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Best approximations and entropy numbers of the classes of periodic functions of many variables

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Abstract: We obtain order estimates for several approximative characteristics of the classes of periodic multivariate functions, that are connected to nonlinear approximation. Namely, we investigate a behaviour of the best orthogonal and M-term trigonometric approximations of the classes of functions with bounded generalized derivative (the classes of Weyl-Nagy type). We indicate cases, when there are advantages of nonlinear methods over the approximation of corresponding functional classes by step hyperbolic Fourier sums and by trigonometric polynomials with "numbers" of harmonics from step hyperbolic crosses.

Further we get the estimates of entropy numbers for the classes of functions with certain restrictions on their modulus of continuity (the classes of Nikol'skyi-Besov type). All the error approximations are measured in a metric of the Lebesgue space.

Resumen: Obtenemos estimaciones de orden para varias características aproximativas de las clases de funciones periódicas multivariantes, que están relacionadas con la aproximación no lineal. Concretamente, investigamos el comportamiento de las mejores aproximaciones ortogonales y trigonométricas de término M de las clases de funciones con derivada generalizada acotada (las clases de tipo Weyl-Nagy). Indicamos los casos, cuando hay ventajas de los métodos no lineales sobre la aproximación de las clases funcionales correspondientes por sumas de Fourier hiperbólicas escalonadas y por polinomios trigonométricos con "números" de armónicos de cruces hiperbólicos escalonados.

Además, obtenemos las estimaciones de los números de entropía para las clases de funciones con ciertas restricciones en su módulo de continuidad (las clases de tipo Nikol'skyi-Besov). Todas las aproximaciones de error se miden en una métrica del espacio de Lebesgue.

Keywords: entropy numbers, best approximation, step hyperbolic cross.

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1. Introduction

Let $\mathbb{R}^d$, $d \geq 1$, be the Euclidean space with elements $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + \ldots + x_d y_d$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. By $L_q := L_q(\pi_d)$, $\pi_d = \prod_{j=1}^d [0, 2\pi]$, $1 \leq q \leq \infty$, we denote the space of functions $f(\boldsymbol{x})$, that are 2π-periodic by each variable, equipped with the usual norm. Suppose further, that for the functions $f \in L_1$ the condition $\int_0^{2\pi} f(\boldsymbol{x})\, \mathrm{d}x_j = 0$, $j = \overline{1, d}$, holds.

Let us consider the Fourier series for $f \in L_1$, i.e.,

$$\sum_{k \in \mathbb{Z}^d} \widehat{f}(k) \mathrm{e}^{\mathrm{i}(k,x)},$$

where $\widehat{f}(\boldsymbol{k}) = (2\pi)^{-d} \int_{\pi_d} f(\boldsymbol{t}) \mathrm{e}^{-\mathrm{i}(\boldsymbol{k},\boldsymbol{t})} \mathrm{d}\boldsymbol{t}$ are the Fourier coefficients of f. Further, let $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d)$, $\psi_j \neq 0$, $j = \overline{1, d}$, be arbitrary sequences of natural argument, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$, $\mathring{\mathbb{Z}}^d = (\mathbb{Z} \setminus \{0\})^d$. Assume that the series

$$\sum_{\boldsymbol{k} \in \mathring{\mathbb{Z}}^d} \prod_{j=1}^d \frac{\mathrm{e}^{\mathrm{i}\frac{\pi \beta_j}{2} \operatorname{sgn} k_j}}{\psi_j(|k_j|)} \widehat{f}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}(\boldsymbol{k},\boldsymbol{x})},$$

where $\mathring{\mathbb{Z}}^d = (\mathbb{Z} \setminus \{0\})^d$, are the Fourier series of some function $f_{\boldsymbol{\beta}}^{\boldsymbol{\psi}}$ summable on π_d.

Let us denote by $L_{\boldsymbol{\beta},p}^{\boldsymbol{\psi}}$, $1 \leq p \leq \infty$, a set of functions f, for which $(\boldsymbol{\psi}, \boldsymbol{\beta})$-derivatives exist, and the condition $\|f_{\boldsymbol{\beta}}^{\boldsymbol{\psi}}\|_p \leq 1$ is satisfied. The univariate classes $L_{\beta,p}^{\psi}$ were introduced in 1983 by A. I. Stepanets [10]. The study of different approximative characteristics on the respective classes of multivariate functions was initiated by A. S. Romanyuk and further continued by his students. Note also that these classes generalize the well-known Weyl-Nagy classes $W_{\boldsymbol{\beta},p}^{\boldsymbol{r}}$, namely, $L_{\boldsymbol{\beta},p}^{\boldsymbol{\psi}} \equiv W_{\boldsymbol{\beta},p}^{\boldsymbol{r}}$ in the case $\psi_j(|\tau|) \equiv |\tau|^{-r_j}$, $\tau \in \mathbb{Z} \setminus \{0\}$, $r_j > 0$, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$.

When investigating best trigonometric approximations on the classes $L_{\boldsymbol{\beta},p}^{\boldsymbol{\psi}}$, we impose some additional conditions on the sequences ψ_i, $i = 1, \ldots, d$. So, let D be a set of functions ψ of natural argument that satisfy the conditions

- ψ are positive and non increasing;
- there exists $M > 0$ such that for all $l \in \mathbb{N}$ we have $\frac{\psi(l)}{\psi(2l)} \leq M$.

Note that to the indicated set belong, in particular, the functions $\phi(|\tau|) = |\tau|^{-r}$; $\phi(|\tau|) = \ln^{\alpha}(|\tau| + 1)$, $\alpha < 0$; $\phi(|\tau|) = \ln^{\alpha}(|\tau| + 1)|\tau|^{-r}$, where $\tau \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathbb{R}$, $r > 0$.

The results are presented in terms of functions

$$\varPhi(n) = \min_{(\boldsymbol{s},\boldsymbol{1})=n} \prod_{j=1}^d \psi_j(2^{s_j}), \qquad \varPsi(n) = \max_{(\boldsymbol{s},\boldsymbol{1})=n} \prod_{j=1}^d \psi_j(2^{s_j}),$$

where the vectors $\boldsymbol{s}, \boldsymbol{1} \in \mathbb{N}^d$. In the case $\psi_j(|\tau|) = |\tau|^{-r}$, $j = \overline{1, d}$, $r > 0$, we have $\varPhi(n) = \varPsi(n) = 2^{-nr}$ and, besides, for $d = 1$ the functions $\varPhi(n)$ and $\varPsi(n)$ coincide and take the form $\psi_1(2^n)$.

In what follows, we also establish estimates for the entropy numbers of the classes $B_{p,\theta}^{\Omega}$ in the metric of the space L_q, $1 \leq q \leq \infty$, under certain conditions imposed on the function Ω and the parameters p, θ. For the first time, the indicated classes with $\theta = \infty$ were considered by N. N. Pustovoitov [6]. In the paper by S. Yongsheng and W. Heping [11], these classes were extended to the case $1 \leq \theta < \infty$. They can be regarded as a generalization of the classes $B_{p,\theta}^{\boldsymbol{r}}$ with respect to a smooth parameter. The classes $B_{p,\theta}^{\Omega}$ are defined with the help of a majorant function $\Omega(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}_+^d$, for the mixed modulus of continuity $\Omega_l(f, \boldsymbol{t})_p$ of order l, $l \in \mathbb{N}$, of the function $f \in L_p(\pi_d)$, $1 \leq p \leq \infty$, and a numerical parameter θ, $1 \leq \theta \leq \infty$.

The results are given in terms of order relations. So, for two nonnegative sequences $\{a(n)\}_{n=1}^{\infty}$ and $\{b(n)\}_{n=1}^{\infty}$, the relation (order inequality) $a(n) \ll b(n)$ means that there exists a constant $C > 0$, independent of n and such that $a(n) \leq Cb(n)$. The relation $a(n) \asymp b(n)$ is equivalent to $a(n) \ll b(n)$ and $b(n) \ll a(n)$.

2. Estimates of the best trigonometric approximation

We now define the approximate characteristics. So, for $f \in L_q$, $1 \le q \le \infty$, the quantity

$$e_M(f)_q = \inf_{\boldsymbol{k}^j, c_j} \Big\| f - \sum_{j=1}^{M} c_j \mathrm{e}^{\mathrm{i}(\boldsymbol{k}^j, \boldsymbol{x})} \Big\|_q \tag{1}$$

is called the best M-term trigonometric approximation of function f, where $\{\boldsymbol{k}^j\}_{j=1}^M$ is the set of vectors $\boldsymbol{k}^j = (k_1^j, \dots, k_d^j) \in \mathbb{Z}^d$, $c_j \in \mathbb{C}$, $j = \overline{1, M}$. For $F \subset L_q$, we put $e_M(F)_q = \sup_{f \in F} e_M(f)_q$ and call it the best M-term trigonometric approximation of the functional class F.

We consider also a close to (1) characteristic $e_M^{\perp}(f)_q$ (respectively, $e_M^{\perp}(F)_q$), where the coefficients c_j of corresponding polynomials are the Fourier coefficients of the function f with respect to the system $\{\boldsymbol{k}^j\}_{j=1}^M$. The detailed history and further references could be found in the monograph by D. Dũng, V. N. Temlyakov and T. Ullrich [2].

Let us formulate some results for the best M-term approximation of the classes $L^{\psi}_{\beta,p}$, that we proved in the papers [7–9]. We considered first the limit case $p = 1$.

Theorem 1. *Let $1 < q \le 2$, $\psi_j \in D$, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$, and $\varepsilon > 0$ be such that $\psi_j(|\tau|)\,|\tau|^{1-1/q+\varepsilon}$, $j = \overline{1, d}$, do not increase. Then, for any natural M and n that satisfy the condition $M = M(n) \asymp 2^n n^{d-1}$, the following relations hold:*

$$\Phi(n) M^{1-1/q} (\log M)^{2(d-1)(1/q-1/2)} \ll e_M\big(L^{\psi}_{\beta,1}\big)_q \ll \Psi(n) M^{1-1/q} (\log M)^{2(d-1)(1/q-1/2)}.$$

Note, that in the univariate case we get an exact-order estimate $e_M\big(L^{\psi_1}_{\beta,1}\big)_q \asymp \psi_1(M) M^{1-1/q}$, $1 < q \le 2$. Analogous estimates (with the same left and right bounds) hold also for the quantity $e_M^{\perp}\big(L^{\psi_1}_{\beta,1}\big)_q$, $1 < q < \infty$.

Theorem 2. *Let $2 < q < \infty$, $\psi_j \in D$, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$, and $\varepsilon > 0$ be such that $\psi_j(|\tau|)\,|\tau|^{1+\varepsilon}$, $j = \overline{1, d}$, do not increase. Then, for any natural M and n that satisfy the condition $M = M(n) \asymp 2^n n^{d-1}$, the following relations hold:*

$$\Phi(n) M^{1/2} \ll e_M\big(L^{\psi}_{\beta,1}\big)_q \ll \Psi(n) M^{1/2}.$$

We see that, in the case $2 < q < \infty$, the best M-term approximation (that uses arbitrary coefficients of approximation polynomials) gives better bounds than the corresponding best orthogonal approximation, while for $1 < q \le 2$ they coincide in order.

In the paper [3], we get the estimates for $e_M\big(L^{\psi}_{\beta,p}\big)_q$ in the case of "small smoothness" of respective functions. We showed, that the best M-term approximation in this case gives better bounds, than the corresponding best orthogonal approximation and the approximation of functions from the class $L^{\psi}_{\beta,p}$ by trigonometric polynomials with numbers of harmonics from the so-called step hyperbolic crosses.

3. Estimates of entropy numbers

Let X be a Banach space, $B_X(\boldsymbol{y}, r)$ be a ball of radius r and center at point $\boldsymbol{y} \in \mathbb{R}^d$. For a compact set $A \subset X$ and $\varepsilon > 0$ by $\varepsilon_k(A, X)$ we denote entropy numbers (see., e.g., [1]) of this set:

$$\varepsilon_k(A, X) = \inf \Big\{ \varepsilon : \ \exists \boldsymbol{y}^1, \dots, \boldsymbol{y}^{2^k} \in X : \ A \subseteq \bigcup_{j=1}^{2^k} B_X(\boldsymbol{y}^j, \varepsilon) \Big\}.$$

In [4, 5], we obtained the estimates for entropy numbers of the classes of periodic multivariate functions $B^{\Omega}_{p,\theta}$, under the so-called Bari-Stechkin (see [1]) conditions (S^{α}) and (S_l) on the function Ω, in the space L_q, $1 \le q \le \infty$. In particular, the following statement holds in the case $d = 2$.

Theorem 3. *Let $d = 2$, $2 \le p \le \infty$, $1 \le \theta \le \infty$, and $\Omega(\boldsymbol{t}) = \omega\Big(\prod_{j=1}^{d} t_j\Big)$, where the function ω satisfies condition (S^{α}) with some $\alpha > 1/2$ and condition (S_l). Then, for any natural M and n such that $M = M(n) \asymp 2^n n$, the following relation holds:*

$$\varepsilon_M\left(B^{\Omega}_{p,\theta}, L_\infty\right) \asymp \omega\left(2^{-n}\right)(\log M)^{1-1/\theta}.$$

Note that we got estimates of the quantity $\varepsilon_M\left(B^{\Omega}_{p,\theta}, L_q\right)$ for diferent relations between p and q.

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