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Computing rotation numbers in the circle with a new algorithm

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Abstract: We present an efficient algorithm to compute rotation intervals of circle maps of degree one. It is based on the computation of the rotation number of a monotone circle map of degree one with a constant section. The main strength of this algorithm is that it computes *exactly* the rotation interval of a natural subclass of the continuous non-invertible degree one circle maps.

We also compare our algorithm with other existing ones by plotting the Devil's Staircase of a one-parameter non-differentiable family of maps, which is out of reach for the existing algorithms that are centred around differentiable maps.

Resumen: Presentamos un algoritmo eficiente para calcular el intervalo de rotación para aplicaciones en el círculo de grado 1. Está basado en el cálculo del número de rotación de aplicaciones en el círculo de grado 1 monótonas que tengan una sección constante. El punto fuerte de este algoritmo es que calcula el intervalo de rotacion de formula *exacta* para una subclasse natural de aplicaciones en el círculo continuas y no invertibles.

También compararemos nuestro algoritmo con otros existentes para dibujar la *Devil's Staircase* de una familia dependiente de un parametro no-diferenciable, fuera del alcance de los algoritmos existentes, centrados en funciones diferenciables.

Keywords: rotation number, circle maps, algorithm.

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1. Statement of the problem

This extended abstract basically summarizes the results in [1]. Most of the preliminary results can be found in [2].

We want to efficiently compute rotation intervals for degree one circle maps, the reason being the theoretical importance it plays on combinatorial dynamics. Many results, ranging from the exact set of periods of the maps to their entropy, use the rotation interval strongly. Now we will introduce the notion of rotation number and interval, and give some important properties relating degree one circle maps and their rotation numbers or intervals. First, let us introduce the notion of degree one map.

Definition 1 (degree one maps). Let $f: \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map and let $F: \mathbb{R} \to \mathbb{R}$ be such that $\exp(2\pi x) \circ f = F \circ \exp(2\pi x)$. We will say that F *is a lifing of* f. We say that f is of degree 1 if $F(1) - F(0) = 1$. ◂

Note that there may be many liftings, but if F and F' are liftings of f, then $F = F' + k$, with $k \in \mathbb{Z}$, hence the property $F(1) - F(0)$ is independent of the choice of lifting. Now let us introduce the stars of the show, the rotation number and rotation interval.

Definition 2 (rotation number and rotation interval). Let f be a map of degree 1 and let F be a lifting. We will define the *rotation number* of F on $x \in \mathbb{R}$ as

$$\rho_F(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}.$$

Note that this number is dependent on x. Moreover we will define the *rotation set of F as*

$$\mathrm{Rot}(F) = \{\rho_F(x) \mid x \in \mathbb{R}\} = \{\rho_F(x) \mid x \in [0,1]\},$$

which is an interval [3]. ◂

Now, let us study some some ways to infer the rotation number from the properties of F.

Lemma 3. *Let $F \in \mathcal{L}_1$. Then, x is an n-periodic* (mod 1) *point of F if and only if there exists $k \in \mathbb{Z}$ such that $F^n(x) = x + k$ but $F^j(x) - x \notin \mathbb{Z}$ for $j = 1, 2, \dots, n-1$. In this case,*

$$\rho_F(x) = \lim_{m\to\infty} \frac{F^m(x) - x}{m} = \frac{k}{n}.$$

Proposition 4. *Let $F \in \mathcal{L}_1$ be non-decreasing. Then, for every $x \in \mathbb{R}$ the limit*

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

exists and is independent of x. In this case we denote the rotation number of the map by ρ_F.

Using this proposition we will compute the rotation interval by just computing the rotation number of two non decreasing maps. However, first we need to introduce these special maps.

Definition 5. We set

$$F_l(x) = \inf\{F(y) : y \geq x\},$$
$$F_u(x) = \sup\{F(y) : y \leq x\},$$

where u stands for *upper* and l for *lower*. ◂

In Figure 1a we show an example of the upper and lower maps. Finally we can show a result relating the rotation interval with the well defined rotation number of two maps.

Theorem 6. *Let F be of degree 1. Then,*

$$\mathrm{Rot}(F) = [\rho_{F_l}, \rho_{F_u}].$$

2. Main result and new algorithm

For a real number x, we will denote the floor of x as $\lfloor x \rfloor$ and the decimal part function as $\{x\}$.

A *constant section of a lifting* F of a circle map is a closed non-degenerate subinterval K of $\mathbb{R}$ such that $F|_K$ is constant. In the special case when $F \in \mathcal{L}_1$, we have that $F(x+1) = F(x) + 1 \neq F(x)$ for every $x \in \mathbb{R}$. Hence, the length of K is less than 1.

The algorithm we propose is based on Lemma 8 but, especially, on the following simple proposition which allows us to compute *exactly* the rotation number of a non-decreasing lifting from $\mathcal{L}_1$ that has a constant section, provided that $F^n(K) \cap (K + \mathbb{Z}) \neq \varnothing$.

Proposition 7. *Let $F \in \mathcal{L}_1$ be non-decreasing and have a constant section K. Assume that there exists $n \in \mathbb{N}$ such that $F^n(K) \cap (K + \mathbb{Z}) \neq \varnothing$, and that n is minimal with this property. Then, there exists $\xi \in \mathbb{R}$ such that $F^n(K) = \{\xi\} \subset K + m$ with $m = \lfloor \xi - \min K \rfloor \in \mathbb{Z}$, ξ is an n-periodic* (mod 1) *point of F, and $\rho_F = \frac{m}{n}$.*

Proof. Since K is a constant section of F, $F(K)$ contains a unique point, and hence there exists $\xi \in \mathbb{R}$ such that $F^n(K) = \{\xi\}$. Then, the fact that $F^n(K) \cap (K + \mathbb{Z}) \neq \varnothing$ implies that $\xi \in K + m$ with $m = \lfloor \xi - \min K \rfloor \in \mathbb{Z}$. Set $\tilde{\xi} := \xi - m \in K$. Then, $\{F^n(\tilde{\xi})\} = F^n(K) = \{\tilde{\xi} + m\}$. Moreover, the minimality of n implies that $F^j(\tilde{\xi}) - \tilde{\xi} \notin \mathbb{Z}$ for $j = 1, 2, \dots, n-1$. So, Lemma 3 tells us that $\tilde{\xi}$ (and hence ξ) is an n-periodic (mod 1) point of F. Thus, $\rho_F = \frac{m}{n}$ by Proposition 4. ∎

Notice that this proposition gives us the backbone for an algorithm to compute rotation numbers for non-decreasing maps with a constant section. What remains to be checked is what happens if the iteration of the constant part K never falls again inside $K + \mathbb{Z}$, or the number of iterates that are required is too large to make it computationally practical. For this, we may use the following lemma.

Lemma 8. *For every non-decreasing lifting $F \in \mathcal{L}_1$ and $n \in \mathbb{N}$ we have*

$$\left| \rho_F - \tfrac{F^n(x) - x}{n} \right| < \frac{1}{n},$$

for every $x \in \mathbb{R}$.

2.1. Algorithm

From Proposition 7 and Lemma 8 we can obtain the following algorithm:

(i) Decide the maximum number of iterates $\mathsf{N} = \mathrm{ceil}\left(\frac{1}{\mathsf{error}}\right)$ to perform in the worst case (i.e., when Proposition 7 does not work).

(ii) Re-parametrize the lifting F so that it has a maximal constant section of the form $[0, \beta]$.

(iii) Initialize $x = 0$ and $m = 0$.

(iv) Compute iteratively $x = \{F^n(0)\}$ and $m = \lfloor F^n(0) \rfloor$ (so that $F^n(0) = x + m$) for $n \leq \mathsf{N}$.

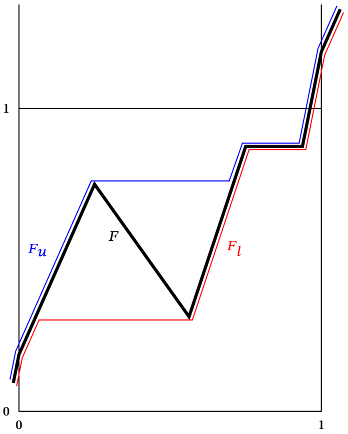
(v) Check whether $x \leq \beta$. On the affirmative we apply the previous proposition, and thus, $\rho_F = \frac{m}{n}$; ⇒ "exact" rotation number.

(vi) If we reach N iterates with $x > \beta$ for every n then, by the Lemma 8

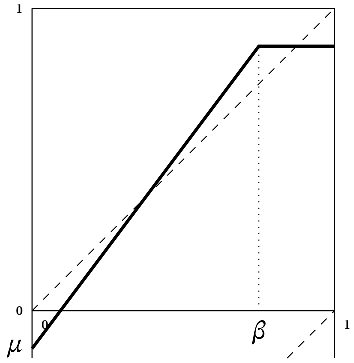
$$\left| \rho_F - \frac{m+x}{\mathsf{N}} \right| = \left| \rho_F - \frac{F^n(0)}{\mathsf{N}} \right| < \frac{1}{\mathsf{N}},$$

and the algorithm returns $\frac{m+x}{\mathsf{N}}$ as an estimate of ρ_F with $\frac{1}{\mathsf{N}}$ as the estimated error bound.

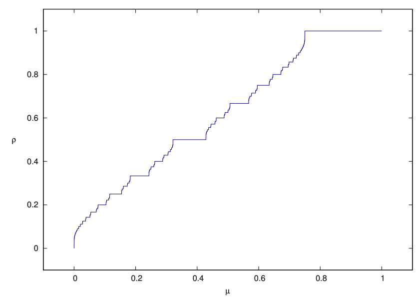
In [1], one can find a slightly more nuanced presentation of the algorithm, taking into account machine and rounding errors, but in spirit they are the same.



(a) An example of a map $F \in \mathcal{L}_1$ with its lower map F_l in red and its upper map F_u in blue.



(b) Plot of F_μ for a general μ



(c) Devil's Staircase plotted using the proposed algorithm

Figure 1: All the figures of the paper.

Table 1: Time taken by both the algorithms studied

Method	Time (s)
Classic	132.418015
Proposed Algorithm	0.003307

3. Testing of the algorithm

To test the algorithm we have plotted the *Devil's Staircase* for the one-parametric family of maps

$$F_\mu(x) = F_\mu|_{[0,1]}(\{x\}) + \lfloor x \rfloor.$$

See Figure 1b for a schematic plot. The so-called Devil's staircase is the result of plotting the rotation number as a function of the parameter μ. It can be proven that this plot will have constant sections for any rational rotation number, hence the "Staircase" in the name.

To conduct the test, we have plotted the Devil's Staircase for F_μ using the proposed algorithm and the algorithm stemming from Lemmas 4 and 8, which tells us that in the non decreasing case we can get the rotation number just by iterating and allow us to control the error. In Figure 1c one can find the plot of the Devil's Staircase plotted with our algorithm and in Table 1 the times each algorithm required to plot such figures. Moreover, the Arnol'd Tongues and the Rotation Intervals have also been computed in [1].

References

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