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# Poisson brackets on the space of solutions of first order Hamiltonian field theories

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**Abstract:** We investigate the existence of a symplectic and, consequently, a Poisson structure on the space of solutions of a first order field theory. We provide an affirmative answer for theories where all constraints can be solved. The analysis for gauge theories is postponed to a more extensive work.

**Resumen:** Se investiga la existencia de una estructura simpléctica y, en consecuencia, de Poisson en el espacio de soluciones de una teoría de campos de primer orden. Se da una respuesta afirmativa para las teorías en las que se pueden resolver todas las restricciones. El análisis para las teorías gauge se pospone a un trabajo más extenso.

**Keywords:** multisymplectic geometry, Poisson geometry, global analysis.

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# Introduction

We aim to analyse geometrical structures needed to give a description of a classical field theory allowing for a formulation of its quantum counterpart being compatible with special relativity. Now, we motivate how, with this problem in mind, it could be of interest to search for a Poisson structure on the space of solutions of the equations of the motion of the classical field theory.

Given a classical dynamical system, a formulation in terms of Poisson geometry is helpful to give a description of its quantum counterpart, if existing. If a Poisson structure exists on the *phase space*[1], say $\mathcal{M}$, of the dynamical system, then a Poisson bracket on the space of smooth functions on $\mathcal{M}$ is defined. Real valued smooth functions on $\mathcal{M}$ are usually interpreted as the observables of the theory. The so called *Dirac's analogy principle* states that in order for the predictions of the classical and the quantum theory to coincide within the energy scale where they are experimentally indistinguishable, then the Poisson structure on the space of classical observables must "come from" a Lie algebra structure on the space of quantum observables, that are modelled as self-adjoint operators on a Hilbert space. This motivates the search of a Poisson description of the classical theory.

However, the very concept of phase space is not compatible with special relativity. Indeed, often, the phase space is the space of configurations and momenta of the dynamical system for a fixed value of the *time* and the definition of a concept of time requires the introduction of a reference frame which splits the relativistic space-time into *space* and *time*. But, after a particular reference frame is introduced, the covariance of the dynamical system[2] under the Lorentz group can not be manifest. Following an idea of Souriau [7], the space of solutions of the equations of the motion seems to be more suitable for a relativistic description. Indeed, the *relativity group* is a group whose action preserves the equations of the motion and, consequently, maps solutions into solutions. Thus, differently from phase space, the space of solutions is actually covariant with respect to the action of the relativity group. With all this in mind, in order to give a description of the quantum counterpart of a classical field theory which is compatible with special relativity, it is of interest to investigate whether and how a Poisson structure can be given on the space of solutions of the classical theory.

This is what we do in this paper, giving an affirmative answer in the case of field theories where all the constraints can be solved and postponing the analysis of gauge theories to a more extensive work.

# 1. Multisymplectic formulation of field theories

We refer to [5, 6] for basic notions, notations and conventions about differential geometry and jet bundles. We adopt the so called multisymplectic formulation of field theories [4]. In this formulation the fields of the theory are modelled as sections of a fibre bundle $(E, \pi, \mathcal{M})$ whose base space $\mathcal{M}$ is a space-time with boundary $\partial\mathcal{M}$. A chart on $\mathcal{M}$ will be denoted by $(U_{\mathcal{M}}, \psi_{\mathcal{M}})$, $\psi_{\mathcal{M}}(m) = (x^{\mu})_{\mu=0,\dots,d}$, with $d+1$ being the dimension of the space-time and $m \in \mathcal{M}$. An adapted fibered chart on $E$ will be denoted by $(U_E, \psi_E)$, $\psi_E(e) = (x^{\mu}, u^a)_{\mu=0,\dots,d;\, a=1,\dots,n}$, with $n$ being the dimension of the fibres of $E$ and $e \in E$. Sections of $\pi$ are the fields of the theory, and we denote them by $\phi^a$. The analogue of the phase space of mechanics is the so called *covariant phase space* which is the affine dual of the first order jet bundle of $\pi$ [4]. It is again a fibre bundle over $\mathcal{M}$, denoted by $(\mathcal{P}(E), \tau_1, \mathcal{M})$, where an adapted fibered chart will be denoted by $(U_{\mathcal{P}}, \psi_{\mathcal{P}})$, $\psi_{\mathcal{P}}(p) = (x^{\mu}, u^a, \rho_a^{\mu})_{\mu=0,\dots,d;\, a=1,\dots,n}$, with $p \in \mathcal{P}(E)$. Sections of $\tau_1$ will be denoted by $\chi = (\phi^a, P_a^{\mu})$, where $P_a^{\mu}$ are the momenta fields conjugate with the fields $\phi^a$. We take actually a subset of suitably regular section admitting a Banach manifold structure, we refer to them as *dynamical fields* of the theory and we denote them as $\mathcal{F}_{\mathcal{P}}$. The particular field theory under investigation is specified by selecting an Hamiltonian function, namely, a real valued function on $\mathcal{P}(E)$, say $H(x, u, \rho)$. As it is explained in [4], when an Hamiltonian is fixed, the covariant phase space has a canonical $(d+1)$-form denoted by

$$\Theta_H = \rho_a^{\mu}\, \mathrm{d}u^a \wedge i_{\mu} vol_{\mathcal{M}} - H\, \nu_{\mathcal{M}},$$

[1] It is a space where each possible configuration of the dynamical system is represented by a point.
[2] We mean the invariance of the equations of the motion as well as the invariance of physical meaningful quantities.

where $i_\mu$ denotes the contraction with the vector field $\frac{\partial}{\partial x^\mu}$ and $\nu_{\mathcal{M}}$ is the volume form on $\mathcal{M}$.

The dynamical content of the theory is encoded in the Schwinger-Weiss variational principle. Trajectories are defined to be the critical points of the following action functional

$$\mathcal{S}[\chi] = \int_{\mathcal{M}} \chi^\star \Theta_H.$$

The critical points of $\mathcal{S}$ are those dynamical fields for which the variation of the action along any direction only depends on boundary terms. Let us clarify what we mean for "variation", "direction" and "boundary term". The space $\mathcal{F}_{\mathcal{P}}$ is a space of sections. A tangent vector at some "point" $\chi$ is defined [4] as a map $m \mapsto X^{(\chi)} \in \mathbf{V}_{\chi(m)}\mathcal{P}(E)$ for all $m \in \mathcal{M}$, namely, as a section of the pull-back bundle of $\mathbf{V}\mathcal{P}(E)$ via $\chi$. Intuitively it is a collection of $\tau_1$-vertical[3] tangent vectors at $\mathcal{P}(E)$ along the map $\chi$. Let us denote as $X$ an extension of $X^{(\chi)}$ to a ($\tau_1$-vertical) vector field on $\mathcal{P}(E)$ defined on a neighborhood of the image of $\chi$. Denote by $F_s^X$ the local flow of $X$. Then, $\chi_s := F_s^X \circ \chi$ is a one-parameter family of sections of $\tau_1$. The *variation* of $\mathcal{S}$ along the *direction* $X^{(\chi)}$ is defined to be

$$\delta_{X^{(\chi)}}\mathcal{S}[\chi] = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \int_{\mathcal{M}} \chi_s^\star \Theta_H = \int_{\mathcal{M}} \chi^\star \left(i_X \, \mathrm{d}\Theta_H\right) + \int_{\partial\mathcal{M}} \chi_{\partial\mathcal{M}}^\star \left(i_X \Theta_H\right), \tag{1}$$

where $\chi_{\partial\mathcal{M}} = \chi\big|_{\partial\mathcal{M}}$ is the dynamical field $\chi$ restricted to $\partial\mathcal{M}$[4]. The first term on the right hand side (r.h.s.) of (1) can be interpreted as the contraction of a differential one-form over $\mathcal{F}_{\mathcal{P}}$, that we denote by $\mathbb{EL}$ and call *Euler-Lagrange form*, with the tangent vector $X^{(\chi)}$. The second term in (1) is a *boundary term* in the sense that it only depends on the restriction of the dynamical fields to the boundary. We are going to denote the space of restrictions of dynamical fields to the boundary by $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$. Therefore, the second term on the r.h.s. can be interpreted as the pull-back of a differential form on $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$ via the restriction map $\Pi_{\partial\mathcal{M}} \colon \mathcal{F}_{\mathcal{P}} \to \mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$. We denote such a differential form by $\Pi_{\partial\mathcal{M}}^\star \alpha^{\partial\mathcal{M}}$, where $\alpha^{\partial\mathcal{M}}$ is a differential one-form on $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$. Thus, following the Schwinger-Weiss principle, trajectories are those $\chi$ for which the first term on the r.h.s. of (1) vanishes for any direction $X^{(\chi)}$. Then, the fundamental lemma of the calculus of variations implies that $\chi$ satisfies the following equations of the motion

$$\mathbb{EL}_\chi(X^{(\chi)}) = 0 \quad \forall X^{(\chi)} \in \mathbf{T}_\chi \mathcal{F}_{\mathcal{P}} \implies \chi^\star \left(i_X \, \mathrm{d}\Theta_H\right) = 0 \quad \forall X \in \mathfrak{X}^v\left(U_{\mathcal{P}}^{(\chi)}\right) \implies \begin{cases} \frac{\partial \phi^a}{\partial x^\mu} = \frac{\partial H}{\partial \rho_a^\mu}(\chi), \\ \frac{\partial P_a^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial u^a}(\chi), \end{cases}$$

$U_{\mathcal{P}}^{(\chi)}$ being an open neighborhood of the image of $\chi$. The space of solutions of the equations of the motion will be denoted by $\mathcal{EL}_{\mathcal{M}}$.

Now, we focus on the role of the differential form $\Pi_{\partial\mathcal{M}}^\star \alpha^{\partial\mathcal{M}}$ within the construction of the Poisson bracket on $\mathcal{EL}_{\mathcal{M}}$. First, its differential gives the following two-form on $\mathcal{F}_{\mathcal{P}}$ being, again, the pull-back of a two-form on $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$

$$\mathrm{d}\Pi_{\partial\mathcal{M}}^\star \alpha^{\partial\mathcal{M}}(X^{(\chi)}, Y^{(\chi)}) =: \Pi_{\partial\mathcal{M}}^\star \Omega^{\partial\mathcal{M}}(X^{(\chi)}, Y^{(\chi)}) = \int_{\partial\mathcal{M}} \chi_{\partial\mathcal{M}}^\star \left(i_X i_Y \, \mathrm{d}\Theta_H\right).$$

It can be proved that [1, 2]

**Proposition 1.** *$\mathcal{EL}_{\mathcal{M}}$ is an isotropic manifold for $\Pi_{\partial\mathcal{M}}^\star \Omega^{\partial\mathcal{M}}$.*

On the other hand, if we consider a block $\mathcal{M}_{12}$ in $\mathcal{M}$ whose boundary is made by two hypersurfaces $\Sigma_1$ and $\Sigma_2$ with opposite orientations, then $\Pi_{\partial\mathcal{M}}^\star \Omega^{\partial\mathcal{M}} = \Pi_{\Sigma_1}^\star \Omega^{\Sigma_1} - \Pi_{\Sigma_2}^\star \Omega^{\Sigma_2}$, where $\Pi_\Sigma^\star \Omega^\Sigma = \int_\Sigma \chi_\Sigma^\star \left(i_X i_Y \, \mathrm{d}\Theta_H\right)$, $\chi_\Sigma$ being the restriction to $\Sigma$ of a dynamical field. Then, because of proposition 1, we have $\Pi_{\Sigma_1}^\star \Omega^{\Sigma_1}\big|_{\mathcal{EL}_{\mathcal{M}}} = \Pi_{\Sigma_2}^\star \Omega^{\Sigma_2}\big|_{\mathcal{EL}_{\mathcal{M}}}$. The same argument for any couple of hypersurfaces in $\mathcal{M}$ gives that the differential two-form $\Pi_\Sigma^\star \Omega^\Sigma$ on the space of restrictions of dynamical fields to a hypersurface $\Sigma$ does not depend on the particular $\Sigma$ if it is evaluated on solutions of the equations of the motion. We are going to denote the equivalence class of all these equivalent $\Pi_\Sigma^\star \Omega^\Sigma$ on $\mathcal{EL}_{\mathcal{M}}$, as $\Pi^\star \Omega$.

[3] They must be vertical in order to ensure their flow to lie in the space of sections.

[4] It is actually a section of the pull-back bundle of $\tau_1$ via $\mathfrak{i}_{\partial\mathcal{M}}$, $\mathfrak{i}_{\partial\mathcal{M}}$ being the canonical immersion of $\partial\mathcal{M}$ into $\mathcal{M}$.

## 2. Construction of the bracket

The crucial point to obtain a Poisson bracket on $\mathcal{EL}_{\mathcal{M}}$ is that $\Pi^\star\Omega$ is a symplectic structure for theories where all constraints can be solved[5].

**Proposition 2.** *$\Pi^\star\Omega$ is a symplectic structure if all constraints can be solved.*

*Proof sketch.* Since $\Pi^\star\Omega$ does not depend on $\Sigma$, one can consider as $\Sigma$ a hypersurface of Cauchy data for the equations on the motion. If all constraints can be solved, an existence and uniqueness theorem for the equations of the motion holds and, thus, the space of Cauchy data is diffeomorphic with the space of solutions, the diffeomorphism denoted by $\Phi_{\mathcal{EL}_{\mathcal{M}}}$. The structure $\Pi^\star\Omega$ restricted to the space of Cauchy data of the equations of the motion is $\Phi^\star_{\mathcal{EL}_{\mathcal{M}}}\Pi^\star_\Sigma\Omega^\Sigma$. Via a direct computation it is easy to prove that $\Phi^\star_{\mathcal{EL}_{\mathcal{M}}}\Pi^\star_\Sigma\Omega^\Sigma$ is symplectic. Therefore, the structure $\Pi^\star\Omega$ on the space of solutions is the pull-back via a diffeomorphism of a symplectic structure, thus, it is symplectic. ∎

With the symplectic structure $\Pi^\star\Omega$ in hand, a Poisson bracket on $\mathcal{EL}_{\mathcal{M}}$ can be defined in the usual way as

$$\{F, G\} = \Pi^\star\Omega(X_F, X_G) = \mathfrak{L}_{X_F}G,$$

$F$ and $G$ being functions on $\mathcal{EL}_{\mathcal{M}}$ and $X_F$ being the Hamiltonian vector field associated with $F$ w.r.t. $\Pi^\star\Omega$, i.e., the one satisfying $i_{X_F}\Pi^\star\Omega = \mathrm{d}F$.

We conclude by mentioning that an easier way to compute the Hamiltonian vector field , and, thus, the bracket, exists[6]. Indeed, the functional $F$ can be restricted to the space of Cauchy data to $f = \Phi^\star_{\mathcal{EL}_{\mathcal{M}}}F$. To $f$ a Hamiltonian vector field, say $X_f$, can be associated via the symplectic structure $\Phi^\star_{\mathcal{EL}_{\mathcal{M}}}\Pi^\star\Omega$, and this is much easier from the computational point of view. Then, it can be proved that the Hamiltonian vector field associated with the original functional $F$ with respect to the structure $\Pi^\star\Omega$ can be recovered by solving the linearization of the equations of the motion with $X_f$ as Cauchy datum.

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[5] For gauge theories the road to obtain a symplectic structure from this two-form is more involved and can not be treated here [3].
[6] We will give an extended proof in a more extensive work [3].