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Factor of iid Schreier decoration of transitive graphs

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Abstract: A Schreier decoration is a combinatorial coding of an action of the free group F_d on the vertex set of a $2d$-regular graph. We investigate whether a Schreier decoration exists on various countably infinite transitive graphs as a factor of iid.

We show that the square lattice and also the three other Archimedean lattices of even degree and $\mathbb{Z}^d$, $d \geq 3$, have finitary factor of iid Schreier decorations, and exhibit examples of transitive graphs of arbitrary even degree in which obtaining such a decoration as a factor of iid is impossible.

We also prove that non-amenable quasi-transitive unimodular $2d$-regular graphs have a factor of iid balanced orientation, meaning each in- and outdegree is equal to d. This result involves extending earlier spectral-theoretic results on Bernoulli shifts to the Bernoulli graphings of quasi-transitive unimodular graphs. Balanced orientation is also obtained for symmetrical planar lattices.

Resumen: Una decoración de Schreier es una codificación combinatoria de una acción del grupo libre F_d en el conjunto de vértices de un grafo $2d$-regular. Investigamos si existe una decoración de Schreier en varios grafos transitivos numerables infinitos como un factor de iid.

Mostramos que el retículo cuadrado y también los otros tres grafos arquimedianos de grado par y $\mathbb{Z}^d$, $d \geq 3$, tienen decoraciones de Schreier de factor finito de iid, y mostramos ejemplos de grafos transitivos de grado par arbitrario en los que la obtención de tal decoración como factor de iid es imposible.

También demostramos que los grafos $2d$-regulares unimodulares cuasi transitivos no amenables tienen un factor de orientación equilibrada iid, lo que significa que cada grado de entrada y salida es igual a d. Este resultado implica la extensión de los resultados espectrales anteriores sobre los desplazamientos de Bernoulli a los grafos de Bernoulli de los grafos unimodulares cuasi-transitivos. También se obtiene la orientación equilibrada para retículos planos simétricos.

Keywords: transitive graph, factor of iid, Schreier graph, site percolation, Archimedean lattice, planar lattice, graphing, spectral gap.

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1. Introduction, notation, and basics

A *Schreier decoration* of a $2d$-regular graph G is a colouring of the edges with d colours together with an orientation such that, at every vertex, there is exactly one incoming and one outgoing edge of each colour. A partial result towards a Schreier decoration is a balanced orientation of the edges. An orientation of a graph with all degrees even is *balanced* if the indegree of any vertex is equal to its outdegree. We investigate whether an invariant random Schreier decoration or at least balanced orientation can be obtained on infinite transitive graphs as a factor of iid.

Informally, a Schreier decoration is a factor of iid if it is produced by a certain randomised local algorithm. To start with, each vertex independently gets a random label from $[0, 1]$. Then it makes a deterministic measurable decision about the decoration of its incident edges, based on the labelled graph that it sees from itself as a root. Adjacent vertices must make a consistent decision regarding the edge between them.

1.1. Schreier graphs, factors of iid and non-examples

Given a group $\Gamma = \langle S\rangle$ and an action $\Gamma \curvearrowright X$, the *Schreier graph* $\mathrm{Sch}(\Gamma \curvearrowright X, S)$ has X as its vertex set, and for every $x \in X$, $s \in S$, we introduce an oriented s-labelled edge from x to $s \cdot x$. A map $\Phi :\ X \to Y$ between two Γ-spaces is a Γ*-factor* if it is measurable and $\gamma \cdot \Phi(x) = \Phi(\gamma \cdot x)$ for every $\gamma \in \Gamma$, $x \in X$.

Definition 1. Let G be a graph and u denote the Lebesgue measure on $[0, 1]$. We endow the space $[0, 1]^{V(G)}$ with the product measure $\mathrm{u}^{V(G)}$. A factor of iid Schreier decoration (respectively, balanced orientation) of G is an $\mathrm{Aut}(G)$-factor $\Phi :\ \left([0, 1]^{V(G)}, \mathrm{u}^{V(G)}\right) \to \mathtt{Sch}(G)$ (respectively, to $\mathtt{BalOr}(G)$). ◂

For simple graphs G_1 and G_2, let the graph $G_1 \times G_2$ be defined by having $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ with vertices (u, v) and (u', v') being adjacent if $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$.

Proposition 2. *Let H be a finite $(2d - 2)$-regular graph with an odd number of vertices. The $2d$-regular graph $H \times P$, where P is the bi-infinite path, has no factor of iid balanced orientation.*

Being quasi-isometric to P is a necessary condition in our non-examples. It is, however, not sufficient.

Proposition 3. *Let H be a finite bipartite $(2d - 2)$-regular graph. Then, the $2d$-regular graph $H \times P$, where P is the bi-infinite path, has a factor of iid Schreier decoration.*

2. Infinite amenable graphs

To obtain Schreier decorations of the Archimedean lattices later in this section, we partition their vertex set V into finite clusters such that, for each cluster C, there is a unique cluster C^+ surrounding it.

Definition 4 (Hierarchy). Let G be a graph and $\mathbf{H}$ a partition of $V(G)$. We say that two distinct parts $C, D \in \mathbf{H}$ are adjacent if and only if there is $u \in C$ and $v \in D$ which are adjacent in G. Then, $\mathbf{H}$ is a *hierarchy on* G if the following hold for every $C \in \mathbf{H}$: 1) C is finite, 2) there is a *unique* $C^+ \in \mathbf{H}$ such that C and C^+ are adjacent and, for all $v \in V(G)$ but finitely many, any path from C to v contains a vertex from C^+, and 3) whenever $B \in \mathbf{H}$ is adjacent to C, either $B = C^+$ or $C = B^+$. ◂

A key feature of our hierarchies is that any two non-adjacent clusters are far from one another. But starting with any hierarchy, we can obtain one in which such clusters are as far from one another as we wish.

Proposition 5. *Let G be a graph and $k \in \mathbb{N}$. A hierarchy $\mathbf{H}$ on G is k*-spaced *if, for all non-adjacent $B, C \in \mathbf{H}$, the graph distance $d(B, C) = \min_{u\in B, v\in C} d_G(u, v)$ is at least k. Suppose there is a factor of iid hierarchy $\mathbf{H}$ on G. Then, for all $k \in \mathbb{N}$, there is a factor of iid k-spaced hierarchy $\mathbf{H}_k$ on G.*

Moreover, for all $c, k \in \mathbb{N}$, there is a factor of iid pair $(J_{c,k}, \eta :\ J_{c,k} \to [c])$ where $J_{c,k}$ is a k-spaced hierarchy and $\eta :\ J_{c,k} \to [c]$ is a colouring with c colours such that, for all $C \in J_{c,k}$, if C has colour i, then C^+ has colour $i + 1 \pmod c$.

For a general planar lattice Λ, we wish to use site percolation to obtain a hierarchy.

Theorem 6 ([1]). *Let Λ be a plane lattice with m-fold symmetry for some $m \geq 2$ and $\Lambda^{\times}$ be its matching lattice, i.e., the graph obtained from Λ by adding all diagonals to all faces of Λ. Then, for every $p \in [0,1]$, the probabilities $\theta^s_{\Lambda}(p)$, $\theta^s_{\Lambda^{\times}}(p)$ satisfy that $\theta^s_{\Lambda}(p) = 0$ or $\theta^s_{\Lambda^{\times}}(p) = 0$. Furthermore, $p^s_H(\Lambda) + p^s_H(\Lambda^{\times}) = 1$.*

So if Λ has m-fold symmetry, we can add a vertex to every non-triangular face and connect it to all the vertices of that face. The resulting lattice $\Lambda^{\bullet}$ also has m-fold symmetry and is self-matching, so Theorem 6 tells us that $p^s_H(\Lambda^{\bullet}) = \frac{1}{2}$ and percolation does not occur at criticality. This gives us a hierarchy on Λ as follows. Colour the vertices of Λ yellow or green uniformly at random. For each face, decide randomly whether either all its yellow or all its green vertices will be treated as if they were connected through the face. This results in a hierarchy, which together with Proposition 5 gives basis for the following.

Theorem 7. *Let Λ be a planar lattice with m-fold symmetry, $m \geq 2$, in which all degrees are even. There is a finitary factor of iid which is a balanced orientation of Λ almost surely.*

2.1. Schreier decorations of Archimedean lattices and $\mathbb{Z}^d$, $d \geq 3$, as factors of iid

Theorem 8. *Let Λ be $\mathbb{Z}^d$, $d \geq 3$, or any of the four Archimedean lattices with even degrees: the square lattice, the triangular lattice, the Kagomé lattice or the $(3,4,6,4)$ lattice. There is a finitary factor of iid which is a.s. a Schreier decoration of Λ. Moreover, it has almost surely no infinite monochromatic paths.*

Our approach is the same throughout. We break the lattices into a hierarchy of finite pieces. Then, for each piece independently, we choose an edge-d-colouring scheme such that we can ensure that every monochromatic connected subgraph is a finite cycle. Each cycle will orient itself strongly.

Both in the case of the triangular lattice and $\mathbb{Z}^d$, $d \geq 3$, once we have a spaced enough hierarchy, we reuse the patterns developed for $\Lambda_{\square}$. Unlike in the proofs for Archimedean lattices, we do not use percolation as our starting point for $\mathbb{Z}^d$, $d \geq 3$, but instead the results of Gao, Jackson, Krohne and Seward [3].

Corollary 9. *For every $d \geq 2$, there is a factor of iid which is a proper $2d$-colouring of the edges of $\mathbb{Z}^d$ a.s. Subsequently, there is a factor of iid which is a perfect matching on $\mathbb{Z}^d$ a.s.*

3. Balanced orientation of non-amenable quasi-transitive graphs

Theorem 10. *Every non-amenable quasi-transitive unimodular $2d$-regular graph G has a factor of iid orientation that is balanced almost surely.*

For example, the regular trees are unimodular. For $d > 1$, the $2d$-regular tree T_{2d} is also non-amenable, so it is covered by Theorem 10. For $d > 2$, Theorem 10 allows us to remark the following too.

Proposition 11. *If T_d has a factor of iid proper edge d-colouring, then T_{2d} has a factor of iid Schreier decoration.*

Despite this connection, it remains open whether there is a factor of iid Schreier decoration of T_{2d}.

To prove Theorem 10, we first reduce a balanced orientation of G to a perfect matching in an auxiliary bipartite graph G^*. Then, we extend earlier matching results on Cayley graphs to the case of unimodular quasi-transitive graphs. The key step for us, as it is for the earlier results [2, 4], is to use spectral theory to show stabilisation of an infinite algorithm.

Theorem 12. *Let G be a connected, unimodular, quasi-transitive graph. If G is non-amenable, then its Bernoulli graphing $\mathcal{G}$ has positive spectral gap.*

The interpretation of spectral gap differs depending on the Bernoulli graphing being bipartite or not. See Theorems 14 and 15 for exact statements.

Corollary 13. *Let G be a connected, unimodular, quasi-transitive non-amenable d-regular bipartite graph. Then, G has a factor of iid subset of the edges which is a perfect matching almost surely.*

3.1. Unimodular quasi-transitive graphs, Bernoulli graphings and spectral gap

Let G be a locally finite quasi-transitive graph, $\Gamma = \mathrm{Aut}(G)$. G is *unimodular* if $|\mathrm{Stab}_\Gamma(x) \cdot y| = |\mathrm{Stab}_\Gamma(y) \cdot x|$ for any $x, y \in V(G)$ that are in the same Γ orbit. Let $T = \{o_1, \dots, o_t\}$ be a set of representatives of the orbits of $\Gamma \curvearrowright V(G)$. We set $p(o_i) = \mu(o_i)^{-1}$ and scale such that $\sum_i p(o_i) = 1$.

The notion of unimodularity comes hand in hand with the Mass Transport Principle, which allows us to set up a Markov chain M_T mimicking the transitions of the random walk on G between Γ-orbits. Let its states be T and transition probabilities $p_{M_T}(o_i, o_j) = \frac{|\{(v,o_i)\in E | v \in \Gamma \cdot o_j\}|}{\deg(o_i)}$.

List the eigenvalues of M_T in decreasing order, $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t$. We say M_T is *bipartite* if $\lambda_t = -1$. When M_T is not bipartite, we set $\rho_T = \max(|\lambda_2|, |\lambda_t|)$. When it is, we set $\rho_T = \max(|\lambda_2|, |\lambda_{t-1}|)$.

Let Ω denote the space of $[0, 1]$-labelled rooted connected graphs. Elements of Ω are of the form (H, u, ω), where (H, u) is a bounded-degree rooted graph and $\omega : V(H) \to [0, 1]$. We connect (H, u, ω) with (H', u', ω') if and only if we can obtain (H', u', ω') from (H, u, ω) by moving the root to one of its neighbours. We denote the resulting edge set by $\mathcal{E}$. To define the probability measure on Ω, pick the rooted graph (G, o_i) with probability $p(o_i)$. Then pick a labelling $\omega \in [0, 1]^{V(G)}$ according to $\mathrm{u}^{V(G)}$. Let ν_G denote the distribution of (G, o_i, ω). Then, the *Bernoulli graphing* of G is $\mathcal{G} = (\Omega, \mathcal{E}, \nu_G)$.

The Markov operator $\mathcal{M}$ is a self-adjoint operator on $L^2(\Omega, \nu_{\mathrm{st}})$. Similarly, denote the Markov operator of G on $\ell^2(G, m_{\mathrm{st}})$ as M. Here ν_{st} and m_{st} denote the degree-biased versions of ν and of the counting measure on $V(G)$. The following two theorems deal with the non-bipartite and bipartite cases separately.

Theorem 14. *Let G be as in Theorem 12, and assume also that M_T is not bipartite. Let $\rho < 1$ denote the spectral radius of G on $\ell^2(G, m_{\mathrm{st}})$. Then, the spectral radius of $\mathcal{M}$ on $L^2_0(\Omega, \nu_{\mathrm{st}})$ is at most* $\max(\rho, \rho_T) < 1$.

Theorem 15. *Let G be as in Theorem 12, and assume that M_T is bipartite. Let $\rho < 1$ denote the spectral radius of G on $\ell^2(G, m_{\mathrm{st}})$. The Bernoulli graphing $\mathcal{G}$ is measurably bipartite, with bipartition $X_1 \cup X_2 = V(\mathcal{G})$. Let $L^2_{00}(\Omega, \nu_{\mathrm{st}})$ denote the orthogonal complement of the subspace generated by the functions $\mathbb{1}_X$ and $\mathbb{1}_{X_1} - \mathbb{1}_{X_2}$. Then, the spectral radius of $\mathcal{M}$ on $L^2_{00}(\Omega, \nu_{\mathrm{st}})$ is at most* $\max(\rho, \rho_T) < 1$.

3.2. Perfect matchings and balanced orientations

To prove Theorem 10, we relate balanced orientations of our $2d$-regular G to perfect matchings of a bipartite graph G^*. G^* is constructed from G by local transformations, which makes sure that it remains unimodular. In particular, the bipartite $G^* = (S, T, E^*)$ is obtained by setting $S = E(G)$ and $T = V(G) \times [d]$. Edges of G^* are defined by connecting $e \in S$ to $(v, i) \in T$ if and only if e is incident to v in G.

Lemma 16. *The graph G^* is quasi-isometric to G. If G is unimodular quasi-transitive, then so is G^*. Crucially, any perfect matching M in G^* defines a balanced orientation of G by orienting the edge $e \in S$ towards its endpoint v if and only if e and (v, i) are matched by M for some $i \in [d]$.*

Proof of Theorem 10. We construct G^*, which Lemma 16 tells us is bipartite, unimodular, quasi-transitive, and quasi-isometric to G. As amenability is a quasi-isometry invariant property, G^* is non-amenable. By Corollary 13, G^* has a perfect matching, which by Lemma 16 gives a balanced orientation of G. ∎

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