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On linear generic inhomogeneous boundary-value problems for differential systems in Sobolev spaces

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Abstract: For the systems of ordinary differential equations of an arbitrary order on a compact interval, we study a character of solvability of the most general linear boundary-value problems in the Sobolev spaces W_p^n, with $n \in \mathbb{N}$ and $1 \le p \le \infty$. We find the indices of these Fredholm problems and obtain a criterion of their well-posedness. Each of these boundary-value problems relates to a certain rectangular numerical characteristic matrix with kernel and cokernel of the same dimension as the kernel and cokernel of the boundary-value problem. The condition for the sequence of characteristic matrices to converge is found. We obtain a constructive criterion under which the solutions to these problems depend continuously on the small parameter ε at $\varepsilon = 0$, and find the degree of convergence of the solutions. Also applications of these results to multipoint boundary-value problems are obtained.

Resumen: Para los sistemas de ecuaciones diferenciales ordinarias de un orden arbitrario en un intervalo compacto, estudiamos un carácter de solubilidad de los problemas lineales de valor límite más generales en los espacios de Sobolev W_p^n, con $n \in \mathbb{N}$ y $1 \le p \le \infty$. Encontramos los índices de estos problemas de Fredholm y obtenemos un criterio de su buena composición. Cada uno de estos problemas de valor límite se relaciona con una cierta matriz característica numérica rectangular con núcleo y cokernel de la misma dimensión que el núcleo y el cokernel del problema de valor límite. Se encuentra la condición para que la secuencia de matrices características converja. Obtenemos un criterio constructivo bajo el cual las soluciones de estos problemas dependen continuamente del pequeño parámetro ε en $\varepsilon = 0$, y encontramos el grado de convergencia de las soluciones. También se obtienen aplicaciones de estos resultados a problemas de valores límite multipunto.

Keywords: differential system, generic boundary-value problem, Sobolev space, operator index, continuity in a parameter.

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1. Generic boundary-value problem

Let a finite interval $[a, b] \subset \mathbb{R}$ and parameters $\{m, n, r, l\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given. By $W_p^{n+r} = W_p^{n+r}([a, b]; \mathbb{C}) := \{y \in C^{n+r-1}[a, b] : y^{(n+r-1)} \in AC[a, b], y^{(n+r)} \in L_p[a, b]\}$ we denote a complex Sobolev space and set $W_p^0 := L_p$. This space is a Banach one with respect to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \|y^{(k)}\|_p + \|y^{(n+r)}\|_p,$$

where $\|\cdot\|_p$ is the norm in the space $L_p([a, b]; \mathbb{C})$. Similarly, by $(W_p^{n+r})^m := W_p^{n+r}([a, b]; \mathbb{C}^m)$ and $(W_p^{n+r})^{m\times m} := W_p^{n+r}([a, b]; \mathbb{C}^{m\times m})$ we denote Sobolev spaces of vector-valued functions and matrix-valued functions, respectively, whose elements belong to the function space W_p^{n+r}.

We consider the following linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a, b), \tag{1}$$

$$By = c, \tag{2}$$

where the matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, the vector-valued function $f(\cdot) \in (W_p^n)^m$, the vector $c \in \mathbb{C}^l$, and the linear continuous operator

$$B : (W_p^{n+r})^m \to \mathbb{C}^l \tag{3}$$

are arbitrarily chosen, and the vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

We represent vectors and vector-valued functions in the form of columns. A solution to the boundary-value problem (1), (2) is understood as a vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ satisfying equation (1) almost everywhere on (a, b) (everywhere for $n \geq 2$) and equality (2) specifying l scalar boundary conditions. The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the boundary condition (2) with continuous operator (3) is the most general condition for this equation.

It includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and many-point problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives (generally fractional) $y^{(k)}(\cdot)$ with $0 < k \leq n + r$.

For $1 \leq p < \infty$, every operator B in (3) admits a unique analytic representation

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\, \mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m,$$

where the matrices $\alpha_k \in \mathbb{C}^{rm\times m}$ and the matrix-valued function $\Phi(\cdot) \in L_{p'}\big([a, b]; \mathbb{C}^{rm\times m}\big)$, $1/p + 1/p' = 1$.

For $p = \infty$ this formula also defines an operator $B : (W_\infty^{n+r})^m \to \mathbb{C}^{rm}$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

With the generic inhomogeneous boundary-value problem (1), (2), we associate a linear continuous operator in pair of Banach spaces

$$(L, B) : (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \tag{4}$$

Recall that a linear continuous operator $T : X \to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If operator T is Fredholm, then its range $T(X)$ is closed in Y and the index

$$\operatorname{ind} T := \dim \ker T - \dim(Y/T(X)) \in \mathbb{Z}$$

is finite.

Theorem 1. *The linear operator (4) is a bounded Fredholm operator with index $mr - l$.*

Theorem 1 allows the next specification.

For each number $k \in \{1, \dots, r\}$, we consider the family of matrix Cauchy problems:

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a, b),$$

with the initial conditions

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \qquad j \in \{1, \dots, r\}.$$

Here, $Y_k(\cdot)$ is an unknown $m \times m$ matrix-valued function, and $\delta_{k,j}$ is the Kronecker symbol.

By $[BY_k]$ we denote the numerical $m \times l$ matrix, in which the j-th column is the result of the action of the operator B on the j-th column of the matrix-valued function $Y_k(\cdot)$.

Definition 2. A block rectangular numerical matrix $M(L, B) := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{mr \times l}$ is characteristic to the inhomogeneous boundary-value problem (1), (2). It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m \times l}$. ◂

Here mr is the number of scalar differential equations of the system (1), and l is the number of scalar boundary conditions.

Theorem 3. *The dimensions of the kernel and cokernel of the operator (4) are equal to the dimensions of the kernel and cokernel of the characteristic matrix $M(L, B)$, respectively.*

Theorem 3 implies a criterion for the invertibility of the operator (4).

Corollary 4. *The operator (L, B) is invertible if and only if $l = mr$ and the matrix $M(L, B)$ is nondegenerate.*

2. Generic boundary-value problem with a parameter

Let us consider, parameterized by number $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, the linear boundary-value problem

$$L(\varepsilon) y(t, \varepsilon) := y^{(r)}(t, \varepsilon) + \sum_{j=1}^{r} A_{r-j}(t, \varepsilon) y^{(r-j)}(t, \varepsilon) = f(t, \varepsilon), \quad t \in (a, b), \tag{5}$$

$$B(\varepsilon) y(\cdot; \varepsilon) = c(\varepsilon), \tag{6}$$

where for every fixed ε the matrix-valued functions $A_{r-j}(\cdot; \varepsilon) \in ({W_p}^n)^{m \times m}$, the vector-valued function $f(\cdot; \varepsilon) \in ({W_p}^n)^m$, the vector $c(\varepsilon) \in \mathbb{C}^{rm}$, $B(\varepsilon)$ is the linear continuous operator $B(\varepsilon)\colon ({W_p}^{n+r})^m \to \mathbb{C}^{rm}$, and the solution (the unknown vector-valued function) $y(\cdot; \varepsilon) \in ({W_p}^{n+r})^m$.

It follows from Theorem 1 that the boundary-value problem (5), (6) is a Fredholm one with index zero.

Definition 5. A solution to the boundary-value problem (5), (6) depends continuously on the parameter ε at $\varepsilon = 0$ if the following two conditions are satisfied:

- there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0, \varepsilon_1)$ and arbitrary chosen right-hand sides $f(\cdot; \varepsilon) \in ({W_p}^n)^m$ and $c(\varepsilon) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot; \varepsilon)$ that belongs to the space $({W_p}^{n+r})^m$;
- the convergence of the right-hand sides $f(\cdot; \varepsilon) \to f(\cdot; 0)$ in $({W_p}^n)^m$ and $c(\varepsilon) \to c(0)$ in $\mathbb{C}^{rm}$ as $\varepsilon \to 0+$ implies the convergence of the solutions $y(\cdot; \varepsilon) \to y(\cdot; 0)$ in $({W_p}^{n+r})^m$. ◂

Consider the following conditions as $\varepsilon \to 0+$:

(i) the limiting homogeneous boundary-value problem

$$L(0) y(t, 0) = 0, \quad t \in (a, b), \quad B(0) y(\cdot, 0) = 0$$

has only the trivial solution;

(ii) $A_{r-j}(\cdot;\varepsilon) \to A_{r-j}(\cdot;0)$ in the space $(W_p^n)^{m\times m}$ for each number $j \in \{1,\dots,r\}$;

(iii) $B(\varepsilon)y \to B(0)y$ in the space $\mathbb{C}^{rm}$ for every $y \in (W_p^{n+r})^m$.

Theorem 6. *A solution to the boundary-value problem (5), (6) depends continuously on the parameter ε at $\varepsilon = 0$ if and only if this problem satisfies condition (i) and the conditions (ii) and (iii).*

We supplement our result with a two-sided estimate of the error $\|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p}$ of the solution $y(\cdot;\varepsilon)$ via its discrepancy

$$\tilde{d}_{n,p}(\varepsilon) := \|L(\varepsilon)y(\cdot;0) - f(\cdot;\varepsilon)\|_{n,p} + \|B(\varepsilon)y(\cdot;0) - c(\varepsilon)\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\cdot;0)$ as an approximate solution to the problem (5), (6).

Theorem 7. *Suppose that the boundary-value problem (5), (6) satisfies conditions (i), (ii) and (iii). Then, there exist positive numbers $\varepsilon_2 < \varepsilon_1$ and γ_1, γ_2 such that, for any $\varepsilon \in (0,\varepsilon_2)$, the following two-sided estimate is true:*

$$\gamma_1\, \tilde{d}_{n,p}(\varepsilon) \le \|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p} \le \gamma_2\, \tilde{d}_{n,p}(\varepsilon),$$

where the quantities ε_2, γ_1, and γ_2 do not depend on $y(\cdot;\varepsilon)$ and $y(\cdot;0)$.

Thus, the error and discrepancy of the solution $y(\cdot;\varepsilon)$ to the boundary-value problem (5), (6) are of the same degree of smallness.

The results are published in the articles by Atlasiuk and Mikhailets [1, 2].

References

[1] Atlasiuk, O. M. and Mikhailets, V. A. "On the solvability of inhomogeneous boundary-value problems in sobolev spaces". In: *Dopovidi Natsionalnoi Akademi Nauk Ukraini. Matematika. Prirodoznavstvo. Tekhnichni Nauki* 11 (2019), pp. 3–7. ISSN: 1025-6415. https://doi.org/10.15407/dopovidi2020.06.003.

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