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On unique-extension renormings

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Abstract: E. Oja, T. Viil, and D. Werner proved that every weakly compactly generated Banach space X having a norm with the property that every linear functional on X has a unique Hahn-Banach extension to its bidual X^{**} (which R. R. Phelps referred to as "X having *property U* in X^{**}") can be renormed to have the stronger property that every linear continuous functional defined on any linear subspace of X has a unique Hahn–Banach extension to X^{**} (the so-called *total smoothness* property of X). We proved that, thanks to a deep theorem of M. Raja, the above result can be obtained even in a stronger form and without any extra conditions on the space X (i.e., omitting the "weakly compactly generated" on the statement). Here we recall this result and present some extensions in the direction of what is called "weak Hahn-Banach smoothnes". This is partially based on a joint work with A. J. Guirao and V. Montesinos.

Resumen: E. Oja, T. Viil, and D. Werner probaron que todo espacio de Banach débilmente compactamente generado que tenga una norma con la propiedad de que todo funcional lineal y continuo en X tenga una única extensión de Hahn-Banach a su bidual X^{**} (es decir, "X tiene la *propiedad U* en X^{**}", en la terminología de R. R. Phelps) puede ser renormado para tener la propiedad más fuerte de que todo funcional lineal y continuo definido en cualquier subespacio lineal de X tiene una única extensión de Hahn-Banach a X^{**} (lo que se conoce como *total suavidad* de X). Probamos que, gracias a un profundo teorema de M. Raja, se puede obtener una versión incluso más fuerte del resultado anterior sin ninguna condición adicional en el espacio X (es decir, omitiendo "débilmente compactamente generado" en el enunciado). Reproducimos el resultado y proponemos algunas extensiones usando el concepto de suavidad Hahn-Banach débil del espacio. Esto está parcialmente basado en un trabajo conjunto con A. J. Guirao y V. Montesinos.

Keywords: renorming, Hahn-Banach extensions, Hahn-Banach smoothness, total smoothness, LUR norm, Kadets-Klee property.

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1. Introduction

The present contribution —based on the joint work [2]— is motivated by a recent paper [4], where it is proved that *every weakly compactly generated Banach space whose norm has the Hahn-Banach smooth property has an equivalent norm with the (stronger) totally smooth property*. This improved an earlier result of Sullivan [8] proving this statement under the assumption of separability. Our main contribution, that solves an open problem in [4], is that no extra requirement —besides the Hahn-Banach property of the norm— on the space is needed. This is a consequence of an important theorem due to M. Raja [7], where the so-called Kadets-Klee property for the w and w^* topologies in the dual of a Banach space allows for a renorming of the space with the local uniformly rotund property of its dual norm. By "renorming" a Banach space we mean defining an equivalent norm on it —naturally seeking for better geometric or analytic properties. Details and definitions needed are given below.

The basic Hahn-Banach theorem does not ensure uniqueness of the existing norm-preserving extension from a subspace to the whole space. This issue was considered by Phelps, who introduced the following definition:

Definition 1 (Phelps). Let $(X, \|\cdot\|)$ be a Banach space, and M a linear (not necessarily closed) subspace of X. We will say that M has *property U* in X if each continuous linear functional on M has a unique norm-preserving extension to X. ◂

We shall consider every Banach space X canonically embedded in its bidual space X^{**}.

Definition 2 (Sullivan). The norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ is said to be *Hahn–Banach smooth* (*HBS*, for short) if $(X, \|\cdot\|)$ has property U in X^{**} (i.e., every $x^* \in X^*$ has a unique norm-preserving extension to X^{**}). ◂

Definition 3. The norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ is said to be *totally smooth* (*TS*, for short) if every linear subspace M of X has property U in X^{**} (i.e., for every linear subspace M of X, every $f \in M^*$ has a unique norm-preserving extension to X^{**}). ◂

Obviously, if a norm has the HBS property, then it has the TS property. Notice that properties U, HBS, and TS, are of isometric nature. Indeed (some needed definitions will appear in Section 2 below),

(i) the Hilbertian norm on a Hilbert space H has property U. However, as happens in every Banach space, H has an equivalent norm $\|\cdot\|$ that fails to be Gâteaux differentiable at some $x_0 \in S_H$. Thus, two different norm-preserving extensions of $x_0^*|_M$ exist, where $M := \operatorname{span}\{x_0\}$ and x_0^* belongs to the subdifferential of $\|\cdot\|$ at x_0. This shows that U is not invariant under renormings.

(ii) On the other hand, it is not hard to prove that a Banach space $(X, \|\cdot\|)$ is reflexive if, and only if, every equivalent norm on X is HBS. If $(X, \|\cdot\|)$ is a Banach space with a separable dual, it is well known (see [3]) that X^* admits an equivalent dual LUR norm $|||\cdot|||^*$, so the topologies w and w^* coincide on S_{X^*}. Proposition 6 below shows then that $|||\cdot|||$ on X is HBS. By a previous observation, if moreover the space X is not reflexive, then it has an equivalent norm $|\cdot|$ which is not HBS. Thus, HBS is not invariant under renormings.

(iii) Finally, Theorem 9 below shows that the property TS of a norm is equivalent to the HBS property plus the strict convexity of its dual norm. Thus, the Hilbertian norm on a Hilbert space H is obviously TS, although H admits a non-rotund equivalent norm (this is a dualization of the argument in (i) above). This shows that TS is also non-invariant under renormings.

The statement of our main result follows. As we mentioned, this improves results in [8] and [4], and solves a problem in [4].

Theorem 4. *Let $(X, \|\cdot\|)$ be a Banach space. Then, the following statements are equivalent:*

(i) *X has an equivalent norm with property HBS.*

(ii) *X^* has an equivalent w^*-w-Kadets-Klee norm.*

(iii) *X has an equivalent norm whose dual norm is LUR.*

(iv) *X has an equivalent norm with property TS.*

The following classes of Banach spaces satisfy one —and then all— of the conditions in Theorem 4: (i) Asplund spaces that are weakly compactly generated, and (ii) spaces whose dual is a subspace of a weakly compactly generated Banach space. In particular, separable Banach spaces satisfy one of the conditions in Theorem 4 if, and only if, they are Asplund.

2. The walkthrough

We shall provide a sketch of the proof of Theorem 4. We shall need some extra definitions. Recall that a norm $\|\cdot\|$ on a Banach space is called *strictly convex* or *rotund* if the unit sphere does not contain non-trivial line segments. It is called *locally uniformly rotund* (*LUR*, for short) if for every $x, x_n \in S_X$, $n \in \mathbb{N}$, such that $\|x + x_n\| \to 2$, then $x_n \to x$.

Definition 5. Let $\|\cdot\|$ be a norm in a Banach space X and $\tau_1 \subset \tau_2 \subset \|\cdot\|$ two vector topologies on X. We say that $\|\cdot\|$ has the ***τ_1-τ_2**-Kadets-Klee property* if the topologies τ_1 and τ_2 coincide on its unit sphere. ◂

Proof of (i)⟺(ii) in Theorem 4: it will follow from Propositions 6 and 7 below.

Proposition 6 (Godefroy). *Let $(X, \|\cdot\|)$ be a Banach space. Then, $x^* \in S_{X^*}$ has a unique norm-preserving extension to X^{**} if, and only if, the w^* and w topologies on S_{X^*} coincide on x^*. In particular, $\|\cdot\|$ is HBS if, and only if, its dual norm has the w^*-w-Kadets-Klee property.*

This last proposition suggests that in order to find an equivalent HBS norm on a Banach space X —if possible—, we should try to renorm the dual space X^* with a w^*-w-Kadets-Klee norm. There is an extra requirement that in some cases is hard to achieve: the equivalent norm on the dual space must be a dual norm. In our situation, this is obtained for free: as a particular case of the following result, any w^*-w-Kadets-Klee norm on X^* is already a dual norm.

Proposition 7. *Let $\|\cdot\|$ be a τ_1-τ_2-Kadets-Klee norm which is τ_2-lower semicontinuous. Then, it is also τ_1-lower semicontinuous.*

(ii)⟺(iii): this is a consequence of the following deep result proved by Raja [6, 7], a landmark in renorming theory.

Theorem 8 (Raja). *Let X be a Banach space. If X admits an equivalent norm whose dual is w^*-w-Kadets-Klee, then it admits an equivalent norm whose dual is LUR.*

That (iv)⟹(i) was already mentioned above.

To finalize the proof of Theorem 4, the only remaining thing is to prove that (iii)⟹(iv). Notice that the norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ has the TS property if, and only if, every linear subspace M of X has property U in X and $\|\cdot\|$ has the HBS property. We need the following result.

Theorem 9 (Taylor-Foguel). *Let $(X, \|\cdot\|)$ be a Banach space. Then, every linear subspace M of X has property U on X if, and only if, the dual norm $\|\cdot\|^*$ is rotund.*

This last theorem and the paragraph above allow us to decompose the TS property in the following way: The norm $\|\cdot\|$ of a Banach space is TS if, and only if, it has HBS property and its dual norm is strictly convex. It is easy to see that the property of having a dual LUR norm is stronger than having those two properties simultaneously. In fact, any LUR norm is already a strictly convex norm, and also, as an easy application of the Riesz lemma, we have that if $\|\cdot\|^*$ is a dual LUR norm then the w^* and the norm topologies (and so, any topology in between them) coincide on its unit sphere. Proposition 6 shows that this implies the HBS property on its predual norm $\|\cdot\|$, and the proof of Theorem 4 is over.

3. Some further topics

We present here some remarks on, and some extensions of the previous results. Most of this can be found in detail in [1].

Remark 10. First of all, it is of key importance to prove that Theorem 4 is a real extension of the result of Oja, Viil and Werner: Indeed, it could happen that all four conditions stated in Theorem 4 would imply the original Banach space X being WCG. However, this is not the case. To see this, it is enough to take any Hausdorff non-Eberlein compact space K such that $K^{\omega_1} = \varnothing$ (for example, $K = [0, \omega_1]$), and consider the corresponding $C(K)$ space. It is proved in [3] that $C(K)^*$ admits a dual LUR norm (in particular, $C(K)$ admits an HBS norm), but it is not a WCG space (as K is not Eberlein). ◂

Some of the work done in Section 2 can be extended to more general cases. For this purpose, we may introduce some extra definitions related to the uniqueness of extensions (if X is a Banach space, then the subset of X^* consisting of all norm-attaining functionals on X will be denoted by $\mathrm{NA}(X)$): (i) If M is a linear (not necessarily closed) subspace of X, we will say that M has *property wU* in X if each element in $\mathrm{NA}(M)$ has a unique norm-preserving extension to X. (ii) The space $(X, \|\cdot\|)$ is said to be *weak Hahn–Banach Smooth* (*wHBS* for short) if $(X, \|\cdot\|)$ has property wU in X^{**} (i.e., every $x^* \in \mathrm{NA}(X)$ has a unique norm-preserving extension to X^{**}). The last definition is due to Sullivan. (iii) Let $\|\cdot\|$ be a norm in a Banach space X, $A \subset X$ be a cone, and $\tau_1 \subset \tau_2 \subset \|\cdot\|$ two vector topologies on X. We say that $\|\cdot\|$ has the ***τ_1-τ_2**-Kadets-Klee property with respect to* $\boldsymbol{A}$ when both topologies τ_1 and τ_2 coincide when restricted to $A \cap S_{(X,\|\cdot\|)}$

These definitions allow us to generalize Proposition 7 and the equivalence (i)⟺(ii) in Theorem 4.

Proposition 11. *Let $\|\cdot\|$ be a norm in the Banach space X that is τ_1-τ_2-Kadets-Klee with respect to a cone $A \subset X$ that satisfies $\overline{A \cap B_{(X,\|\cdot\|)}}^{\|\cdot\|}$. Then, if the norm is τ_2-lower semicontinuous, it is also τ_1-lower semicontinuous.*

Proposition 12. *Let $(X, \|\cdot\|)$ be a Banach space. Then, X admits a wHBS norm if, and only if, X^* admits a norm which is w^*-w-Kadets-Klee with respect to $NA(X)$.*

There are further similarities between the two properties HBS and wHBS. For example, it can be proved that a norm $\|\cdot\|$ is very smooth if, and only if, $\|\cdot\|$ its simultaneously Gâteaux smooth and wHBS, and this scheme is the analogous version of the TS decomposition above, but for the unique extension of the norm-attaining elements. It is natural to ask if wHBS on X also implies the existence of dual norm on X^* with good convexity properties, just as HBS implies the dual LUR norm on X^*. However, this is far from being true, since Talagrand proved that there are some spaces ($C([0, \mu])$ with uncountable μ that admit a Fréchet smooth equivalent norm (a much stronger property than being wHBS and even very smooth) but its dual spaces do not admit a dual strictly convex norm (see [3]).

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