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On the growth and fixed points of solutions of linear differential equations with entire coefficients

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Abstract: In this paper, we use a generalized concept of order called the φ-order to investigate the growth and the oscillation of fixed points of solutions of higher order complex linear differential equations with entire coefficients. We describe the relationship between the solutions and the entire coefficients in terms of φ-order and φ-convergence exponent. We extend and improve some earlier results due to Cao, Xu, and Chen, Chyzhykov and Semochko, Kara and Belaïdi.

Resumen: En este trabajo, utilizamos un concepto generalizado de orden llamado φ-orden para investigar el crecimiento y la oscilación de los puntos fijos de las soluciones de las ecuaciones diferenciales lineales complejas de orden superior con coeficientes enteros. Describimos la relación entre las soluciones y los coeficientes enteros en términos del φ-orden y del φ-exponente de convergencia. Ampliamos y mejoramos algunos resultados anteriores de Cao, Xu y Chen, Chyzhykov y Semochko, Kara y Belaïdi.

Keywords: entire function, meromorphic function, φ-order, φ-exponent of convergence, linear differential equation.

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1. Introduction

Throughout this paper, we use the standard notations of Nevanlinna value distribution theory such as $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ (see [5]). The term "meromorphic function" will mean meromorphic in the whole complex plane $\mathbb{C}$. For $k \geq 2$, we consider the following linear differential equations:

(1) $$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = 0,$$

(2) $$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = F(z).$$

It is well know that, if the coefficients F and $A_0(z), \dots, A_{k-1}$ are entire functions, all solutions of (1) and (2) are entire. Cao, Xu, and Chen [1] have investigated the growth of meromorphic solutions of equations (1) and (2) when the coefficients are meromorphic functions of finite iterated order. Chyzhykov and Semochko [2] used a more general concept called the φ-order which can cover an arbitrary growth of fast growing functions. They obtained the precise estimates for φ-order of entire solutions of (1) when the coefficient A_0 strictly dominates the growth of coefficients. Later, the authors [3] investigated equations (1) and (2) when the coefficients are meromorphic functions with finite φ-order.

Definition 1 ([2]). Let φ be an increasing unbounded function on $(0, +\infty)$. The φ-orders of a meromorphic function f are defined by

$$\rho^0_\varphi(f) = \limsup_{r\to+\infty} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r}, \qquad \rho^1_\varphi(f) = \limsup_{r\to+\infty} \frac{\varphi\left(T(r,f)\right)}{\log r}.$$ ◂

Definition 2 ([3]). Let φ be an increasing unbounded function on $(0, +\infty)$. We define the φ-convergence exponents of the sequence of zeros of a meromorphic function f by

$$\lambda^0_\varphi(f) = \limsup_{r\to+\infty} \frac{\varphi\left(e^{N(r,1/f)}\right)}{\log r}, \qquad \lambda^1_\varphi(f) = \limsup_{r\to+\infty} \frac{\varphi\left(N\left(r, \frac{1}{f}\right)\right)}{\log r}.$$

Similarly, if we replace $N(r, 1/f)$ by $\overline{N}(r, 1/f)$, we obtain $\overline{\lambda}^0_\varphi(f)$ and $\overline{\lambda}^1_\varphi(f)$, which denote the φ-convergence exponents of the sequence of distinct zeros of f. ◂

Let Φ denotes the class of positive unbounded increasing functions on $(0, +\infty)$ such that $\varphi(e^t)$ grows slowly, i.e., for all $c > 0$ we have $\lim_{t\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1$. For instance, $\log\log(\cdot) \in \Phi$, while $\log(\cdot) \notin \Phi$.

Proposition 3 ([2, 4]). *Let $\varphi \in \Phi$ and let f_1, f_2 be two meromorphic functions. Then, for $j = 0, 1$ we have*

$$\max\left\{\rho^j_\varphi(f_1 + f_2), \rho^j_\varphi(f_1 f_2)\right\} \leq \max\left\{\rho^j_\varphi(f_1), \rho^j_\varphi(f_2)\right\}.$$

Moreover, if $\rho^j_\varphi(f_1) < \rho^j_\varphi(f_2)$, then $\rho^j_\varphi(f_1 + f_2) = \rho^j_\varphi(f_1 f_2) = \rho^j_\varphi(f_2)$.

Proposition 4 ([3, 4]). *Let $\varphi \in \Phi$ and let f be a meromorphic function. Then,*

(i) *$\rho^j_\varphi(f') = \rho^j_\varphi(f)$ for $j = 0, 1$,*

(ii) *if $\rho^0_\varphi(f) < +\infty$, then $\rho^1_\varphi(f) = 0$.*

Theorem 5 ([2]). *Let $\varphi \in \Phi$ and $A_0, A_1, \dots, A_{k-1}$ be entire functions satisfying*

$$\max\{\rho^0_\varphi(A_j), j = 1, \dots, k-1\} < \rho^0_\varphi(A_0).$$

Then, every solution $f \not\equiv 0$ of (1) satisfies $\rho^1_\varphi(f) = \rho^0_\varphi(A_0)$.

Theorem 6 ([3]). *Let $\varphi \in \Phi$ and let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If f is a meromorphic solution of (2) satisfying for $i = 0, 1$*

$$\max\left\{\rho^i_\varphi(F), \rho^i_\varphi(A_j) : j = 0, 1, \dots, k-1\right\} < \rho^i_\varphi(f),$$

then $\overline{\lambda}^i_\varphi(f) = \lambda^i_\varphi(f) = \rho^i_\varphi(f)$.

2. Main results

This paper is concerned with the properties of growth and oscillation of fixed points of entire solutions of equations (1) and (2) involving the concept of φ-order. We list here our main results.

Theorem 7. *Under the hypothesis of Theorem 5, if $A_1(z) + zA_0(z) \not\equiv 0$, then every solution $f \not\equiv 0$ of (1) satisfies $\overline{\lambda}^{1}_{\varphi}(f - z) = \rho^1_\varphi(f) = \rho^0_\varphi(A_0)$.*

Proof. Let $f \not\equiv 0$ be an entire solution of (1). Set $g = f - z$. Clearly,

$$\overline{\lambda}^{1}_{\varphi}(g) = \overline{\lambda}^{1}_{\varphi}(f - z) \quad \text{and} \quad \rho^1_\varphi(g) = \rho^1_\varphi(f - z) = \rho^1_\varphi(f). \tag{3}$$

From (1), we get

$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \ldots + A_1(z)g' + A_0(z)g = -[A_1(z) + zA_0(z)]. \tag{4}$$

By Theorem 5, we have $\rho^1_\varphi(f) = \rho^0_\varphi(A_0)$. Since $A_1(z) + zA_0(z) \not\equiv 0$ is also an entire function, it follows from Proposition 4 that

$$\max\left\{\rho^1_\varphi(-A_1 - zA_0), \rho^1_\varphi(A_j) \,:\, j = 0, 1, \ldots, k-1\right\} < \rho^1_\varphi(f) = \rho^1_\varphi(g).$$

Thus, by applying Theorem 6 to (4), we obtain $\overline{\lambda}^{1}_{\varphi}(g) = \rho^1_\varphi(g)$. Therefore, $\overline{\lambda}^{1}_{\varphi}(f - z) = \rho^1_\varphi(f) = \rho^0_\varphi(A_0)$. ∎

By using analogous proofs of Theorem 1 and Theorem 4 in Kara and Belaïdi [3], we can easily obtain the following two results.

Theorem 8. *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying*

$$\max\left\{\rho^0_\varphi(A_j) \,:\, j = 0, 1, \ldots, k-1 \ (j \neq s)\right\} < \rho^0_\varphi(A_s) < +\infty.$$

Then, every transcendental solution f of (1) satisfies $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s) \le \rho^0_\varphi(f)$. Furthermore, there exists at least one solution satisfying $\rho^1_\varphi(f) = \rho^0_\varphi(A_s)$.

Theorem 9. *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be entire functions satisfying*

$$\max\left\{\rho^1_\varphi(F), \rho^0_\varphi(A_j) \,:\, j = 1, \ldots, k-1\right\} < \rho^0_\varphi(A_0) < +\infty.$$

Then, every solution f of (2) satisfies $\overline{\lambda}^{1}_{\varphi}(f) = \lambda^1_\varphi(f) = \rho^1_\varphi(f) = \rho^0_\varphi(A_0)$ with at most one exceptional solution satisfying $\rho^1_\varphi(f) < \rho^0_\varphi(A_0)$.

By replacing the dominant coefficient A_0 in Theorem 7 by an arbitrary coefficient A_s ($s \in \{0, 1, \ldots, k-1\}$), we obtain the following result.

Theorem 10. *Under the hypothesis of Theorem 8, if $A_1(z) + zA_0(z) \not\equiv 0$, then every transcendental solution f of (1) such that $\rho^0_\varphi(f) > \rho^0_\varphi(A_s)$ satisfies $\overline{\lambda}^{0}_{\varphi}(f - z) = \rho^0_\varphi(f)$. Moreover, there exists at least one solution f_1 satisfying $\overline{\lambda}^{1}_{\varphi}(f_1 - z) = \rho^1_\varphi(f_1) = \rho^0_\varphi(A_s)$.*

Proof. By Theorem 8, we have $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s)$. Assume that $\rho^0_\varphi(A_s) < \rho^0_\varphi(f)$ and, since $A_1(z) + zA_0(z) \not\equiv 0$, then

$$\max\left\{\rho^0_\varphi(-A_1 - zA_0), \rho^0_\varphi(A_j) \,:\, j = 0, 1, \ldots, k-1\right\} = \rho^0_\varphi(A_s) < \rho^0_\varphi(f).$$

Thus, by using the fact that $\rho^0_\varphi(f) = \rho^0_\varphi(g) = \rho^0_\varphi(f - z)$, and applying Theorem 6 to (4), we obtain $\overline{\lambda}^{0}_{\varphi}(g) = \rho^0_\varphi(g)$, i.e., $\overline{\lambda}^{0}_{\varphi}(f - z) = \rho^0_\varphi(f)$. Again, by Theorem 8, there exists a solution f_1 of (1) such that $\rho^1_\varphi(f_1) = \rho^0_\varphi(A_s)$. We deduce from Proposition 4 that

$$\max\left\{\rho^1_\varphi(-A_1 - zA_0), \rho^1_\varphi(A_j) \,:\, j = 0, 1, \ldots, k-1\right\} < \rho^1_\varphi(f_1) = \rho^1_\varphi(f_1 - z).$$

Hence, it follows from Theorem 6 and (4) that $\overline{\lambda}^{1}_{\varphi}(f_1 - z) = \rho^1_\varphi(f_1 - z)$. Therefore, $\overline{\lambda}^{1}_{\varphi}(f_1 - z) = \rho^1_\varphi(f_1) = \rho^0_\varphi(A_s)$. ∎

Theorem 11. *Under the hypothesis of Theorem 9, if* $F(z) - [A_1(z) + zA_0(z)] \not\equiv 0$, *then every solution* f *of (2) such that* $\rho^1_\varphi(f) = \overline{\lambda}^1_\varphi(f)$ *satisfies* $\overline{\lambda}^1_\varphi(f - z) = \rho^1_\varphi(f)$.

Proof. Let f be a solution of (2) such that $\rho^1_\varphi(f) = \overline{\lambda}^1_\varphi(f)$. Set $g = f - z$. Equation (2) becomes

(5) $$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g = F(z) - [A_1(z) + zA_0(z)].$$

It follows from Theorem 9 that every solution f of (2) satisfies $\rho^1_\varphi(f) = \rho^1_\varphi(A_0) = \overline{\lambda}^1_\varphi(f)$ with at most one exceptional solution satisfying $\rho^1_\varphi(f) < \rho^0_\varphi(A_0)$. Hence, by Proposition 4 and since $F(z) - [A_1(z) + zA_0(z)] \not\equiv 0$, we obtain

$$\max\left\{\rho^1_\varphi(F - A_1 - zA_0), \rho^1_\varphi(A_j) \,:\, j = 0, 1, \ldots, k-1\right\} < \rho^1_\varphi(f) = \rho^1_\varphi(g).$$

Thus, by applying Theorem 6 to (5), we obtain $\overline{\lambda}^1_\varphi(g) = \rho^1_\varphi(g)$. Therefore, $\overline{\lambda}^1_\varphi(f - z) = \rho^1_\varphi(f)$. ∎

3. Future aspects

This paper rises many interesting questions, such as the following:

Question 1: Can we obtain similar results if the coefficients of equations (1) and (2) are meromorphic functions?

Question 2: What can be said if we consider equations (1) and (2) with analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} \,:\, |z| < 1\}$?

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