3rd *BYMAT Conference* (2020)

On the connection between Hardy kernels and reproducing kernels

Jesús Oliva-Maza
Universidad de Zaragoza
joliva@unizar.es

Abstract: Hardy kernels are known for being a useful tool to construct bounded operators on $L^p(\mathbb{R}^+)$ spaces, property which follows from Hardy's inequality. Even more, recently Hardy kernels have also been used to define bounded operators on Hardy spaces on the half plane $H^p_a(\mathbb{C}^+)$. In this work, the range spaces in $L^p(\mathbb{R}^+)$ and $H^p_a(\mathbb{C}^+)$ of such operators are analysed. We focus on the case $p = 2$, where under some circumstances, these range spaces arise as reproducing kernel Hilbert spaces. We show that in the $L^2(\mathbb{R}^+)$ case, the reproducing kernels of these spaces turn out to be Hardy kernels as well, whereas in the $H^2_a(\mathbb{C}^+)$ setting, their reproducing kernels are holomorphic extensions of Hardy kernels. We also present how the Laplace transform connects the real and complex settings of this family of range spaces.

Resumen: Los núcleos de Hardy son conocidos por ser una herramienta útil para construir operadores acotados en los espacios $L^p(\mathbb{R}^+)$, hecho que se sigue de la desigualdad de Hardy. Además, los núcleos de Hardy han sido recientemente utilizados para construir operadores acotados en los espacios de Hardy del semiplano $H^p_a(\mathbb{C}^+)$. En este trabajo, se analizan los espacios rango de dichos operadores en $L^p(\mathbb{R}^+)$ y $H^p_a(\mathbb{C}^+)$. En particular, nos centramos en el caso $p = 2$, en el que, bajo determinadas condiciones, estos espacios rango son de hecho espacios de Hilbert con núcleo reproductor. Demostramos que, en el caso de $L^2(\mathbb{R}^+)$, los núcleos reproductores de dichos espacios son a su vez núcleos de Hardy, y que en el caso de $H^2_a(\mathbb{C}^+)$, los núcleos reproductores vienen dados por extensiones holomorfas de núcleos de Hardy. Por último, mostramos cómo la transformada de Laplace conecta los escenarios real y complejo de esta familia de espacios rango.

Keywords: Hausdorff operators, reproducing kernel Hilbert spaces, Hardy kernels.

MSC2010: 46E22, 47G10.

Acknowledgements: The author thanks J. E. Galé for several ideas, comments and additional information, which have led to considerably improving the quality of this work. This research has been partially supported by Project MTM2016-77710, fondos FEDER, of MCYTS, Spain; and by BES-2017-081552, MINECO, Spain and FSE.

Reference: Oliva-Maza, Jesús. "On the connection between Hardy kernels and reproducing kernels". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 187-190. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p187.

1. Banach algebra of Hardy kernels

Set $\mathbb{R}^+ := (0,\infty)$ and $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \Re z > 0\}$.

Definition 1. Let $1 \le p < \infty$ and let $H\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ be a measurable map. H is said to be a Hardy kernel of index p if the following conditions hold.

(i) H is homogeneous of degree -1; that is, for all $\lambda > 0$, $H(\lambda r, \lambda s) = \lambda^{-1} H(r,s)$ for all $r, s > 0$.
(ii) $\int_0^\infty H(1,s) s^{-1/p}\,\mathrm{d}s < \infty$.

Let us denote by $\mathfrak{H}_p$ the set of Hardy kernels of index p. ◂

For $1 \le p < \infty$, let $L^p(\mathbb{R}^+)$ denote the classical Lebesgue space on the positive real line, and $H^p_a(\mathbb{C}^+)$ the Hardy space on the right complex half plane. Given a Hardy kernel of index p, one can construct bounded operators A_H and D_H on $L^p(\mathbb{R}^+)$ and $H^p(\mathbb{C}^+)$ respectively, which are given by

$$(A_H f)(r) := \int_0^\infty H(r,s) f(s)\,\mathrm{d}s, \quad \text{for a.e. } r > 0,\ f \in L^p(\mathbb{R}^+),$$
$$(D_H F)(z) := \int_0^\infty H(|z|,s) F(s\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}s, \quad z = |z|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}^+,\ F \in H^p(\mathbb{C}^+).$$

The boundedness of $A_H \in \mathcal{B}(L^p(\mathbb{R}^+))$ follows from Hardy's inequality [3, Theorem 319], and the boundedness of $D_H \in \mathcal{B}(H^p(\mathbb{C}^+))$ was shown in the recent work about Hausdorff operators [5]. In fact, these families of operators $(A_H)_{H\in\mathfrak{H}_p}$ and $(D_H)_{H\in\mathfrak{H}_p}$ may be labelled as Hardy-Hausdorff operators since they are a particular case of Hausdorff operators, see the survey article [6] for more details.

It is part of folklore that the family of operators given by $(A_H)_{H\in\mathfrak{H}_p}$ can be described as convolution operators by identifying a Hardy kernel H with a Lebesgue integrable function $g_H \in L^1(\mathbb{R})$, see for example the paper about the spectra of A_H [1]. More precisely, let H be a Hardy kernel of index p, and set $g_H(t) := H(1,\mathrm{e}^{-t})\mathrm{e}^{-t/p'}$ for all $t \in \mathbb{R}$, where p' is such that $1/p + 1/p' = 1$. It is readily seen that $g_H \in L^1(\mathbb{R})$, with $\|g_H\|_1 = \int_0^\infty |H(1,s)| s^{-1/p}\,\mathrm{d}s$. Moreover, if one takes certain equivalence classes on $\mathfrak{H}_p$, it is straightforward to obtain that the mapping $H \mapsto g_H$ is a bijection from $\mathfrak{H}_p$ onto $L^1(\mathbb{R})$, see the forthcoming paper [8] for more details. Therefore, one obtains that this set of equivalence classes of Hardy kernels of index p entails a commutative Banach algebra structure, isomorphic to $L^1(\mathbb{R})$, whose norm and product are given, respectively, by

$$\|H\|_{\mathfrak{H}_p} := \int_0^\infty |H(1,s)| s^{-1/p}\,\mathrm{d}s, \quad H \in \mathfrak{H}_p,$$
$$(H \bullet G)(r,s) := \int_0^\infty H(r,t) G(t,s)\,\mathrm{d}t, \quad r,s > 0,\ H, G \in \mathfrak{H}_p.$$

Notice that the multiplication $\bullet$ resembles typical formulas about the construction of reproducing kernel Hilbert spaces, see for example the expression [9, (2.1)].

For the purposes of this work, two subsets of $\mathfrak{H}_p$ must be pointed out. First, set $\mathcal{I}_p := \{H \in \mathfrak{H}_p \mid g_H \in L^{p'}(\mathbb{R})\}$, which is a dense ideal of $\mathfrak{H}_p$. Second, let $\mathfrak{H}_p^{Hol}$ denote the subspace of $\mathfrak{H}_p$ of Hardy kernels H of index p that admit a (unique) extension H^{Hol} from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{C}^+ \times \mathbb{C}^+$ such that $H^{Hol}(z,w)$ is holomorphic in z and anti-holomorphic in w. The following will also be needed.

Definition 2. Let $1 < p < \infty$, and $H \in \mathfrak{H}_p$. Set $H^t(r,s) := H(s,r)$ and $H^*(r,s) := \overline{H(s,r)}$ for all $r,s > 0$, where $\overline{z}$ denotes the conjugate of a complex number z. ◂

It is readily seen that both H^t, H^* belong to $\mathfrak{H}_{p'}$.

2. Range spaces of Hardy-Hausdorff operators

In this section, we proceed to study the range spaces of Hardy-Hausdorff operators on $L^2(\mathbb{R}^+)$ and $H^2_a(\mathbb{C}^+)$ as reproducing kernel Hilbert spaces, that is, Hilbert spaces of functions for which point evaluations define continuous functionals. This is partly motivated by the forthcoming work [2], where range spaces of fractional Cesàro operators are analysed. Let us recall that if Ω is a reproducing kernel Hilbert space (RKHS from now on) of complex functions with domain X, its reproducing kernel K is given by

$$K(x,y) = k_y(x), \quad x, y \in X,$$

where $k_y \in \Omega$ is such that $f(y) = \langle f | k_y \rangle$ for all $f \in \Omega$. One of the main interesting properties of reproducing kernels is that one can recover the whole Hilbert space Ω from its reproducing kernel K, see for example Chapter I in [7].

2.1. Hardy kernels as reproducing kernels on $\mathbb{R}^+ \times \mathbb{R}^+$

First, we shall study the $L^p(\mathbb{R}^+)$ scenario. Recall that $L^p(\mathbb{R}^+)$ denotes the Banach space of functions f defined a.e. on $\mathbb{R}^+$, such that $\|f\|_{L^p} := \left(\int_0^\infty |f(r)|^p \, \mathrm{d}r\right)^{1/p} < \infty$.

Definition 3. Let $1 \leq p < \infty$, and $H \in \mathfrak{H}_p$. Set the range space $\mathcal{A}(H) := A_H(L^p(\mathbb{R}^+))$, and endow it with the canonical Banach space structure $\mathcal{A}(H) \cong L^p(\mathbb{R}^+)/\ker A_H$. ◂

Notice that since $\mathcal{A}(H) \subset L^p(\mathbb{R}^+)$, one cannot guarantee that point evaluations are well defined on $\mathcal{A}(H)$. Let $C(\mathbb{R}^+)$ denote the set of complex continuous functions on $\mathbb{R}^+$.

Lemma 4. *Let $1 \leq p < \infty$ and let $H \in \mathcal{I}_p \subset \mathfrak{H}_p$. One has that $\mathcal{A}(H) \subset C(\mathbb{R}^+)$, in the sense that, if $f \in \mathcal{A}(H)$, then there is a (unique) continuous function $g \in C(\mathbb{R}^+)$ such that $f = g$ a.e.*

Therefore, if $H \in \mathcal{I}_p$ and $r > 0$, one can define point evaluations on $\mathcal{A}(H)$ by $f(r) := g(r)$, where $f \in \mathcal{A}(H)$ and $g \in C(\mathbb{R}^+)$ are as in the lemma above. The proposition below shows that these are all the Hardy kernels for which one can define continuous point evaluations on $\mathcal{A}(H)$.

Proposition 5. *Let $1 \leq p < \infty$, and $H \in \mathfrak{H}_p$. Then, one can define continuous point evaluations on $\mathcal{A}(H)$ if and only if $H \in \mathcal{I}_p$. If this is the case, it follows that for all $f \in \mathcal{A}(H)$*

$$|f(r)| \leq r^{-1/p} \|g_H\|_{L^{p'}} \|f\|_{\mathcal{A}(H)}, \qquad r > 0.$$

Next we give the reproducing kernel of this family of range spaces with continuous point evaluations.

Theorem 6. *Let $H \in \mathfrak{H}_2$. Then, $\mathcal{A}(H)$ is a RKHS if and only if $H \in \mathcal{I}_2$, and in this case its reproducing kernel K_H is separately continuous and given by*

$$K_H(r,s) = \int_0^\infty H(r,t)\overline{H(s,t)} \, \mathrm{d}t, \quad \text{for } r, s > 0.$$

It follows that K_H defines a Hardy kernel, satisfying $K_H = H \bullet H^*$.

2.2. Hardy kernels as reproducing kernels on $\mathbb{C}^+ \times \mathbb{C}^+$

Now we focus on the Hardy spaces of the half plane $H_a^p(\mathbb{C}^+)$, which are formed by all holomorphic functions F on $\mathbb{C}^+$ such that $\|F\|_{H^p} := \sup_{x>0} \left(\int_{-\infty}^\infty |f(x+\mathrm{i}y)|^p \, \mathrm{d}y\right)^{1/p} < \infty$. It is well known that these spaces present continuous point evaluations, so in particular $H_a^2(\mathbb{C}^+)$ is a RKHS whose reproducing kernel $\mathcal{K}$ is given by $\mathcal{K}(z,w) = (z + \overline{w})^{-1}$ for all $z, w \in \mathbb{C}^+$, see for example Proposition 1.8 in the notes [4]. Notice that, if one restricts $\mathcal{K}$ to $\mathbb{R}^+ \times \mathbb{R}^+$, one obtains the Stieltjes kernel $\mathcal{S}$, which is a Hardy kernel of index 2 given by $\mathcal{S}(r,s) = (r+s)^{-1}$ for all $r, s > 0$.

Definition 7. Let $H \in \mathfrak{H}_p$. Set $\mathcal{D}(H) := D_H(H_a^p(\mathbb{C}^+)) \subset H_a^p(\mathbb{C}^+)$ and endow $\mathcal{D}(H)$ with the canonical structure of a Banach space by $\mathcal{D}(H) \cong H_a^p(\mathbb{C}^+)/\ker D_H$. ◂

It is readily seen that point evaluations are continuous functionals on $\mathcal{D}(H)$ for all $H \in \mathfrak{H}_p$. The following theorem gives the reproducing kernel $\mathcal{K}_H$ of $\mathcal{D}(H)$, where $H \in \mathfrak{H}_2$, and which is given by the holomorphic extension of a Hardy kernel.

Theorem 8. *Let $H \in \mathfrak{H}_2$. One has that $H \bullet \mathcal{S} \bullet H^* \in \mathfrak{H}_2^{Hol}$, and that $\mathcal{D}(H)$ is a RKHS continuously embedded into $H_a^2(\mathbb{C}^+)$ whose reproducing kernel $\mathcal{K}_H$ is given by*

$$\mathcal{K}_H = (H \bullet \mathcal{S} \bullet H^*)^{Hol}.$$

2.3. A Paley-Wiener result

Next, we analyse the connection between the real and complex settings. First of all, recall that the classical Paley-Wiener theorem states that the Laplace transform $\mathcal{L}$, given by $(\mathcal{L}f)(z) := \int_0^\infty e^{-rz} f(r)$ for all $z \in \mathbb{C}^+$, defines an isometric isomorphism from $L^2(\mathbb{R}^+)$ onto $H_a^2(\mathbb{C}^+)$. The results below show how the Laplace transform connects the range spaces presented in subsections above.

Proposition 9. *Let $H \in \mathfrak{H}_2$. It follows that $\mathcal{L}A_H = D_{H^t}\mathcal{L}$.*

Theorem 10. *Let $H \in \mathfrak{H}_2$. The Laplace transform $\mathcal{L}$ restricted to $\mathcal{A}(H)$ is an isometric isomorphism onto $\mathcal{D}(H^t)$, $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H^t)$.*

Corollary 11. *Let $H \in \mathfrak{H}_2$. Either if H is symmetric, that is, $H = H^t$, or if H is real-valued, one obtains that $\mathcal{D}(H) = \mathcal{D}(H^t)$ as RKH spaces. Thus, the Laplace tranform $\mathcal{L}$ restricts to an isometric isomorphism from $\mathcal{A}(H)$ onto $\mathcal{D}(H)$, $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H)$.*

One may ask whether there exists an isometric isomorphism from $\mathcal{A}(H)$ onto $\mathcal{D}(H)$ for a general $H \in \mathfrak{H}_2$. This question is answered in the forthcoming work [8], where two mappings $\mathcal{P}, \mathcal{S} : L^2(\mathbb{R}^+) \to H_a^2(\mathbb{C}^+)$ are given such that they define isometric isomorphisms from $\mathcal{A}(H)$ onto $\mathcal{D}(H)$ for all $H \in \mathfrak{H}_2$.

As a final note, we refer the reader again to the upcoming work [8], where the proofs of all the results presented here, as well as a bunch of new results about this topic, are given in detail.

References

[1] Fabes, E. B.; Jodeit Jr, M., and Lewis, J. E. "On the spectra of a Hardy kernel". In: *J. Funct. Anal.* 21.2 (1976), pp. 187–194.

[2] Galé, J. E.; Miana, P. J., and Sánchez-Lajusticia, L. "RKH spaces of Brownian type defined by Cesàro-Hardy operators". Submitted. 2021.

[3] Hardy, G. H.; Littlewood, J. E., and Pólya, G. *Inequalities.* London/New York: Cambridge U. P., 1934.

[4] Hilgert, J. "Reproducing kernels in representation theory". In: *Symmetries in Complex Analysis, Contemporary Math* 468 (2005), pp. 1–98.

[5] Hung, H. D.; Ky, L. D., and Quang, T. T. "Hausdorff operators on holomorphic Hardy spaces and applications". In: *Proc. Roy. Soc. Edinburgh Sect. A* 150.3 (2020), pp. 1095–1112.

[6] Liflyand, E. R. "Hausdorff operators on Hardy spaces". In: *Eurasian Math. J.* 4.4 (2013), pp. 101–141.

[7] Neeb, K. H. *Holomorphy and convexity in Lie theory.* Vol. 28. Berlin: Walter de Gruyter, 2011.

[8] Oliva-Maza, J. "Range spaces of Hardy-Hausdorff operators as reproducing kernel Hilbert spaces (tentative title)". Preprint. 2021.

[9] Saitoh, S. et al. "Integral transforms in Hilbert spaces". In: *Proc. Japan Acad.* 58.8 (1982), pp. 361–364.