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Free Banach lattices

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Abstract: In this paper we introduce the free Banach lattices generated by certain structures, such as Banach spaces and lattices, and show some of their properties.

Resumen: En este artículo introducimos los retículos de Banach libres generados por determinadas estructuras, tales como los espacios de Banach y los retículos, y mostramos algunas de sus propiedades.

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1. Introduction

We all know that the starting point of functional analysis was the investigation of the classical function spaces, which provide its most important applications. However, the natural order in these spaces was neglected almost completely. A first attempt to include a compatible order structure in the study of linear and normed spaces was due to F. Riesz, H. Freudenthal and L. V. Kantorovič in the mid-thirties. In the following years, schools of research on vector lattices were subsequently founded and these investigations were continued by various mathematicians in the Soviet Union (B. Z. Vulikh, A. G. Pinsker, A. I. Judin), in Japan (H. Nakano, T. Ogasawara, K. Yosida), and in United States (G. Birkhoff, S. Kakutani, H. F. Bohnenblust, M. H. Stone).

L. V. Kantorovič and his school first recognized the importance of studying vector lattices in connection with Banach's theory of normed spaces; they investigated normed vector lattices as well as order-related linear operators between such vector lattices.

This paper is about the basic theory of free Banach lattices. We define the free Banach lattices generated by a set, by a Banach space, and by a lattice, and show some of their properties. We refer the reader to [1–9] for more background on free Banach lattices.

2. Free Banach lattices

Recall that a *Banach lattice* is a Banach space $(X, \|\cdot\|)$ together with a partial order $\leq$ with the following properties:

(i) For every pair of elements $x, y \in X$ there exist $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$.

(ii) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in X$,

(iii) $0 \leq x$ implies $0 \leq tx$ for every $x \in X$ and $t \in \mathbb{R}^+$,

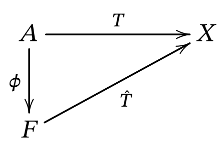
(iv) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for every $x, y \in X$, where $|x| := x \vee (-x)$.

Properties (i), (ii) and (iii) together can be read as $(X, \leq)$ is a *vector lattice*, while property (iv) means that $\|\cdot\|$ is a *lattice norm*.

The natural morphisms in this category are those maps that preserve the structure of Banach space and vector lattice. A map $T\colon X \to Y$ between two Banach lattices, X and Y, is said to be a *Banach lattice homomorphism* if it is a bounded linear operator and preserves the lattice operations. If T is also bijective and T^{-1} is a Banach lattice homomorphism, we say that T is a *Banach lattice isomorphism*. If moreover, T preserves the norm (that is, $\|T(x)\| = \|x\|$ for every $x \in X$), we say that T is a *Banach lattice isometry*.

The first authors who introduced the concept of free object within the category of Banach lattices were B. de Pagter and A. W. Wickstead in 2015, who defined and studied properties about the free Banach lattice generated by a set [9].

Definition 1. Let A be a non-empty set. A *free Banach lattice over* or *generated by* A is a Banach lattice F together with a bounded map $\phi\colon A \to F$ with the property that for every Banach lattice X and every bounded map $T\colon A \to X$ there is a unique Banach lattice homomorphism $\hat{T}\colon F \to X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.



Here, the norm of T is $\|T\| := \sup\{\|T(a)\| : a \in A\}$, while the norm of $\hat{T}$ is the usual for Banach spaces.

This property uniquely determines F up to Banach lattices isometries, and so we can speak of *the* free Banach lattice generated by A, denoted by $FBL(A)$.

Now, the question is whether such an object exists. The answer is affirmative. B. de Pagter and A. W. Wickstead prove it in [9], but A. Avilés, J. Rodríguez and P. Tradacete give an alternative and more tangible way of constructing it in [4]. They describe it as a space of functions:

For $a \in A$, let $\delta_a : [-1,1]^A \to \mathbb{R}$ be the evaluation function given by $\delta_a(x^*) = x^*(a)$ for every $x^* \in [-1,1]^A$, and for $f : [-1,1]^A \to \mathbb{R}$ define

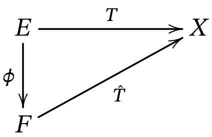
$$\|f\| = \sup\left\{\sum_{i=1}^{n} |f(x_i^*)| \,:\, n \in \mathbb{N},\ x_1^*, \dots, x_n^* \in [-1,1]^A,\ \sup_{a\in A} \sum_{i=1}^{n} |x_i^*(a)| \leq 1\right\}.$$

Theorem 2 ([4, Corollary 2.9]). *The free Banach lattice generated by a set A is the closure of the vector lattice generated by $\{\delta_a : a \in A\}$ under the above norm inside the Banach lattice of all functions $f \in \mathbb{R}^{[-1,1]^A}$ with $\|f\| < \infty$, endowed with the norm $\|\cdot\|$, the pointwise order and the pointwise operations.*

The natural identification of A inside $FBL(A)$ is given by the map $\phi : A \to FBL(A)$ where $\phi(a) = \delta_a$ for every $a \in A$. Since every function in $FBL(A)$ is a uniform limit of such functions, they are all continuous (in the product topology) and positively homogeneous (that is, $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in [-1,1]^A$ and for every $\lambda \geq 0$ such that $\lambda x^* \in [-1,1]^A$, or equivalently, $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in [-1,1]^A$ and for every $0 \leq \lambda \leq 1$).

This definition was soon generalized by A. Avilés, J. Rodríguez and P. Tradacete in [4] to the free Banach lattice generated by a Banach space E in the following sense:

Definition 3. Let E be a Banach space. A *free Banach lattice over* or *generated by E* is a Banach lattice F together with a bounded operator $\phi : E \to F$ with the property that for every Banach lattice X and every bounded operator $T : E \to X$ there is a unique Banach lattice homomorphism $\hat{T} : F \to X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.



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This property uniquely determines F up to Banach lattices isometries, and so we can speak of *the* free Banach lattice generated by E, denoted by $FBL[E]$. This definition generalizes the notion of the free Banach lattice generated by a set A in the sense that the free Banach lattice generated by a set A is the free Banach lattice generated by the Banach space $\ell_1(A)$ (see [4, Corollary 2.9]).

This definition is not very friendly to work with it. However, similar to the previous case, it is possible to give an explicit description of it as a space of functions:

Let us denote by $H[E]$ the vector subspace of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions $f : E^* \to \mathbb{R}$ (that is, all functions that satisfy $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in E^*$ and for every $\lambda \geq 0$). For any $f \in H[E]$ let us define

$$\|f\| = \sup\left\{\sum_{i=1}^{n} |f(x_i^*)| \,:\, n \in \mathbb{N},\ x_1^*, \dots, x_n^* \in E^*,\ \sup_{x\in B_E} \sum_{i=1}^{n} |x_i^*(x)| \leq 1\right\}.$$

Let us take $H_0[E] = \{f \in H[E] : \|f\| < \infty\}$. It is easy to check that $H_0[E]$ is a Banach lattice when equipped with the norm $\|\cdot\|$ and the pointwise order.

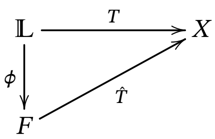
Now, given $x \in E$, let $\delta_x : E^* \to \mathbb{R}$ be the evaluation function given by $\delta_x(x^*) = x^*(x)$ for every $x^* \in E^*$.

Theorem 4 ([4, Theorem 2.5]). *The free Banach lattice generated by a Banach space E is the closure of the vector lattice generated by $\{\delta_x : x \in E\}$ under the above norm inside $H_0[E]$.*

The natural identification of E inside $FBL[E]$ is given by the map $\phi : E \to FBL[E]$ where $\phi(x) = \delta_x$ for every $x \in E$ (it is a linear isometry between E and its image in $FBL[E]$). Moreover, all the functions in $FBL[E]$ are weak*-continuous when restricted to the closed unit ball B_{E^*} (see [4, Lemma 4.10]).

On the other hand, the notion of the free Banach lattice generated by a lattice $\mathbb{L}$ is due to A. Avilés and J. D. Rodríguez Abellán [5].

Definition 5. Given a lattice $\mathbb{L}$, a *free Banach lattice over* or *generated by* $\mathbb{L}$ is a Banach lattice F together with a bounded lattice homomorphism $\phi\colon \mathbb{L} \to F$ with the property that for every Banach lattice X and every bounded lattice homomorphism $T\colon \mathbb{L} \to X$ there is a unique Banach lattice homomorphism $\hat{T}\colon F \to X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.



Here, the norm of T is $\|T\| \coloneqq \sup\{\|T(x)\| : x \in \mathbb{L}\}$, while the norm of $\hat{T}$ is the usual for Banach spaces.

This definition determines a Banach lattice that we denote by $FBL\langle\mathbb{L}\rangle$ in an essentially unique way. When $\mathbb{L}$ is a distributive lattice (which is a natural assumption in this context, see [5, Section 3]) the function ϕ is injective and, loosely speaking, we can view $FBL\langle\mathbb{L}\rangle$ as a Banach lattice which contains a subset lattice-isomorphic to $\mathbb{L}$ in a way that its elements work as free generators modulo the lattice relations on $\mathbb{L}$.

In order to give an explicit description of it similar to the previous cases, define

$$\mathbb{L}^* = \{x^* : \mathbb{L} \to [-1,1] : x^* \text{ is a lattice homomorphism}\}.$$

For every $x \in \mathbb{L}$ consider the evaluation function $\delta_x\colon \mathbb{L}^* \to \mathbb{R}$ given by $\delta_x(x^*) = x^*(x)$, and for $f \in \mathbb{R}^{\mathbb{L}^*}$, define

$$\|f\| = \sup\left\{\sum_{i=1}^{n} |f(x_i^*)| : n \in \mathbb{N},\ x_1^*, \dots, x_n^* \in \mathbb{L}^*,\ \sup_{x\in\mathbb{L}} \sum_{i=1}^{n} |x_i^*(x)| \le 1\right\}.$$

Theorem 6 ([5, Theorem 1.2]). *Consider F to be the closure of the vector lattice generated by $\{\delta_x : x \in \mathbb{L}\}$ under the norm $\|\cdot\|$ inside the Banach lattice of all functions $f \in \mathbb{R}^{\mathbb{L}^*}$ with $\|f\| < \infty$, endowed with the norm $\|\cdot\|$, the pointwise order and the pointwise operations. Then F, together with the assignment $\phi(x) = \delta_x$, is the free Banach lattice generated by $\mathbb{L}$.*

References

[1] Avilés, A.; Martínez-Cervantes, G., and Rodríguez Abellán, J. D. "On projective Banach lattices of the form $C(K)$ and $FBL[E]$". In: *J. Math. Anal. Appl.* 489.124129 (2020).

[2] Avilés, A.; Martínez-Cervantes, G., and Rodríguez Abellán, J. D. "On the Banach lattice c_0". In: *Rev. Mat. Complut.* 34 (2021), pp. 203–213.

[3] Avilés, A.; Plebanek, G., and Rodríguez Abellán, J. D. "Chain conditions in free Banach lattices". In: *J. Math. Anal. Appl.* 465 (2018), pp. 1223–1229.

[4] Avilés, A.; Rodríguez, J., and Tradacete, P. "The free Banach lattice generated by a Banach space". In: *J. Funct. Anal.* 274 (2018), pp. 2955–2977.

[5] Avilés, A. and Rodríguez Abellán, J. D. "The free Banach lattice generated by a lattice". In: *Positivity* 23 (2019), pp. 581–597.

[6] Avilés, A. and Rodríguez Abellán, J. D. "Projectivity of the free Banach lattice generated by a lattice". In: *Archiv der Mathematik* 113 (2019), pp. 515–524.

[7] Dantas, S.; Martínez-Cervantes, G.; Rodríguez Abellán, J. D., and Rueda Zoca, A. "Octahedral norms in free Banach lattices". In: *RACSAM* 115.6 (2021).

[8] Dantas, S.; Martínez-Cervantes, G.; Rodríguez Abellán, J. D., and Rueda Zoca, A. "Norm-attaining lattice homomorphisms". In: *Rev. Mat. Iberoam.* (to appear).

[9] Pagter, B. de and Wickstead, A. W. "Free and projective Banach lattices". In: *Proc. Royal Soc. Edinbrgh Sect. A* 145 (2015), pp. 105–143.