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Additivities of the families of Darboux-like functions

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Abstract: It is well known that every continuous function from $\mathbb{R}$ to $\mathbb{R}$ maps connected sets to connected sets. However, the converse is not true in general, that is, the family of real functions that map connected sets to connected sets (known as Darboux functions) strictly contains the family of continuous functions. This method of considering necessary but not sufficient conditions for continuous functions lead us to obtain the families of functions known as Darboux-like functions. In this expository paper we will study the main results related to the inclusions and set operations between the classical families of Darboux-like functions, and also analyze the cardinal coefficient known as additivity of these families.

Resumen: Es bien conocido que toda función continua de $\mathbb{R}$ en $\mathbb{R}$ lleva conjuntos conexos en conjuntos conexos. Sin embargo, el recíproco no es cierto en general, es decir, la familia de funciones que llevan conjuntos conexos en conjuntos conexos (conocidas como funciones Darboux) contiene estrictamente a la familia de funciones continuas. Este método de considerar condiciones necesarias pero no suficientes para las funciones continuas nos lleva a obtener las familias de funciones conocidas como funciones de tipo Darboux. En este artículo expositivo estudiaremos los resultados principales relacionados con las inclusiones y operaciones de conjuntos entre las familias clásicas de funciones de tipo Darboux, y también analizaremos el coeficiente cardinal conocido como aditividad de estas familias.

Keywords: Additivity, Darboux-like functions, lineability.

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1. Introduction and preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. We will denote for the rest of this paper the set of functions from $\mathbb{R}$ to $\mathbb{R}$ and the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ by $\mathbb{R}^{\mathbb{R}}$ and C, respectively.

Let us begin with some historical background. The Intermediate Value Theorem (in its classical formulation) is a well-known result on continuous functions proven by Bolzano in 1817. It states that if $f \in C$, then f maps intervals to intervals, that is, f maps connected sets to connected sets. We say that a function $f \in \mathbb{R}^{\mathbb{R}}$ satisfies the Intermediate Value Property (IVP) if f maps connected sets to connected sets. Around 1875, famous mathematician Darboux studied the IVP proving, for instance, that the derivative of every differentiable function in $\mathbb{R}^{\mathbb{R}}$ satisfies the IVP. However, not every function that maps connected sets to connected sets is continuous, as shown by the following classical example:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We say that $f \in \mathbb{R}^{\mathbb{R}}$ is *Darboux* (in honor of Jean-Gaston Darboux) if f maps connected sets to connected sets. We will denote by $\mathcal{D}$ the family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ that map connected sets to connected sets. Notice that $C \subsetneq \mathcal{D}$. This idea of considering necessary but not sufficient conditions for continuous functions has been thoroughly studied throughout the past century by many mathematicians, leading to what are known as *Darboux-like* functions (or generalized continuous functions).

In this paper we will study how the classical families of Darboux-like functions are related by inclusions and by set operations, showing that they form an algebra of sets, and also we will show the cardinal coefficient known as additivity of these families. The paper is arranged as follows. In Section 2 we will define all the classical families of Darboux-like functions as well as how they are related by inclusions and intersections. This section will also show that these families form an algebra of sets, providing the atoms that generate the algebra as well. In Section 3 we will define the concept of additivity of a family of real functions and its relation with the field of lineability. We will finish this section by providing the known additivities of (i) the classical families of Darboux-like functions, (ii) the complements of the classical families of Darboux-like functions and (iii) the atoms that form the algebra of sets.

To finish this section we will introduce standard notations and definitions from set theory that will be used for the rest of this paper. The symbol $|X|$ will denote the cardinality of the set X. If $f \in \mathbb{R}^{\mathbb{R}}$ and $X \subseteq \mathbb{R}$, then $f \restriction X$ denotes the restriction of f to X. The successor of a cardinal number λ and its cofinality will be denoted by λ^+ and $\mathrm{cof}(\lambda)$, respectively. We say that a cardinal number λ is regular if $\mathrm{cof}(\lambda) = \lambda$. Given a set X and a cardinal number λ, we denote by $[X]^{<\lambda}$ and $[X]^{\lambda}$ the sets of all subsets of X of cardinality less than λ and equal to λ, respectively. Let $\omega_1 = |\mathbb{N}|$, $\omega_2 = \omega_1^+$ and $\mathfrak{c} = |\mathbb{R}|$. We define also $\mathfrak{c}_-$ as κ when $\kappa = \mathfrak{c}^+$ and as $\mathfrak{c}$ otherwise. Finally, let T be a theory and A an additional axiom. Then, A is consistent with T (or A is relatively consistent with T) if it can be proved that if T is consistent (does not entail contradiction), then $T + A$ is consistent.

2. Darboux-like functions and their relations

There are eight classical families of Darboux-like functions (one of them being $\mathcal{D}$). They are defined and denoted as follows:

- PC - family of all *peripherally continuous* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that for every $x \in \mathbb{R}$, there exist two sequences $s_n \nearrow x$ and $t_n \searrow x$ with $\lim_{n\to\infty} f(s_n) = f(x) = \lim_{n\to\infty} f(t_n)$.
- Conn - family of all *connectivity* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that the graph of f restricted to any connected $C \subseteq \mathbb{R}$ is a connected subset of $\mathbb{R}^2$.
- AC - family of all *almost continuous* functions $f \in \mathbb{R}^{\mathbb{R}}$ (in the sense of Stallings), that is, such that every open subset of $\mathbb{R}^2$ containing the graph of f contains also the graph of function in C.

PR - family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ with *perfect road*, that is, such that for every $x \in \mathbb{R}$ there exists a perfect $P \subseteq \mathbb{R}$ having x as a bilateral limit point such that $f \restriction P$ is continuous at x.

CIVP - family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ with the *Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set C between p and q such that $f[C] \subseteq K$.

SCIVP - family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ with the *Strong Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set C between p and q such that $f[C] \subseteq K$ and $f \restriction C$ is continuous.

Ext - family of all *extendable* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that there exists a connectivity function $g\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ with $f(x) = g(x, 0)$ for all $x \in \mathbb{R}$.

We will denote by $\mathbb{D}$ the set of all classical families of Darboux-like functions. The families in $\mathbb{D}$ are related in terms of sets by inclusions. Figure 1 shows all the strict inclusions of the families in $\mathbb{D}$. Moreover, the families in $\mathbb{D}$ still have the containment relations as in Figure 1 even when we consider the intersections between the families. We refer the interested reader to [2] and the references therein for the proofs of the containment relations.

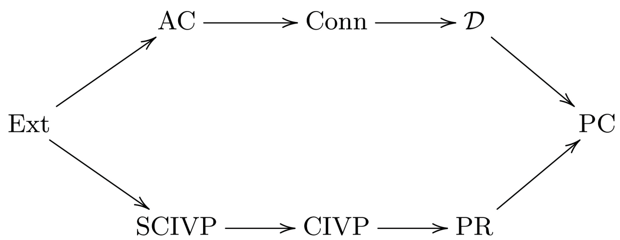

Figure 1: All strict inclusions, indicated by arrows, among the families in $\mathbb{D}$.

Therefore, the families in $\mathbb{D}$ form an algebra of sets, which will be denoted by $\mathcal{A}(\mathbb{D})$, generated by the following 17 sets: Ext, PC $\setminus$ (PR $\cup$ $\mathcal{D}$), PR $\setminus$ (CIVP $\cup$ $\mathcal{D}$), CIVP $\setminus$ (SCIVP $\cup$ $\mathcal{D}$), SCIVP $\setminus$ $\mathcal{D}$, $\mathcal{D}$ $\setminus$ (PR $\cup$ Conn), $\mathcal{D}\cap$PR$\setminus$(CIVP$\cup$Conn), $\mathcal{D}\cap$CIVP$\setminus$(SCIVP$\cup$Conn), $\mathcal{D}\cap$SCIVP$\setminus$Conn, Conn$\setminus$(PR$\cup$AC), Conn$\cap$PR$\setminus$(CIVP$\cup$AC), Conn$\cap$CIVP$\setminus$(SCIVP$\cup$AC), Conn$\cap$SCIVP$\setminus$AC, AC$\setminus$PR, AC$\cap$PR$\setminus$CIVP, AC$\cap$CIVP$\setminus$SCIVP and AC$\cap$SCIVP$\setminus$Ext.

3. Additivities of the algebra of Darboux-like functions

We begin this section by defining the additivity of a family of real functions.

Definition 1 (Additivity). Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$. The additivity of $\mathcal{F}$ is the following cardinal number:

$$\mathrm{A}(\mathcal{F}) = \min\left(\{|F| : F \subset \mathbb{R}^{\mathbb{R}},\ \forall g \in \mathbb{R}^{\mathbb{R}},\ g + F \nsubseteq \mathcal{F}\} \cup \{(2^{\mathfrak{c}})^{+}\}\right).$$ ◂

Although the additivity is interesting from the point of view of set theoretical real analysis, it can also be used in the field of lineability to find vector spaces of certain dimension. For more information about this field we refer the reader to [1].

Definition 2 (α-lineable). Let X be a vector space, A a subset of X and α a cardinal number. We say that A is α-lineable if $A \cup \{0\}$ contains a vector space of dimension α. ◂

Theorem 3 (Gámez, Muñoz and Seoane [3]). *Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be star-like, that is, $a\mathcal{F} \subseteq \mathcal{F}$ for all $a \in \mathbb{R}$. If $\mathfrak{c} < \mathrm{A}(\mathcal{F}) \leq 2^{\mathfrak{c}}$, then $\mathcal{F}$ is $\mathrm{A}(\mathcal{F})$-lineable.*

Now, one of the properties that additivity has is the following (see [2]): given $\mathcal{F}, \mathcal{G} \subseteq \mathbb{R}^{\mathbb{R}}$ with $\mathcal{F} \subseteq \mathcal{G}$, we have that $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}(\mathcal{G})$. Hence, notice that if we know the additivities of the atoms of $\mathcal{A}(\mathbb{D})$, then we have the lower bounds of all the additivities of the families in $\mathcal{A}(\mathbb{D})$. We proceed to show the main results of this expository paper. We first show the additivities of the families in $\mathbb{D}$ and their complements (see [2] and the references therein). Let us define the following cardinal numbers:

$$\begin{aligned} e_{\mathfrak{c}} &= \min\{|F| : F \subset \mathbb{R}^{\mathbb{R}}, \forall g \in \mathbb{R}^{\mathbb{R}}, \exists f \in F \text{ such that } |f \cap g| < \mathfrak{c}\}, \\ d_{\mathfrak{c}} &= \min\{|F| : F \subset \mathbb{R}^{\mathbb{R}}, \forall g \in \mathbb{R}^{\mathbb{R}}, \exists f \in F \text{ such that } |f \cap g| = \mathfrak{c}\}, \text{ and} \\ d^*_{\mathfrak{c}} &= \min\{|F| : F \subset \mathbb{R}^{\mathbb{R}}, \forall G \in [\mathbb{R}^{\mathbb{R}}]^{\mathfrak{c}}, \exists f \in F \text{ such that } \forall g \in G,\ |f \cap g| = \mathfrak{c}\}. \end{aligned}$$

Theorem 4 (Ciesielski, Miller, Gámez, Muñoz, Seoane, Mazza, Recław, Jordan, Natkaniec - 1994/95, 1996/97, 2010, 2017). *We have the following results:*

(a) $\mathfrak{c}^+ \le \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{Conn}) = \mathrm{A}(\mathcal{D}) = e_{\mathfrak{c}} \le 2^{\mathfrak{c}}$*, and this is all that can be proved in ZFC.*

(b) $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$.

(c) $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$.

(d) $\mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathrm{PC}) = \omega_1$.

(e) $\mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathrm{Ext}) = \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathrm{PR}) = 2^{\mathfrak{c}}$.

(f) $d_{\mathfrak{c}} \le \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{D}) \le \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathrm{Conn}) \le \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathrm{AC}) \le d^*_{\mathfrak{c}}$.
If $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$*, then* $\mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{D}) = \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathrm{Conn}) = \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathrm{AC}) = d_{\mathfrak{c}} = d^*_{\mathfrak{c}}$.
If $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$ *and* $\mathfrak{c} = \lambda^+$*, then* $d_{\mathfrak{c}} \le e_{\mathfrak{c}}$.
Moreover, $\mathfrak{c}^+ \le d_{\mathfrak{c}} \le 2^{\mathfrak{c}}$ *and*

(f_1) *For every cardinals* $\lambda \ge \kappa \ge \omega_2$ *such that* $\mathrm{cof}(\lambda) > \omega_1$ *and* κ *is regular, it is relatively consistent with ZFC+CH that* $2^{\mathfrak{c}} = \lambda$ *and* $d_{\mathfrak{c}} = e_{\mathfrak{c}} = \kappa$*. In particular,* $\mathfrak{c}^+ < d_{\mathfrak{c}} = \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{D}) = \mathrm{A}(\mathcal{D}) = e_{\mathfrak{c}} < 2^{\mathfrak{c}}$ *is consistent with ZFC+CH.*

(f_2) *For every cardinal* $\lambda > \omega_2$ *such that* $\mathrm{cof}(\lambda) > \omega_1$*, it is relatively consistent with ZFC+CH that* $\mathfrak{c}^+ = \omega_2 = \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{D}) = d_{\mathfrak{c}} < e_{\mathfrak{c}} = \mathrm{A}(\mathcal{D}) = 2^{\mathfrak{c}} = \lambda$.

Finally we present the known additivities of the atoms of $\mathcal{A}(\mathbb{D})$ (for more details, see [2]).

Theorem 5 (Ciesielski, Natkaniec, Rodríguez, Seoane [2]). *We have the following results:*

(a) $d_{\mathfrak{c}} \le \mathrm{A}(\mathrm{PC} \setminus (\mathrm{PR} \cup \mathcal{D})) \le d^*_{\mathfrak{c}}$.

(b) $\mathrm{A}(\mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathcal{D})) = \mathrm{A}(\mathrm{CIVP} \setminus (\mathrm{SCIVP} \cup \mathcal{D})) = \mathrm{A}(\mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}) = \mathrm{A}(\mathrm{AC} \cap \mathrm{CIVP} \setminus \mathrm{SCIVP}) = \mathfrak{c}^+$.

(c) $\mathrm{A}(\mathrm{SCIVP} \setminus \mathcal{D}) = \mathrm{A}(\mathcal{D} \cap \mathrm{SCIVP} \setminus \mathrm{Conn}) = \mathrm{A}(\mathrm{Conn} \cap \mathrm{SCIVP} \setminus \mathrm{AC}) = 2$.

(d) $\mathrm{A}(\mathrm{AC} \setminus \mathrm{PR}) = e_{\mathfrak{c}}$.

(e) $\omega_1 \le \mathrm{A}(\mathrm{Conn} \setminus (\mathrm{PR} \cup \mathrm{AC})),\ \mathrm{A}(\mathrm{Conn} \cap \mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathrm{AC})),\ \mathrm{A}(\mathrm{Conn} \cap \mathrm{CIVP} \setminus (\mathrm{SCIVP} \cup \mathrm{AC})) \le \mathfrak{c}$.

(f) $\mathfrak{c}_- \le \mathrm{A}(\mathcal{D} \setminus (\mathrm{PR} \cup \mathrm{Conn})),\ \mathrm{A}(\mathcal{D} \cap \mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathrm{Conn})),\ \mathrm{A}(\mathcal{D} \cap \mathrm{CIVP} \setminus (\mathrm{SCIVP} \cup \mathrm{Conn})) \le \mathfrak{c}$.

(g) $2 \le \mathrm{A}(\mathrm{AC} \cap \mathrm{SCIVP} \setminus \mathrm{Ext}) \le \mathfrak{c}$.

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