3rd *BYMAT Conference* (2020)

Subharmonics in a class of planar periodic predator-prey models

Eduardo Muñoz-Hernández
Universidad Complutense de Madrid
Instituto de Matemática Interdisciplinar
eduardmu@ucm.es

Abstract: This contribution studies the existence of positive subharmonics of arbitrary order in the planar periodic Volterra predator-prey model. When the model is *non-degenerate*, in the sense that the birth rate of the prey intersects the support of the death rate of the predator, as in [8], then the existence of positive subharmonics can be derived from the Poincaré–Birkhoff theorem version [3]. Nevertheless, in the *degenerate* case when these supports do not intersect, then, the Poincaré–Birkhoff theorem fails in general. Still in these degenerate situations, the techniques of [7] provide us with the existence of positive subharmonics of arbitrary order.

This is based on a joint work with Julián López-Gómez (UCM) and Fabio Zanolin (UNIUD).

Resumen: Este trabajo analiza la existencia de subarmónicos positivos de orden arbitrario en el modelo plano periódico de presa y depredador de Volterra. Cuando el modelo es *no degenerado*, en el sentido de que la tasa de natalidad de la presa interseca el soporte de la tasa de mortalidad del depredador, como en [8], entonces la existencia de subarmónicos positivos puede ser derivada mediante un la versión del teorema de Poincaré–Birkhoff que se establece en [3]. Sin embargo, en el caso *degenerado* cuando los soportes no intersecan, el teorema de Poincaré–Birkhoff no puede aplicarse directamente. En estos casos, las técnicas de [7] nos proporcionan la existencia de subarmónicos positivos de orden arbitrario.

Esta colaboración está basada en un trabajo conjunto con Julián López-Gómez (UCM) y Fabio Zanolin (UNIUD).

Keywords: periodic predator-prey model of Volterra type, subharmonic coexistence states, Poincaré–Birkhoff twist theorem, degenerate versus non-degenerate models.

MSC2010: 34C25, 37B55, 37E40.

Acknowledgements: This note has been written under the auspices of the the Contract CT42/18-CT43/18 of Complutense University of Madrid. The author would like to thank his PhD advisors Julián López-Gómez and Fabio Zanolin for their constant support and the beautiful mathematics they have taught him.

Reference: Muñoz-Hernández, Eduardo. "Subharmonics in a class of planar periodic predator-prey models". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 31-34. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p31.

1. Introduction

In this contribution, we analyze the existence of positive subharmonics of arbitrary order (nT-periodic coexistence states) of the periodic Volterra predator-prey model

$$\begin{cases} u' = \lambda\alpha(t)u(1-v), \\ v' = \lambda\beta(t)v(-1+u), \end{cases} \tag{1}$$

where $\lambda > 0$ is a real parameter, and, for some $T > 0$, $\alpha(t)$ and $\beta(t)$ are T-periodic real continuous functions. We set

$$A := \int_0^T \alpha(s)\,\mathrm{d}s \quad \text{and} \quad B := \int_0^T \beta(s)\,\mathrm{d}s.$$

They can arise two different cases according to whether, or not, the following condition holds

$$\operatorname{supp}\alpha \cap \operatorname{supp}\beta \neq \emptyset. \tag{2}$$

In this *non-degenerate* situation the existence of subharmonics of arbitrary order can be obtained through an updated version of the celebrated Poincaré–Birkhoff twist theorem for sufficiently large λ. Nevertheless, in the *degenerate* case when the next condition holds

$$\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset \tag{3}$$

the Poincaré–Birkhoff theorem is unable to provide, in general, with subharmonics of arbitrary order, unless $\alpha(t)$ and $\beta(t)$ have some special nodal structure.

2. The non-degenerate case

The non-degenerate case when (2) is satisfied has been studied in [8] by adapting some original ideas in [3] (later revised and applied in [2]), where a Poincaré–Birkhoff version for Hamiltonian systems was presented. Through the change of variables

$$x = \log u, \qquad y = \log v,$$

(1) is transformed into the planar Hamiltonian system

$$\begin{cases} x' = -\lambda\alpha(t)(\mathrm{e}^y - 1), \\ y' = \lambda\beta(t)(\mathrm{e}^x - 1). \end{cases} \tag{4}$$

The [3] version of the Poincaré–Birkhoff twist theorem that we will use reads as follows:

Theorem 1 (Poincaré–Birkhoff). *Assume that there exist $0 < \varrho_0 < \varrho_1$ and a positive integer ω such that*

$$\operatorname{rot}_{\varrho_0}[(x_0, y_0); [0, nT]] > \omega \quad \textit{and} \quad \operatorname{rot}_{\varrho_1}[(x_0, y_0); [0, nT]] < \omega,$$

where

$$\operatorname{rot}_{\rho}[(x_0, y_0); [0, nT]] = \frac{\theta(nT) - \theta(0)}{2\pi}$$

with $\|(x_0, y_0)\| = \rho$, $\theta(t)$ being the angular polar coordinate of the solution starting at (x_0, y_0), $(x(t), y(t))$. Then, (4) admits, at least, two nT-periodic solutions lying in different periodicity classes with rotation number ω.

As a consequence of Theorem 1, we get the next result:

Theorem 2. *Assume (2). Then, for every positive integers ω and n, there exists $\lambda_n^{\omega} > 0$ such that (4) possesses, at least, two nT-periodic solutions with rotation number ω for every $\lambda > \lambda_n^{\omega}$.*

Proof. First, we focus attention into the small solutions of (4). There exists $\varepsilon > 0$ such that

$$(\mathrm{e}^{\xi} - 1)\xi \geq \frac{\xi^2}{2} \quad \text{if } |\xi| < \varepsilon. \tag{5}$$

It can be chosen an initial data $(x(0), y(0)) = (x_0, y_0)$ sufficiently close to $(0, 0)$, say $|(x_0, y_0)| \leq \varrho_0$, so that the solution of (4), $(x(t), y(t))$, satisfy $|(x(t), y(t))| < \varepsilon$ for all $t \in [0, nT]$. This is possible by continuous dependence on the initial conditions. According to (2), there are $\tau \in (0, T)$ and $\delta > 0$ such that $\alpha(t)\beta(t) > 0$ for every $t \in [\tau - \delta, \tau + \delta] \subsetneq [0, T]$. Thus,

$$\zeta := \min_{t\in[\tau-\delta,\tau+\delta]} \{\alpha(t), \beta(t)\} > 0. \tag{6}$$

Consequently, due to (4), (5) and (6), we obtain that, for every $t \in [0, nT]$,

$$\theta'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x^2(t) + y^2(t)} = \frac{\lambda\beta(t)(\mathrm{e}^{x(t)} - 1)x(t) + \lambda\alpha(t)(\mathrm{e}^{y(t)} - 1)y(t)}{x^2(t) + y^2(t)} \geq \frac{\lambda\zeta}{2}. \tag{7}$$

Hence, owing to (7),

$$\mathrm{rot}_{\varrho_0}[(x_0, y_0); [0, nT]] = \frac{\theta(nT) - \theta(0)}{2\pi} = \frac{1}{2\pi}\int_0^{nT} \theta'(s)\,\mathrm{d}s \geq \frac{n}{2\pi}\int_{\tau-\delta}^{\tau+\delta} \theta'(s)\,\mathrm{d}s \geq \frac{n\lambda\zeta 2\delta}{2\pi} > \omega$$

if $\lambda > \frac{\pi\omega}{n\zeta\delta} =: \lambda_n^{\omega}$.

On the other hand, solutions with sufficiently large initial data do not rotate (see, for further details, Theorem 2.2 of [8]). Hence, the hypothesis of Theorem 1 holds for every $\lambda > \lambda_n^{\omega}$, which ends the proof. ∎

3. The degenerate case

To analyze the problem (1) under the condition (3), we suppose that

$$\operatorname{supp}\alpha \subseteq [0, \tfrac{T}{2}] \quad \text{and} \quad \operatorname{supp}\beta \subseteq [\tfrac{T}{2}, T]. \tag{8}$$

In case (8), introduced in [5], we have that, for every $t \in [0, T]$,

$$u(t) = u_0\mathrm{e}^{(1-v_0)\lambda\int_0^t \alpha(s)\,\mathrm{d}s}, \quad v(t) = v_0\mathrm{e}^{(u(T)-1)\lambda\int_0^t \beta(s)\,\mathrm{d}s},$$

Hence, the T-time Poincaré map is

$$(u_1, v_1) := \mathcal{P}_1(u_0, v_0) := (u(T), v(T)) = (u_0\mathrm{e}^{(1-v_0)\lambda A}, v_0\mathrm{e}^{(u_1-1)\lambda B}).$$

Consequently, iterating n times this map, it becomes apparent that

$$\begin{aligned}(u_n, v_n) := \mathcal{P}_n(u_0, v_0) = \mathcal{P}_1^n(u_0, v_0) &:= (u(nT), v(nT)) = (u_{n-1}\mathrm{e}^{(1-v_{n-1})\lambda A}, v_{n-1}\mathrm{e}^{(u_n-1)\lambda B}) \\ &= (u_0\mathrm{e}^{(n-v_0-v_1-\cdots-v_{n-1})\lambda A}, v_0\mathrm{e}^{(u_1+u_2+\cdots+u_n-n)\lambda B}).\end{aligned} \tag{9}$$

By the uniqueness for the underlying Cauchy problem, the nT-periodic coexistence states of (1) are given by the positive fixed points of $\mathcal{P}_n$. Thus, by (9), we are driven to solve the system

$$\begin{cases} n &= u_0 + u_1 + \cdots + u_{n-1}, \\ n &= v_0 + v_1 + \cdots + v_{n-1}. \end{cases} \tag{10}$$

The next result proves the existence and multiplicity of nT-periodic coexistence states of (1) when $n \geq 2$ in case (8). To get it, we impose the following condition:

$$A = B \quad \text{and} \quad u_0 = v_0 = x. \tag{11}$$

Theorem 3. *Assume (8) and (11). Then, for every $\lambda > \frac{2}{A}$, (1) admits, at least, n coexistence states with period nT if n is even, and $n-1$ coexistence states with period nT if n is odd.*

Proof. By (11), it turns out that, given $\varphi_1(x) = x - 1$,

$$\varphi_n(x) = \varphi_{n-1}(x) - 1 + E_{n-1}(x),$$

is the map whose zeros provide us with the nT-periodic coexistence states of (1), where $E_n(x)$ is a sequence of exponential functions. In order to obtain some information concerning the nT-periodic coexistence states of (1), we analyze the variational equations of these maps at the trivial curve $(\lambda, 1)$,

$$p_n(\lambda) := \frac{\partial \varphi_n}{\partial x}(\lambda, 1).$$

It is easy to prove that $p_n(\lambda)$ is a sequence of polynomials in the indeterminate λ that satisfy the recursive formula

$$p_n(\lambda) = [2 - (-1)^n A\lambda] p_{n-1}(\lambda) - p_{n-2}(\lambda),$$

where $p_1(\lambda) = 1$ and $p_2(\lambda) = 2 - A\lambda$. From this recursive formula, it can be shown that any root of p_n is real and algebraically simple. Thanks to these facts, for any given $r \in p_n^{-1}(0)$, the transversality condition of Crandall-Rabinowitz [1] holds. Thus, for any given $r \in p_n^{-1}(0)$, the algebraic multiplicity of Esquinas and López-Gómez [4] equals one at every point $(r, 1)$. So, according to Crandall and Rabinowitz [1, Th. 1.7], a local bifurcation occurs at every point $(r, 1)$. Moreover, by the unilateral theorem of López-Gómez [6, Th. 6.4.3], the underlying subcomponents of nT-periodic coexistence states are unbounded in λ. As the number of positive roots of $p_n(\lambda)$ equals $\frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd, the result holds. This ends the proof. ∎

References

[1] Crandall, Michael G. and Rabinowitz, Paul H. "Bifurcation from simple eigenvalues". In: *J. Functional Analysis* 8 (1971), pp. 321–340. https://doi.org/10.1016/0022-1236(71)90015-2.

[2] Ding, Tongren and Zanolin, Fabio. "Periodic solutions and subharmonic solutions for a class of planar systems of Lotka-Volterra type". In: *World Congress of Nonlinear Analysts '92, Vol. I–IV (Tampa, FL, 1992)*. de Gruyter, Berlin, 1996, pp. 395–406.

[3] Ding, Wei Yue. "Fixed points of twist mappings and periodic solutions of ordinary differential equations". In: *Acta Mathematica Sinica. Shuxue Xuebao* 25.2 (1982), pp. 227–235. issn: 0583-1431.

[4] Esquinas, Jesús and López-Gómez, Julián. "Optimal multiplicity in local bifurcation theory. I. Generalized generic eigenvalues". In: *Journal of Differential Equations* 71.1 (1988), pp. 72–92. issn: 0022-0396. https://doi.org/10.1016/0022-0396(88)90039-3.

[5] López-Gómez, Julián. "A bridge between operator theory and mathematical biology". In: *Operator theory and its applications (Winnipeg, MB, 1998)*. Vol. 25. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 2000, pp. 383–397.

[6] López-Gómez, Julián. *Spectral theory and nonlinear functional analysis.* Vol. 426. Chapman & Hall/CRC Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2001. https://doi.org/10.1201/9781420035506.

[7] López-Gómez, Julián and Muñoz-Hernández, Eduardo. "Global structure of subharmonics in a class of periodic predator-prey models". In: *Nonlinearity* 33.1 (2020), pp. 34–71. issn: 0951-7715. https://doi.org/10.1088/1361-6544/ab49e1.

[8] López-Gómez, Julián; Muñoz-Hernández, Eduardo, and Zanolin, Fabio. "On the applicability of the Poincaré-Birkhoff twist theorem to a class of planar periodic predator-prey models". In: *Discrete and Continuous Dynamical Systems. Series A* 40.4 (2020), pp. 2393–2419. issn: 1078-0947. https://doi.org/10.3934/dcds.2020119.

[9] López-Gómez, Julián; Ortega, Rafael, and Tineo, Antonio. "The periodic predator-prey Lotka-Volterra model". In: *Advances in Differential Equations* 1.3 (1996), pp. 403–423. issn: 1079-9389.