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What is Sparse Domination and why is it so plentiful?

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Abstract: Many operators in analysis are non-local, in the sense that a perturbation of the input near a point modifies the output everywhere; consider for example the operator that maps the initial data to the corresponding solution of the heat equation.

Sparse Domination consists in controlling such operators by a sum of positive, local averages. This allows to derive plenty of estimates, which are often optimal.

In this work we will introduce this concept, and we will discuss the case of operators that are beyond Calderón–Zygmund theory.

Resumen: Muchos operadores en análisis son no locales, en el sentido de que una perturbación de la entrada cerca de un punto modifica la salida en todas partes; consideremos, por ejemplo, el operador que mapea los datos iniciales a la solución correspondiente de la ecuación del calor.

La dominación dispersa consiste en controlar estos operadores mediante una suma de medias locales positivas. Esto permite derivar multitud de estimaciones que a menudo son óptimas.

En este trabajo introduciremos este concepto y discutiremos el caso de los operadores que están más allá de la teoría de Calderón–Zygmund.

Keywords: sparse domination, weighted estimates, $T(1)$ theorem, elliptic operators.

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1. Weighted estimates

In Harmonic Analysis, weighted inequalities allow us to better understand the action of operators on different domains, and they have many applications to PDEs [12], approximation theory, complex analysis and operator theory [1].

We call *weight* a positive, locally integrable function. You might be interested in understanding how your operator depends on the weight in the underlying measure. Given a (sub)linear operator T from L^p to itself, you can start asking the following questions:

(i) For which weights w is the operator bounded from $L^p(w)$ to $L^p(w)$?

(ii) Can we characterise all such weights?

(iii) How does the operator norm depend on the weight?

Since the '90s there have been a lot of efforts towards answering these questions and to quantify this dependence, also in relation to a problem in quasiconformal theory [2, 17].

A key step towards this goal was a representation of the action of the operator in terms of simpler dyadic operators. This representation was first obtained for the Hilbert transform [16], and later for general Calderón–Zygmund operators [13]. Today we know that a *domination*, rather than a representation, is often enough for deriving optimal weighted estimates with less effort. Such domination is popular as *sparse domination*.

Sparse domination is having a tremendous impact on Harmonic Analysis [4, 7–10]. It has simplified the proof of the A_2-conjecture [13], has found applications beyond the classical theory [4], in the discrete setting [10], and has resolved long-standing questions in operator theory [1].

1.1. Maximal operators

The questions posed above were first answered by Benjamin Muckenhoupt [15] for the maximal operator:

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over cubes Q with sides parallel to the coordinate axis.

If we could control an operator T by the maximal operator M, we could derive weighted estimates for T from the ones for M. Unfortunately, contrarily to maximal operators, singular integral operators are not bounded in L^∞. Thus we cannot hope to control them (pointwise) by a single maximal average.

Sparse domination consists in controlling non-local operators by a *sum* of positive averages. This allows to derive plenty of unweighted, weighted, and vector valued estimates (which are often optimal) from weighted L^p estimates for maximal operators, while the L^∞-norm is still allowed to blow up.

This domination can be performed by constructing a *sparse* family of cubes for a given input function. Roughly speaking, a sparse family is a collection of cubes having a subcollection of sets that are disjoint and not too small. More precisely, for a fixed $\tau \in (0, 1)$ we say that:

Definition 1. A collection of dyadic cubes $\mathcal{S}$ is τ-sparse if for any $Q \in \mathcal{S}$ there exists a subset $E_Q \subset Q$ such that $\{E_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint and the ratio $|E_Q|/|Q|$ is bounded below by τ. ◂

Given a function f and a cube Q_0, one can construct a sparse collection inside Q_0 by selecting maximal cubes covering the superlevel set:

$$F(Q_0) = \left\{ x \in Q_0 \, : \, Mf(x) > \lambda \frac{1}{|Q_0|} \int_{Q_0} |f| \right\}.$$

The weak boundedness of M ensures that we can choose $\lambda > 0$ so that the measure of the complement $E_{Q_0} := Q_0 \setminus F(Q_0)$ is not too small. By iterating this procedure, one obtains a collection of nested cubes organised in generations.

Then, our operator is controlled by the desired average on E_{Q_0}. At each iteration, the remaining area shrinks geometrically, leading to the pointwise domination for $x \in Q_0$

$$|Tf(x)| \le C \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \mathbb{1}_Q(x), \tag{1}$$

where the collection $\mathcal{S}$ is the union of all maximal cubes in each iteration, and it is *sparse* as in definition 1.

The same method can be used to bound bilinear expressions, leading to a domination by a *sparse bilinear form*:

$$\Big| \int_{Q_0} Tf \cdot g \, \mathrm{d}x \Big| \le C \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \left(\frac{1}{|Q|} \int_Q |g| \right) |Q|. \tag{2}$$

The sparse collections $\mathcal{S}$ in (1) and (2) depend on the input functions.

How does one recover bounds in terms of the maximal function? When we integrate sparse operators, the sparseness property allows to reduce the sum over $\mathcal{S}$ to a sum over disjoint sets. The averages are then controlled by the maximal averages. For example, for a $\frac{1}{2}$-sparse family $\mathcal{S}$ we have that $|Q| \le 2|E_Q|$, so

$$\sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) |Q| \le 2 \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) |E_Q| \le 2 \sum_{Q \in \mathcal{S}} \int_{E_Q} Mf \le 2 \int_{Q_0} Mf.$$

In a similar fashion, one recovers L^p estimates from L^p bounds of the maximal operator. One can then take the supremum over all possible $\frac{1}{2}$-sparse collections $\mathcal{S}$ obtaining the same weighted estimates.

2. Further applications

2.1. Sparse $T1$ theorems

In the ’80s, David and Journé [11] showed that L^2-boundedness of singular integral operators follows from the uniform boundedness on characteristic functions. This result is known as the “$T(1)$ theorem”, as the operator is tested on constant functions. The analogous condition for classical square functions is a Carleson measure condition [6].

These classical results have recently been recast to give a sparse domination [5, 14]. Thus, instead of just L^2 boundedness, these theorems imply all weighted L^p-bounds with optimal dependence on the weight.

2.2. Sparse domination of non-integral operators

Classical operators in Harmonic Analysis come with an integral representation and a kernel. On the other hand, many operators coming from elliptic PDEs are “non-integral”, in the sense that they do not possess such an explicit representation.

Recently, the sparse paradigm has been successfully applied also in this context [4], where the usual assumptions on the kernel are replaced by hypotheses on the action of the operator on the semigroup e^{-tL} generated by the elliptic operator L.

For non-integral square functions, optimal weighted estimates are deduced form a different sparse form [3], which reflects the quadratic nature of these operators. This quadratic sparse domination yields estimates for square functions associated with divergence forms and Laplace–Beltrami operators on Riemannian manifolds as particular examples.

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