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A graph equation between the line graph and the edge-complement graph

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Abstract: From a graph G related graphs can be constructed, such as its line graph $L(G)$ and its edge-complement graph $\overline{G}$. After showing how properties of G imply properties of $L(G)$, we ask how different the concepts of the line graph $L(G)$ and that of the edge-complement graph $\overline{G}$ are, by solving the equation $L(G) \simeq \overline{G}$. We show that the equation has only two solutions. The proof uses an argument on the degree of the vertices of a graph that allows to reduce the number of possible solutions until they can be checked algorithmically. This gives an alternative proof to the one by Aigner [1].

Resumen: A partir de un grafo G se pueden construir grafos relacionados, como su grafo de líneas $L(G)$ y su grafo complemento de aristas $\overline{G}$. Después de mostrar cómo las propiedades de G implican propiedades de $L(G)$, nos preguntamos cuán diferentes son los conceptos del grafo lineal $L(G)$ y el del grafo complemento de aristas $\overline{G}$, resolviendo la ecuación $L(G) \simeq \overline{G}$. Demostramos que la ecuación tiene solo dos soluciones. La prueba utiliza un argumento sobre el grado de los vértices de un grafo que permite reducir el número de posibles soluciones hasta poder comprobarlas algorítmicamente. Esto da una prueba alternativa a la de Aigner [1].

Keywords: line graph, edge-complement graph, graph equation.

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1. Introduction

Line graphs as well as edge-complement graphs allow to restate graph questions in sometimes easier versions. In the following section, some relations between the properties of a graph and the respective properties of its line graph are shown. The last section studies the graph equation $L(G) \simeq \overline{G}$ in order to compare the line graph with the edge-complement graph. We find that there are exactly two graphs whose line graphs and edge-complement graphs coincide. This result was first shown by Aigner [1], whose argument uses the existence of a unique cycle in a possible solution. Here, we present an alternative proof that is based on the degree of the vertices of a solution G. The possible degrees of vertices restrict the number of vertices of a graph that is a solution to $L(G) \simeq \overline{G}$. The finite number of remaining cases are checked in an algorithmic way, resulting in exactly two graphs whose line graphs and edge-complement graphs are isomorphic.

2. Line graph: definition and properties

Definition 1. Let G be a graph. The *line graph* $L(G)$ of G is the graph with vertex set $V(L(G)) = E(G)$ and two vertices $u, v \in V(L(G))$ are connected by an edge in $L(G)$ if and only if their corresponding edges share a common vertex in G. ◂

Example 2. A graph G (left) and its line graph $L(G)$ (right) are shown in Figure 1. Edges of G and their corresponding vertices in $L(G)$ are shown in the same colour. ◂

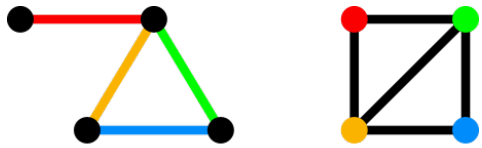

Figure 1: A graph G and its line graph $L(G)$.

The following proposition is an immediate consequence of Definition 1. It relates the number of edges $|E(L(G))|$ and vertices $|V(L(G))|$ in $L(G)$ to the number of edges $|E(G)|$ and vertices $|V(G)|$ in G.

Proposition 3. *Let G be a graph. The* degree *of a vertex is the number of edges attached to that vertex. It holds that $|V(L(G))| = |E(G)|$ and $|E(L(G))| = \frac{1}{2}\sum_{v\in V(G)} \deg v^2 - |E(G)|$.*

Definition 4. A property $\mathcal{P}$ is *preserved under the line graph operation* if it follows from the graph G having property $\mathcal{P}$ that its line graph $L(G)$ also has property $\mathcal{P}$. ◂

The following proposition shows that several properties of graphs are preserved under the line graph operation. We refer to the first chapter of the book [2] for the definitions.

Proposition 5. *Let G be a graph. The following implications are true:*

(i) *If G is connected, then $L(G)$ is connected.*

(ii) *If G is a k-regular graph, then $L(G)$ is a $2(k-1)$-regular graph.*

(iii) *Assume that G and H are two simple graphs. If H is a graph quotient of G via the action of a group $\mathcal{A}$, then $L(H)$ is a graph quotient of $L(G)$ via the action of the same group $\mathcal{A}$.*

Proof. The proofs of the first two statements are direct consequences of Definition 1. For the third statement, note that, by Definition 1, the vertices of $L(G)$ are the edges of G. From this and the assumption that the graph G is simple, it follows that the group $\mathcal{A}$ acts freely on $L(G)$. On the other hand, the assumption that H is a simple graph implies that the action of $\mathcal{A}$ on G and on $L(G)$ is essentially the same. Therefore, a graph morphism is defined between $L(H)$ and $L(G)/\mathcal{A}$. It is straightforward to prove that the morphism is indeed a graph isomorphism. ∎

3. The graph equation $L(G) \simeq \overline{G}$

In this section we compare the line graph with the edge-complement of a graph. We find that, except for two graphs, the line graph is different from the edge-complement.

Definition 6. Let G be a simple graph. The *edge-complement* graph $\overline{G}$ of G is the graph that has the same vertex set as G and two vertices $u, v \in V(\overline{G})$ are connected by an edge in $\overline{G}$ if and only if they are not connected by an edge in G. ◂

Example 7. Figure 2 shows an example of a graph G (left) and its edge-complement $\overline{G}$ (right). ◂

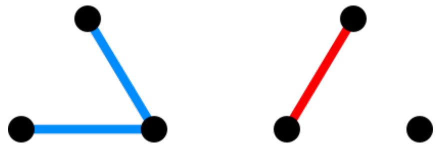

Figure 2: A graph G and its edge-complement $\overline{G}$.

A result similar to Proposition 3 is the following, whose proof follows directly from Definition 6.

Proposition 8. *Let G be a simple graph such that $|V(G)| = n$. If G is not connected, then $\overline{G}$ is connected. Moreover, $|E(\overline{G})| = \binom{n}{2} - |E(G)|$.*

In order to study the relations that exist between the line graph and the edge-complement operations, we focus our attention on the following question: do there exist graphs G with non-empty vertex set which satisfy the equation

$$L(G) \simeq \overline{G}\,? \tag{1}$$

The set of solutions for (1) is not empty, since it is easily found that $G = C_5$, which is the cycle with 5 vertices, is isomorphic to both its line graph and its edge-complement (see Figure 3). In fact, $G = C_5$ is the only regular graph that is a solution to (1).

Figure 3: The graph C_5 fulfills $L(C_5) \simeq C_5 \simeq \overline{C_5}$.

Theorem 9. *The only solutions to the graph equation $L(G) \simeq \overline{G}$ are $G = C_5$ and the graph with six vertices that is drawn left in Figure 4.*

Proof. It follows from the properties of propositions 3, 5 and 8 that a candidate G for a solution to (1) must be connected and must have as many vertices as edges, say $|V(G)| = |E(G)| = n$. That is, if the number of vertices of G grows, the edge-complement graph $\overline{G}$ will have a high number of edges, while the line graph $L(G)$ will not. Thus, focusing on the degrees of vertices of G allows to limit the maximum number of vertices and edges that a G that satisfies (1) is allowed to have.

Indeed, it follows from Definition 6 that G cannot have vertices of degree $n - 1$. If we assume G to be k-regular, from propositions 3 and 8 we obtain that $k^2 n = n(n-1)$ and $nk = 2n$. These two equations are satisfied only if $k = 2$ and $n = 5$. Therefore, the only regular graph which is solution to (1) is C_5.

Thus, we can assume that G is not regular. This assumption implies that G must contain at least one vertex of degree 1. Indeed, if it was not the case, then G would have all vertices of degree at least 2 and at least one vertex of degree at least 3 (because G cannot be regular). This is a contradiction to the handshake lemma (see [2, Theorem 1.1.1]).

The existence of at least one vertex of degree 1 in G implies that $L(G)$ must have at least one vertex of degree $n-2$. For $L(G)$ to contain a vertex of degree $n-2$, there must exist an edge in G with endpoints u and w such that $\deg u + \deg w = n$. However, G has only n edges and vertices, and each vertex is at least of degree 1 since G is connected. Therefore, we are left with two cases: either G has $n-3$ vertices of degree 1 and one vertex of degree 3 in addition to the vertices u and w, or G has $n-4$ vertices of degree 1 and two vertices of degree 2 in addition to u and w. In both cases, since G cannot have vertices of degree $n-1$, it is impossible for G to have more than 7 vertices. This leaves us with a finite number of graphs that are potential solutions and they can be checked individually as shown in the next section for graphs with 6 vertices. It is found that the graph that is drawn left in Figure 4 is the only non-regular solution to equation (1). ∎

3.1. An algorithm for the remaining cases

Graphs with n vertices that are solutions to equation (1) can be found algorithmically as outlined below.

For $L(G) \simeq \overline{G}$ to hold, the number of edges of the graphs must be equal. Propositions 3 and 8 obtain $|E(L(G))|$ and $|E(\overline{G})|$ from $|E(G)|$. Equating these two expressions, one obtains that G must satisfy the following equation:

$$\sum_{v\in V(G)} \binom{\deg v}{2} = \frac{n^2-3n}{2}. \tag{2}$$

From Definition 1, it follows that a vertex of degree d in G corresponds to $\binom{d}{2}$ edges in the line graph $L(G)$. This observation together with formula (2) allows to list all combinations of degrees of vertices that respect (2) for a fixed n. It is then easy to check whether the resulting graphs are solutions to equation (1).

Example 10. This example shows how the algorithm works for $n = 6$. This case will give the only other solution besides $G = C_5$ to the graph equation (1). First, we determine combinatorially the fourteen degree combinations of 6 vertices that satisfy (2), i.e., $\sum_{i=1}^{6} \frac{(\deg v_i)^2 - \deg v_i}{2} = 9$. Among these, it is possible to remove immediately all combinations which contain a zero, as G must be a connected graph (as argued in the proof of Theorem 9). Hence, only four possible combinations of degrees are left:

$$(1): \{4,3,1,1,1,1\} \quad (2): \{4,2,2,2,1,1\} \quad (3): \{3,3,3,1,1,1\} \quad (4): \{3,3,2,2,2,1\}.$$

There is no graph with vertices of the degrees of (1) or (4), because these would require an odd number of vertices of odd degree. From the remaining two cases, only the graph with degrees of (3) is a solution to equation (1). The graph is shown in Figure 4. ◂

Figure 4: The graph G on the left, its edge-complement $\overline{G}$ in the middle, its line graph $L(G)$ on the right. This is the only graph on six vertices with $L(G) \simeq \overline{G}$.

References

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