_3}_{(02)}$ and $y'_0 = \Delta^{Q_3}_{(20)}$.*

References

[1] Albeverio, S.; Pratsiovytyi, M. V., and Torbin, G. "Fractal probability distributions and transformations preserving the Hausdorff-Besicovitch dimension". In: *Ergodic Theory and Dynamical Systems* 24.1 (2004), pp. 1–16. issn: 0143-3857. https://doi.org/10.1017/S0143385703000397.

[2] Massopust, P. R. *Fractal functions, fractal surfaces, and wavelets.* 2nd ed. Elsevier/Academic Press, London, 2016. https://doi.org/10.1016/C2015-0-01882-2.

[3] Pratsiovytyi, M. V. *Fractal approach to the investigation of singular distributions.* View of the National Pedagogical Dragomanov University, Kyiv, 1998.

[4] Zamrii, I. V. and Pratsiovytyi, M. V. "Singularity of the Digit Inversor for the Q_3-Representation of the Fractional Part of a Real Number, Its Fractal and Integral Properties". In: *Journal of Mathematical Sciences* 215.3 (2016), pp. 323–340. issn: 1072-3374. https://doi.org/10.1007/s10958-016-2841-y.