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Numerical index of absolute symmetric norms on the plane

Alicia Quero
Universidad de Granada
aliciaquero@ugr.es

Abstract: We give a lower bound for the numerical index of two-dimensional real spaces with absolute and symmetric norm. This allows us to compute the numerical index of the two-dimensional real L_p-space for $3/2 \leq p \leq 3$.

Resumen: Damos una cota inferior del índice numérico para espacios reales dos-dimensionales con normas absolutas y simétricas. Esto nos permite calcular el índice numérico del espacio real dos-dimensional L_p para $3/2 \leq p \leq 3$.

Keywords: numerical range, numerical radius, numerical index, absolute symmetric norm, L_p-spaces..

MSC2010: 46B20, 47A12.

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1. Introduction

The numerical index of a Banach space is a constant relating the norm and the numerical range of bounded linear operators on the space. Given a Banach space X, we will write X^* for its topological dual and $\mathcal{L}(X)$ for the Banach algebra of all (bounded linear) operators on X. For an operator $T \in \mathcal{L}(X)$, its *numerical range* is defined as

$$V(T) := \{x^*(Tx) \,:\, x^* \in X^*, x \in X, \|x^*\| = \|x\| = x^*(x) = 1\},$$

and its *numerical radius* is

$$v(T) := \sup\{|\lambda| \,:\, \lambda \in V(T)\}.$$

Clearly, v is a seminorm on $\mathcal{L}(X)$ satisfying $v(T) \leq \|T\|$ for every $T \in \mathcal{L}(X)$. The *numerical index* of X is the constant given by

$$n(X) := \inf\{v(T) \,:\, T \in \mathcal{L}(X), \|T\| = 1\},$$

or, equivalently, $n(X)$ is the greatest constant $k \geq 0$ satisfying $k\,\|T\| \leq v(T)$ for every $T \in \mathcal{L}(X)$. There has been a deep development of this field of study with the contribution of several authors. The state of the art on the subject can be found in the survey paper [5] and references therein.

It is clear that $0 \leq n(X) \leq 1$ for every Banach space X. There are some classical Banach spaces for which the numerical index has been calculated. If H is a Hilbert space of dimension greater than one, then $n(H) = 0$ in the real case and $n(H) = 1/2$ in the complex case. Besides, $n(L_1(\mu)) = 1$ and the same happens to all its isometric preduals. In particular, it follows that $n(C(K)) = 1$ for every compact K.

2. Numerical index of absolute symmetric norms and L_p-spaces

The problem of computing the numerical index of the L_p-spaces has been latent since the beginning of the theory [4]. In order to present the known results on this matter we need to fix some notation. For $1 < p < \infty$, we write ℓ_p^m for the m-dimensional L_p-space, $q = p/(p-1)$ for the conjugate exponent to p, and

$$M_p := \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = \max_{t\geq 1} \frac{|t^{p-1} - t|}{1 + t^p},$$

which is the numerical radius of the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defined on the real space ℓ_p^2. This can be found in [7, Lemma 2]. However, some results have been obtained on the numerical index of the L_p-spaces [1–3, 7, 8], we summarize them in the following list.

(i) The sequence $(n(\ell_p^m))_{m\in\mathbb{N}}$ is decreasing.

(ii) $n(L_p(\mu)) = \inf\{n(\ell_p^m) \,:\, m \in \mathbb{N}\}$ for every measure μ such that $\dim(L_p(\mu)) = \infty$.

(iii) In the real case, $n(L_p[0,1]) \geq M_p/12$.

(iv) In the real case, $\max\left\{\frac{1}{2^{1/p}}, \frac{1}{2^{1/q}}\right\} M_p \leq n(\ell_p^2) \leq M_p$.

The presence of the numerical radius of the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the value of the numerical index of L_p-spaces is not a coincidence. For those two-dimensional real spaces with absolute and symmetric norm whose numerical index is known, it coincides with the numerical radius of the mentioned operator. This happens, for instance, to a family of octagonal norms and to the spaces whose unit ball is a regular polygon, see [6, Theorem 2 and Theorem 5]. In the paper [9], it is shown that the same happens for many absolute and symmetric norms on $\mathbb{R}^2$, this is the content of Theorem 1. We say that a norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ is *absolute* if $\|(1,0)\| = \|(0,1)\| = 1$ and

$$\|(a,b)\| = \|(|a|,|b|)\|,$$

for every $a, b \in \mathbb{R}$, and that the norm is *symmetric* if $\|(b,a)\| = \|(a,b)\|$ for every $a, b \in \mathbb{R}$. Some of the most important examples of absolute and symmetric norms are ℓ_p-norms on $\mathbb{R}^2$.

If X is $\mathbb{R}^2$ endowed with an absolute and symmetric norm, the existence of a basis of the space of operators $\mathcal{L}(X)$ formed by onto isometries is particularly useful:

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad I_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first main result of the paper [9] is the following.

Theorem 1. [9, Theorem 2.2] *Let X be $\mathbb{R}^2$ endowed with an absolute and symmetric norm. Let $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ be such that $|x_0^*(I_4 x_0)| = v(I_4)$ and write $c_j = |x_0^*(I_j x_0)|$ for every $j = 1, \dots, 4$. If $c_4 = 0$, then $n(X) = 0$. If, otherwise, $c_4 > 0$, then*

$$n(X) \geq \min\left\{ c_4, \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}} \right\}.$$

Moreover, if the inequality $c_4\left(1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \leq 1$ holds, then

$$n(X) = v(I_4).$$

As a major consequence, the numerical index of ℓ_p^2 for $3/2 \leq p \leq 3$ is calculated, which improves partially [7, Theorem 1] and throws some light to the long standing problem of computing the numerical index of L_p-spaces.

Theorem 2. [9, Theorem 2.3] *Let $p \in \left[\frac{3}{2}, 3\right]$. Then,*

$$n(\ell_p^2) = M_p = \sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

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