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Recent results on interpolation by minimal surfaces

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Abstract: Complex analysis and minimal surfaces are strongly connected via the Weierstrass representation formula. This fact has been exploited recently to construct lots of examples of such surfaces with different properties. We would present the first results dealing with interpolation in the setting of minimal surfaces. These results are inspired by classical Weierstrass Interpolation theorem for holomorphic functions and are proved using techniques coming from complex analysis.

More concretely, given an open Riemann surface M, we would construct conformal minimal immersions $M \to \mathbb{R}^n$, $n \geq 3$, such that the values of the immersion at some points of M are prescribed.

Resumen: El análisis complejo y las superficies mínimas están fuertemente relacionados a través de la fórmula conocida como representación de Weierstrass. Esta relación ha permitido recientemente construir muchos ejemplos de tales superficies con diferentes propiedades. A continuación presentamos los primeros resultados sobre interpolación en el ambiente de superficies mínimas. Estos resultados están inspirados en el teorema clásico de interpolación de Weierstrass para funciones holomorfas y se prueban utilizando técnicas provenientes del análisis complejo.

Concretamente, dada una superficie de Riemann abierta M, construiremos inmersiones mínimas conformes $M \to \mathbb{R}^n$, $n \geq 3$, de manera que los valores de la inmersión en algunos puntos de M estén prescritos.

Keywords: minimal surface, Weierstrass theorem, Riemann surface.

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1. Introduction

An immersed surface in the Euclidean space of dimension $n \geq 3$ is called a *minimal surface* if it is locally area-minimizing, that is, small pieces of it are the ones with least area among all the surfaces with the same boundary. Minimal surfaces are usually defined as those surfaces with vanishing mean curvature vector field; which is equivalent to the previous definition. In the classical theory of minimal surfaces in $\mathbb{R}^n$, we may point out the so-called *Enneper-Weierstrass representation formula*. This formula provides any minimal surface in $\mathbb{R}^n$ in terms of holomorphic data defined on an open Riemann surface.

Let M be an open Riemann surface and $X = (X_1, \dots, X_n) \colon M \to \mathbb{R}^n$ a conformal minimal immersion, denoting by ∂ the complex linear part of the exterior differential $d = \partial + \overline{\partial}$ on M (here $\overline{\partial}$ denotes the antilinear part), we have that the 1-form $\partial X = (\partial X_1, \dots, \partial X_n)$, assuming values in $\mathbb{C}^n$, is holomorphic, has no zeros, and satisfies $\sum_{j=1}^n (\partial X_j)^2 = 0$. Furthermore, its real part $\Re(\partial X)$ is an exact 1-form on M.

Conversely, every holomorphic 1-form $\Phi = (\phi_1, \dots, \phi_n)$ with values in $\mathbb{C}^n$, vanishing nowhere on M, satisfying the nullity condition $\sum_{j=1}^n (\phi_j)^2 = 0$ everywhere on M, and whose real part $\Re(\Phi)$ is exact on M, determines a conformal minimal immersion $X \colon M \to \mathbb{R}^n$ by the classical Enneper-Weierstrass (or simply Weierstrass) representation formula:

$$X(p) = x_0 + \int_{p_0}^{p} \Re(\Phi), \quad p \in M,$$

for any fixed base point $p_0 \in M$ and initial condition $X(p_0) = x_0 \in \mathbb{R}^n$. This formula yields minimal surfaces in $\mathbb{R}^n$ from holomorphic 1-forms assuming values in the complex subvariety of $\mathbb{C}^n$ determined by $\mathfrak{A}_* := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 0\} \setminus \{0\}$.

Weierstrass representation formula has provided powerful tools coming from complex analysis in one and several variables to the study of minimal surfaces in $\mathbb{R}^n$. In particular, Runge-Mergelyan theorem for open Riemann surfaces (see [9, 11]) has resulted very useful in the study of minimal surfaces in the Euclidean space. For instance, the pioneer works of Jorge and Xavier [8] or Nadirashvili [10] combined the classical Runge approximation theorem with the Weierstrass formula to refute the belief that hyperbolic Riemann surfaces play a marginal role in the global theory of minimal surfaces. An open Riemann surface is hyperbolic, by definition, if it carries nonconstant negative subharmonic functions.

However, the most recent results that combine complex analysis and Weierstrass representation formula in this setting use methods coming from modern Oka theory. Roughly speaking, Oka manifolds are natural target for holomorphic functions; the key is that the punctured null quadric $\mathfrak{A}_*$ is an Oka manifold and hence Oka theory applies. A detailed explanation may be seen at the survey [3].

2. Interpolation results for conformal minimal immersions

General existence results for minimal surfaces in $\mathbb{R}^n$ have been proved using Oka theory. Further, one may add very interesting global properties to the solutions. In the following sections, we are going to show some of these results concerning interpolation. In particular we show in §2.1 those of interpolation for conformal minimal immersions in $\mathbb{R}^n$, $n \geq 3$. Next, we state in §2.2 the corresponding analogues for minimal surfaces of finite total curvature in $\mathbb{R}^3$. Finally, we show some applications in §2.3 to the construction of examples.

2.1. Results for conformal minimal immersions in any dimension $n \geq 3$

Approximation by holomorphic functions began with the classical Runge Theorem. It gives a topological characterization of those subsets of $\mathbb{C}$ for which any holomorphic function on them may be uniformly approximated by entire functions. Interpolation by holomorphic functions is another main research topic in Complex Analysis. It began with the classical Weierstrass Interpolation Theorem that ensures that one may prescribe the values of an entire function on a discrete subset of $\mathbb{C}$. Both results have been generalized to the framework of maps from Stein manifolds into Oka manifolds, and in particular for functions from open Riemann surfaces. Recall that any open Riemann surface is a Stein manifolds and that the null quadric is an Oka manifold.

Focusing on minimal surfaces, Alarcón, Forstnerič, and López have developed an uniform approximation theory for conformal minimal immersions in $\mathbb{R}^n$, $n \geq 3$ and more general families of holomorphic immersions in $\mathbb{C}^n$; see [4, 5]. Concerning interpolation for conformal minimal immersion the author in collaboration with Alarcón proved the following analogue to the Weierstrass interpolation theorem for conformal minimal immersions in $\mathbb{R}^n$. This result is proved in [1].

Theorem 1. (Weierstrass Interpolation Theorem for conformal minimal surfaces). *Let Λ be a closed discrete subset of an open Riemann surface, M, and let $n \geq 3$ be an integer. Every map $\Lambda \to \mathbb{R}^n$ extends to a conformal minimal immersion $M \to \mathbb{R}^n$.*

The assumptions on Λ in Theorem 1 are necessary since Λ has no accumulation point by the Identity Principle for harmonic maps. We obtain in that paper a much more general result which ensures not only interpolation but also *jet-interpolation of given finite order*, *uniform approximation on Runge compact subsets*, *control on the flux*, and global properties such as completeness and, under natural assumptions, properness and injectivity; see [1, Theorem 1.2] for the detailed statement and the necessary definitions. In addition, an analogue for directed holomorphic curves in $\mathbb{C}^n$ is proved, see [1, Theorem 1.3].

2.2. Conformal minimal immersions of finite total curvature in dimension $n = 3$

One of the main topic of research in the global theory of minimal surfaces in $\mathbb{R}^3$ are complete minimal surfaces with finite total curvature. We recall that a conformal minimal immersion $X\colon M \to \mathbb{R}^3$ has finite total curvature if

$$TC(X) \coloneqq \int_M K \, \mathrm{d}s^2 = -\int_M |K| \, \mathrm{d}s^2 > -\infty,$$

here $\mathrm{d}s^2$ is the area element of the surface and K denotes the Gauss curvature of $(M, \mathrm{d}s^2)$. These surfaces have the simplest topological, conformal, and asymptotic geometry. They are intimately related to meromorphic functions and 1-forms on compact Riemann surfaces. Indeed, given an open Riemann surface M and a complete conformal minimal immersion $X\colon M \to \mathbb{R}^3$ with finite total curvature, there are a compact Riemann surface Σ and a finite subset $\varnothing \neq E \subset \Sigma$ such that M is biholomorphic to $\Sigma \setminus E$.

The author in collaboration with Alarcón and López proved the following interpolation result for complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature. It is proved in [2].

Theorem 2. (Weierstrass Interpolation Theorem for conformal minimal immersions with finite total curvature). *Let Σ be a compact Riemann surface with empty boundary and let $E \neq \varnothing$ and Λ be disjoint finite sets in Σ. Every map $\Lambda \to \mathbb{R}^3$ extends to a complete conformal minimal immersion $\Sigma \setminus E \to \mathbb{R}^3$ with finite total curvature.*

We shall obtain a more general result providing also *uniform approximation*, *jet-interpolation of given finite order*, and *control on the flux*, see [2, Theorem 3.1] for details and definitions.

2.3. Applications and other results

Finally, we finish with some applications to the construction of examples. As we said before, an uniform approximation theory on compact subset have been developed for conformal minimal immersions, analogous to the one of holomorphic functions ([4, 5]). Continuing a natural sequence of approximation results, one may ask whether Carleman approximation theorem holds for minimal surfaces. Carleman theorem for holomorphic functions asserts that one may approximate any continuous function $\mathbb{R} \to \mathbb{C}$ by entire functions better than any given positive function. Next result is an analogue for conformal minimal immersions and it is proved in [7].

Theorem 3 (Carleman Theorem for conformal minimal immersions). *Let M be an open Riemann surface and let $R \subset M$ be a proper embedded curve. Let $f\colon R \to \mathbb{R}^n$, $n \geq 3$, and $\epsilon\colon M \to \mathbb{R}_+$ be continuous maps. There exists a complete conformal minimal immersion $X\colon M \to \mathbb{R}^n$ such that $\|X(p) - f(p)\| < \epsilon(p)$, $p \in M$. Furthermore, if $n \geq 5$, then X may be chosen to be injective.*

Similarly to the previous results, in collaboration with Chenoweth we proved an analogue to holomorphic directed immersions which is stated in [7, Theorem 1.2]. Furthermore, the solutions may be chosen to be complete and proper under natural assumptions, see [7, Theorems 1.3 and 1.4].

On the other hand, the next interpolation result ensures that one may construct minimal surfaces with all coordinates prescribed but two. The theorem is proved in [6].

Theorem 4. *Let M be an open Riemann surface and $n \geq 3$ be an integer. Let $\Lambda \subset M$ be a closed discrete subset and let $h\colon M \to \mathbb{R}^{n-2}$ be a nonconstant harmonic map. For any map $F\colon \Lambda \to \mathbb{R}^2$, there is a complete conformal minimal immersion $X = (X_1, X_2, \dots, X_n)\colon M \to \mathbb{R}^n$ such that $(X_1, X_2)|_\Lambda = F$ and $(X_3, \dots, X_n) = h$.*

As a consequence of the previous result, it is shown on [6] that we may interpolate by minimal surfaces in $\mathbb{R}^n$, $n \geq 3$, whose *generalized Gauss map* G_X is nondegenerate and fails to intersect n hyperplanes in general position. In dimension $n = 3$, we have the following.

Corollary 5. *Let M be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Any map $\Lambda \to \mathbb{R}^3$ extends to a complete nonflat conformal minimal immersion $X\colon M \to \mathbb{R}^3$ whose Gauss map $M \to \mathbb{S}^2$ omits two (antipodal) values of the sphere $\mathbb{S}^2$.*

For the general statement of the previous result and the necessary definitions, see [6, Theorem 1.1].

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