3rd *BYMAT Conference* (2020)

Model categories and homotopy theories

Daniel Miguel
Universidad de La Rioja
daniel.miguelt@unirioja.es

Abstract: Model categories are a category theoretic tool defined by Daniel Quillen with the aim of generalizing the homotopy theory built for topological spaces. The main goal of this text is to give an introduction to them, following an article by Dwyer and Spalinski. Just by its definition, it is almost inmediate that we can generalize analogues of well-known notions such as cylinder spaces, path spaces and homotopies. We use these tools to build a homotopy theory on a model category. Moreover, we will give some examples of different model structures over some categories, such as the expected category of topological spaces and the category of chain complexes of modules over a ring. Concerning the second one, we will also speak a little about spectral sequences and about the related model structure for filtered chain complexes.

Resumen: Las categorías de modelos son un concepto categórico teórico que fue definido por Daniel Quillen con el objetivo de generalizar la teoría de homotopía ya existente para espacios topológicos. Tan solo a partir de su definición, es casi inmediato que podemos generalizar nociones bien conocidas como son los espacios cilíndricos, los espacios de caminos y las homotopías. Utilizaremos estas herramientas para construir una teoría de homotopía en una categoría de modelos. Además, daremos algunos ejemplos de diferentes estructuras de modelos para diversas categorías, como es la esperada categoría de espacios topológicos o como la categoría de complejos de cadenas de módulos sobre un anillo. Con respecto a éste último, también hablaremos un poco sobre sucesiones espectrales y, en relación con ellas, sobre una estructura de modelos para complejos de cadenas filtrados.

Keywords: homotopy theories, homotopy categories, model categories fibrations, cofibrations, weak equivalences, spectral sequences, spectral systems.

MSC2010: 18N40.

Acknowledgements: I would like to acknowledge the tutoring and teaching of Enrique Arrondo, Andrea Guidolin and Ana Romero.

Reference: Miguel, Daniel. “Model categories and homotopy theories”. In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 223-226. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p223.

1. Introduction

Model categories were introduced by Daniel Quillen in [8], looking for a generalization of the classic homotopic tools that we knew for topological spaces. As he explains there, he did that in sight of some work by Dold and Kan where some sort of "homotopic methods" were used succesfully in the context of derived categories.

To generalize it, he defined what he called *model categories*. A model structure over a category is defined by distinguishing some classes of maps and imposing some axioms over them. These axioms, which resemble basic homotopy properties, turn out to be enough to define an equivalence relation for the maps of this category. This relation is called *homotopy relation*, and it is what will give us the homotopy theory (also called rational homotopy theory). From then on, several authors have proved different categories to fulfill the axioms for some classes of maps, and also have found different structures for a particular category.

2. The definition of model categories

The first step to define a model structure over a category $\mathcal{C}$ (following [2, Section 3]) is to distinguish three classes of maps, all of them closed under composition:

- *Weak equivalences*, of which we may think of as weak homotopy equivalences (maps that induce isomorphisms over all the homotopy groups).
- *Fibrations*, which can correspond to Serre fibrations and include covering maps.
- *Cofibrations*, which are dual to fibrations, and in the case of topological spaces can correspond to retracts of maps that obtain a space from another one by attaching cells.

Also, we ask them to fulfill the following axioms:

MC1 Finite limits and colimits exist in $\mathcal{C}$.

MC2 If f and g are maps in $\mathcal{C}$ such that its composition $g \circ f$, and two out of the three of them are weak equivalences, then so is the third one.

MC3 If f is a retract of g and g is a fibration, a cofibration or a weak equivalence, then so is f.

MC4 Let us consider the commutative diagram on the right. If i is a cofibration and p is a fibration and a weak equivalence (called acyclic fibration), or if i is an acyclic cofibration and p a fibration, then there exists a lift for the diagram (that is, a map $l\colon B \to X$ that commutes with the other arrows of the diagram).

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

MC5 Any morphism f can be factored (maybe with a functorial factorization) as $f = pi$, where i is a cofibration and p is an acyclic fibration, or where i is an acyclic cofibration and p is a fibration.

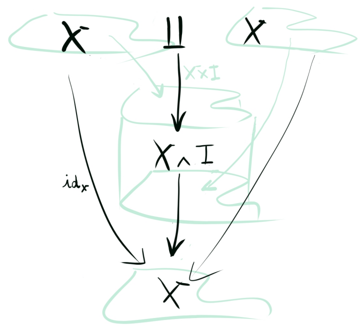
MC1 is purely technical, and is related to the existence of initial and terminal objects. MC2 tells us about the good behaviour of weak equivalences with respect to compositions. MC3 and MC4 ask our classes to behave well with respect to retracts (of maps), extensions and liftings. We notice that those two and MC2 resemble topological spaces, homotopy liftings and composition of weak homotopy equivalences.

To understand MC5, we have to introduce the so called *cofibrant* and *fibrant* objects. These are objects for which, respectively, the map from the initial object is a cofibration and the map to the terminal object is a fibration. In the case of topological spaces, all objects can be fibrants, whereas the cofibrant objects can be the retracts of cell-complexes. Using now MC5, and given an object X, we can factor those maps as

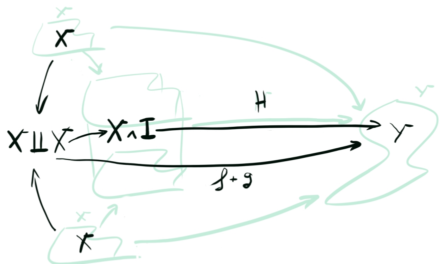
$$\begin{array}{ccccccc} \varnothing \xrightarrow{i} X & \dashrightarrow & \varnothing \longrightarrow F \xrightarrow{\sim} X \\ X \xrightarrow{p} * & \dashrightarrow & X \xrightarrow{\sim} C \longrightarrow * \end{array}$$

In other words, MC5 means that we can find some sort of CW-approximation for any object in our category. These kinds of objects are important because they behave very well with respect to the homotopy relations.

Next, we define a *cylinder object* on a model category $\mathcal{C}$ to be an object $X \wedge I$ that factors the map $\mathrm{id}_X + \mathrm{id}_X \colon X \coprod X \to X$ in such a way that the map $X \wedge I \to X$ is a weak equivalence. Looking at the diagram on the right, we can see that this definition tries to capture the topological idea of cylinder, with inclusion of both bases into a topological cartesian product $X \times [0,1]$ together with the projection onto X. However, it throws away all the geometric or topological information, and keeps only the maps.



Now, if we take two maps, $f, g \colon X \to Y$, and a cylinder object for X, $X \wedge I$, then we define a *left homotopy* between f and g via $X \wedge I$ to be a map $H \colon X \wedge I \to Y$ that extends the sum $f + g \colon X \coprod X \to Y$. In that case, we say that f and g are homotopic, and we call this relation "left homotopy relation". This obviously reminds of the usual notion of homotopy, as illustrated by the diagram below.



We can also define dual notions of cylinders and left homotopies, which are called *path spaces* and *right homotopies*. The key point here is that not only fibrant and cofibrant objects give us the desired lifting properties for homotopies, but also make left and right homotopy relations equivalent.

Using MC5 as we did before, we take those CW-approximations and define with them a unique homotopy relation. Therefore, the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the category with the same objects of $\mathcal{C}$ and with morphisms the equivalence classes of maps, between the fibrant and cofibrant replacements, by the homotopy relation previously defined. Specifically, this means that we can work there "up to homotopy", and that we have a functor $\gamma \colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ that inverts all the maps that we distinguished as weak equivalences.

3. Examples

As we mentioned previously, one can easily find different examples of homotopy theories over different categories. We will comment the ones that are mentioned in [2] and that we studied in [6].

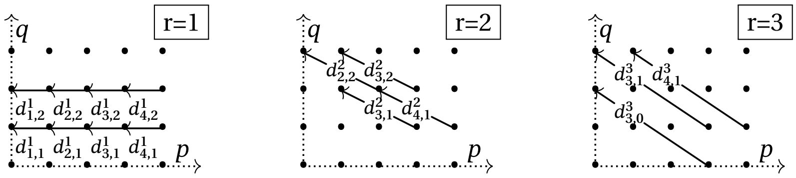
As we have been mentioning previously, *topological spaces* admit a model structure taking the class of weak equivalences to be weak homotopy equivalences, the class of fibrations to be Serre fibrations and the class of cofibrations to be the retracts of maps that attach cells on a given space.

However, this is not the only way to do this. If we look at the class of weak equivalences, we could ask ourselves if it is possible to build a model structure where the weak equivalences are the homotopy equivalences. Strom, in [9], answered this question by building such a model structure, taking Hurewicz fibrations and closed Hurewicz cofibrations. The difference between these structures lies in the fact that there are maps that are weak homotopy equivalences but not homotopy equivalences (see for example [6, Section 3.1]). However, they are the same for CW-complexes, as Whitehead's Theorem states.

There are several categories of *chain complexes* that admit a model structure, and several ways to define one over each of them ([3, Chapter 2]). In particular, we have the so-called projective model structure, which is built by taking as weak equivalences the maps that induce isomorphisms between the homology groups, as cofibrations the monomorphisms with projective kernel and as fibrations the epimorphisms. Also, it is worth mentioning that there are model structures that take as weak equivalences the usual chain homotopy equivalences (called Hurewicz model structure).

Filtered chain complexes. Spectral sequences

Spectral sequences are families $(E^r, d^r)_{r\geq 1}$ of bigraded modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ for each r (the number r is called page). The $d^r_{p,q}$ are maps of bidegree $(-r, r-1)$ that are called differentials (see [5] for more about them). We can obtain each page computing the homology of the previous one.



Given a filtered chain complex $(F_k C_*, d)_{p\in\mathbb{Z}}$, one defines its associated spectral sequence (which is a progressive approximation of homology groups by pages) by taking the quotient of the so called almost-cycles ($Z^r_{p,q}$) and almost-boundaries ($B^r_{p,q}$) as follows:

$$Z^r_{p,q} = \frac{A^r_{p,q} + F_{p-1}C_n}{F_{p-1}C_n}, \quad B^r_{p,q} = \frac{\mathrm{d}(A^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_n}{F_{p-1}C_n} \quad \text{and} \quad E^r_{p,q} := \frac{Z^r_{p,q}}{B^r_{p,q}},$$

where $n = p+q$, $A^r_{p,q} = \{c \in F_pC_n \mid \mathrm{d}(c) \in F_{p-r}C_{n-1}\}$, and the differentials are induced by the ones of the complex. Noticing that a map of filtered chain complexes induces a map of spectral sequences, and looking at the previous example of model structure, one could ask if we can take as weak equivalences the maps that induce a spectral sequences isomorphism from a certain page. The answer is positive, and it is given by Joana Cirici [1]. Moreover, there exists a generalization of spectral sequences for generalized filtered chain complexes, called *spectral systems*, and introduced in [4]. An open problem is to define a model structure for generalized filtered chain complexes by taking the class of weak equivalences to be the maps that induce isomorphisms between certain terms of the associated spectral system.

4. Conclusion

There are more examples that we could mention, such as the classic Kan complexes and the category of simplicial sets. However, there exist more “unexpected” examples, such as [7], concernig schemes. One can apply this in many different areas, and work with generalized homotopy notions that can be thought intuitively but that have also proved themselves useful. Consequently, its study is really encouraging.

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