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Crossing limit cycles for piecewise linear differential centers separated by a reducible cubic curve

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Abstract: As for the general planar differential systems, one of the main problems for the piecewise linear differential systems is to determine the existence and the maximum number of crossing limits cycles that these systems can exhibit. But in general to provide a sharp upper bound on the number of crossing limit cycles is a very difficult problem. In this work we study the existence of crossing limit cycles and their distribution for piecewise linear differential systems formed by linear differential centers and separated by a reducible cubic curve, formed either by a circle and a straight line, or by a parabola and a straight line.

Resumen: Así como para los sistemas diferenciales planares, uno de los principales problemas para los sistemas diferenciales planares lineares por partes es determinar la existencia y el número máximo de ciclos límite de cruce que estos sistemas pueden tener. Pero, en general, proporcionar una cota superior ajustada para ese número máximo de ciclos límite es un problema difícil. En este trabajo estudiamos la existencia de ciclos límite y su distribución para sistemas diferenciales lineales por partes formados por centros diferenciales lineales y separados por una curva cúbica reducible, formada o por un círculo y una línea recta o por una parábola y una línea recta.

Keywords: discontinuous piecewise linear differential centers, limit cycles, conics.

MSC2010: 34C05, 34C07, 37G15.

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1. Introduction and main statements

One of the main problems for the piecewise linear differential systems (pwls) is to determine the existence and the maximum number of limits cycles that these systems can exhibit. That is the version of Hilbert's 16th problem for pwls. In the plane the class of pwls separated by a straight line is apparently the simplest class to study, and has been studied in several papers, see [1, 2] and references quoted therein, but it is still an open problem to know if three is the maximum number of crossing limit cycles that this class can have. In particular when the class of pwls separated by a straight line is formed by linear differential centers we know that these systems have no crossing limit cycles (clc), see [4]. However, there are more recent works which study planar discontinuous piecewise linear differential centers (pwlc) where the curve of discontinuity is not a straight line, see [5], there it was proved that there are clc in those systems. Moreover in the paper [3] it was provided the maximum number of clc for pwlc separated by any conic. In this work we study the existence of crossing limit cycles for piecewise linear differential systems formed by linear differential centers and separated by a reducible cubic curve, formed by a parabola and a straight line. Namely

$$\Sigma_k = \big\{(x, y) \in \mathbb{R}^2 : (y-k)(y-x^2) = 0,\ k \in \mathbb{R}\big\}.$$

Let $\mathcal{F}_{\Sigma_{k-}}$ be the family of pwlc separated by Σ_k with $k < 0$. In this case, we have the following three regions in the plane:

$$R^1_{\Sigma_{k-}} = \{y > x^2\}, \quad R^2_{\Sigma_{k-}} = \{k < y < x^2\}, \quad R^3_{\Sigma_{k-}} = \{y < x^2, y < k\}.$$

Let $\mathcal{F}_{\Sigma_0}$ be the family of pwlc separated by Σ_k with $k = 0$. Here we have the four regions

$$R^1_{\Sigma_0} = R^1_{\Sigma_{k-}}, \quad R^2_{\Sigma_0} = \{0 < y < x^2,\ x < 0\}, \quad R^3_{\Sigma_0} = R^3_{\Sigma_{k-}}, \quad R^4_{\Sigma_0} = \{0 < y < x^2, x > 0\}.$$

Here we have two types of clc, first clc of type 4 formed by parts of orbits of the four regions, see Figure 1b. Second clc of type 5, see Figure 1c, which intersect $R^1_{\Sigma_0}, R^3_{\Sigma_0}$ and $R^4_{\Sigma_0}$. Let $\mathcal{F}_{\Sigma_{k+}}$ be the family of pwlc separated by Σ_k with $k > 0$, here we have the five regions

$$R^1_{\Sigma_{k+}} = \{k < y < x^2, x > \sqrt{k}\}, \qquad R^2_{\Sigma_{k+}} = \{y > x^2, y > k\},$$
$$R^3_{\Sigma_{k+}} = \{k < y < x^2, x < -\sqrt{k}\}, \qquad R^4_{\Sigma_{k+}} = R^3_{\Sigma_{k-}}, \qquad R^5_{\Sigma_{k+}} = \{x^2 < y < k\}.$$

Here we have six types of clc. First we have clc such that are formed by parts of orbits of $R^1_{\Sigma_{k+}}, R^2_{\Sigma_{k+}}, R^5_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$, or clc formed by parts of $R^2_{\Sigma_{k+}}, R^3_{\Sigma_{k+}}, R^4_{\Sigma_{k+}}$ and $R^5_{\Sigma_{k+}}$, namely clc of type 6^+ and clc of type 6^-, respectively, see Figure 1d. Second we have clc of type 7, see Figure 1e, which intersect $R^2_{\Sigma_{k+}}$, $R^5_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$. Third we have the clc of type 8, see Figure 1f, which intersect $R^1_{\Sigma_{k+}}, R^2_{\Sigma_{k+}}, R^3_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$. And finally we have the clc such that intersect $R^1_{\Sigma_{k+}}, R^2_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$, or clc formed by parts of orbits of $R^2_{\Sigma_{k+}}, R^3_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$, namely clc of type 9^+ and clc of type 9^-, respectively, see Figure 1g. In what follows we exhibit the main results and their respective configurations.

Theorem 1. *The following statements hold.*

- (i) *There are pwlc in $\mathcal{F}_{\Sigma_{k-}}$ that have 4 clc that intersect Σ_k, see Figure 1a.*
- (ii) *There are pwlc in $\mathcal{F}_{\Sigma_0}$ that have 4 clc of type 4, see Figure 1b.*
- (iii) *There are pwlc in $\mathcal{F}_{\Sigma_0}$ that have 3 clc of type 5, see Figure 1c.*
- (iv) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 5 clc of type 6^+, see Figure 1d.*
- (v) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 3 clc of type 7, see Figure 1e.*
- (vi) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 4 clc of type 8, see Figure 1f.*
- (vii) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 3 clc of type 9^+, see Figure 1g.*

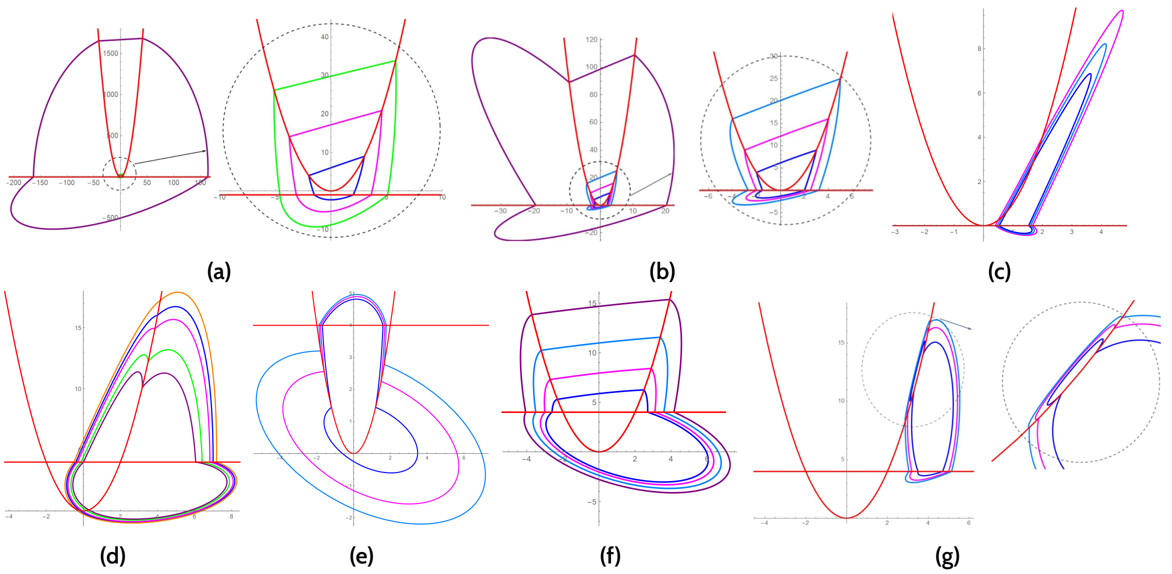

Figure 1: (a) 4 clc with four points on Σ_k. (b) 4 clc of type 4. (c) 3 clc of type 5. (d) 5 clc of type 6^+. (e) 3 clc of type 7. (f) 4 clc of type 8. (g) 3 clc of type 9^+. These limit cycles are traveled in counterclockwise.

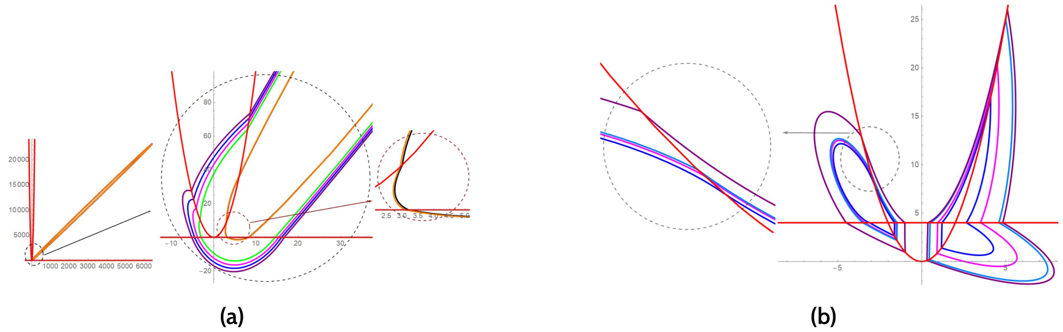

Figure 2: (a) 4 clc of type 4 and 2 clc of type 5. (b) 4 clc of type 6^+ and 4 clc of type 6^-. These limit cycles are traveled in counterclockwise.

In the following theorem, we study the pwlc in the families $\mathcal{F}_{\tilde{\Sigma}_k}$, $k \in \mathbb{R}$, with two and three clc of different types, simultaneously.

Theorem 2. *The following statements hold.*

(i) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_0}$ that have simultaneously* 4 *clc of type* 4 *and* 2 *clc of type* 5*, see Figure* 2a.

(ii) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 4 *clc of type* 6^+ *and* 4 *clc of type* 6^-*, see Figure* 2b.

(iii) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 4 *clc of type* 6^+ *and* 2 *clc of type* 7*, see Figure* 3a.

(iv) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 3 *clc of type* 6^+ *and* 4 *clc of type* 8*, see Figure* 3b.

(v) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 4 *clc of type* 6^+ *and* 2 *clc of type* 9^+*, see Figure* 3c.

(vi) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 3 *clc of type* 7 *and* 4 *clc of type* 8*, see Figure* 3d.

(vii) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 4 *clc of type* 8 *and* 2 *clc of type* 9^+*, see Figure* 3e.

(viii) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 2 *clc of type* 6^+*,* 2 *clc of type* 7 *and* 4 *clc of type* 8*, see Figure* 3*f*.

(ix) *There are pwlc in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously* 4 *clc of type* 6^+*,* 3 *clc of type* 8 *and* 2 *clc of type* 9^+*, see Figure* 3g.

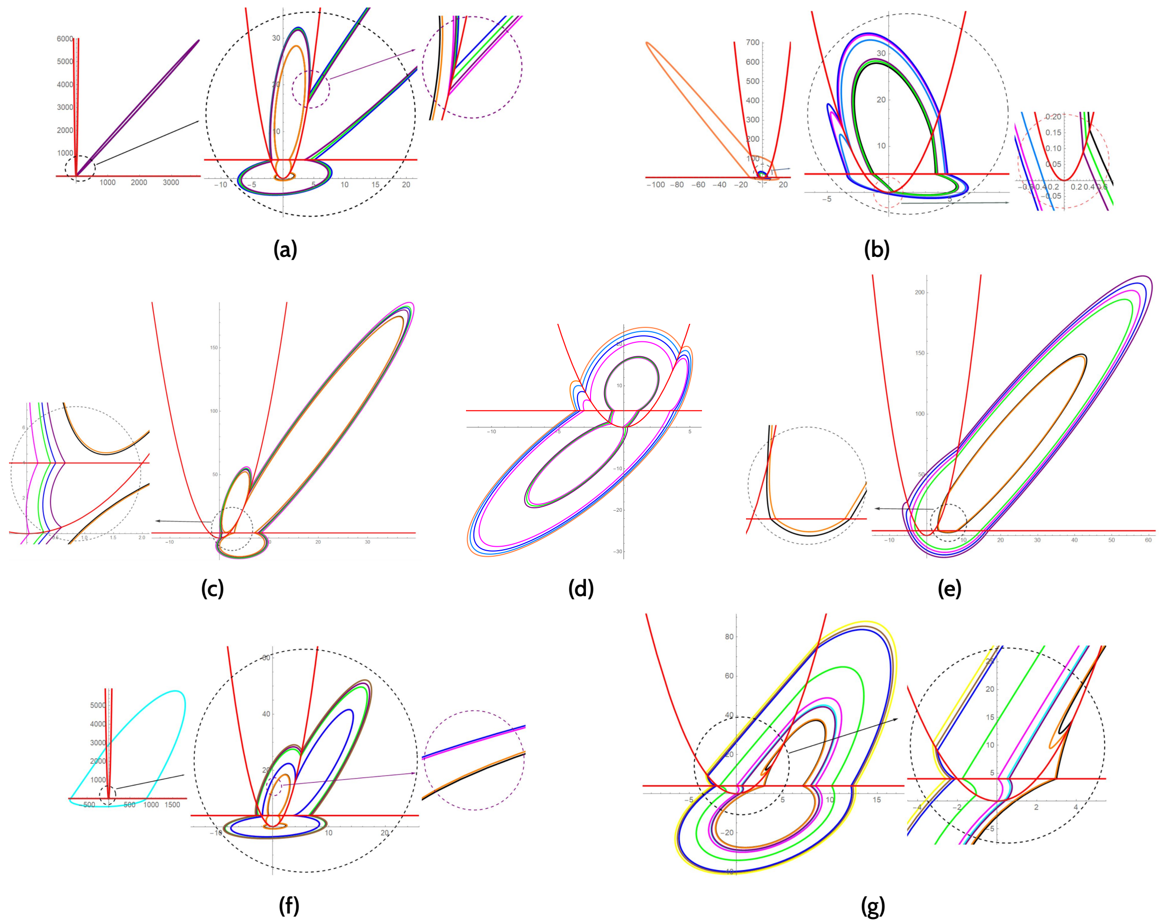

Figure 3: (a) 4 clc of type 6^+ and 2 clc of type 7. (b) 3 clc of type 6^+ and 4 clc of type 8. (c) 4 clc of type 6^+ and 2 clc of type 9^+. (d) 3 clc of type 7 and 4 clc of type 8. (e) 4 clc of type 8 and 2 clc of type 9^+. (f) 2 clc of type 6^+, 2 clc of type 7 and 4 clc of type 8. (g) 4 clc of type 6^+, 3 clc of type 8 and 2 clc of type 9^+. These limit cycles are traveled in counterclockwise.

References

[1] Freire, E.; Ponce, E., and Torres, F. “A general mechanism to generate three limit cycles in planar Fillipov systems with two zones”. In: *Nonlinear Dynam.* 78 (2014), pp. 251–263.

[2] Huan, S. M. and Yang, X. S. “On the number of limit cycles in general planar piecewise linear systems”. In: *Disc. Cont. Dyn. Syst.* 32.6 (2012), pp. 2147–2164.

[3] Jimenez, J. J.; Llibre, J., and Medrado, J. C. “Crossing limit cycles for a class of piecewise linear differential centers separated by a conic”. In: *Electron. J. Differential Equations* 41 (2020), 36pp.

[4] Llibre, J.; Novaes, D. D., and Teixeira, M. A. “Maximum number of limit cycles for certain piecewise linear dynamical systems”. In: *Nonlinear Dyn.* 82 (2015), pp. 1159–1175.

[5] Llibre, J. and Teixeira, M. A. “Limit cycles in Filippov systems having a circle as switching manifold”. In: *Nonlinear Dyn.* 82 (2018), pp. 1159–1175.