

# Integrability and rational soliton solutions for gauge invariant derivative nonlinear Schrödinger equations

✉ Paz Albares

Universidad de Salamanca

paz.albares@usal.es

**Abstract:** The present work addresses the study and characterization of the integrability of three famous nonlinear Schrödinger equations with derivative-type nonlinearities in  $1 + 1$  dimensions. Lax pairs for these three equations are successfully obtained by means of a Miura transformation and the singular manifold method. After implementing the associated binary Darboux transformations, we are able to construct rational soliton-like solutions for those systems.

**Resumen:** El presente trabajo aborda el estudio y caracterización de la integrabilidad de tres célebres ecuaciones diferenciales tipo Schrödinger no lineal con no linealidades que incluyen términos en derivadas. Mediante el método de la variedad singular, junto con una transformación de Miura, se obtienen las expresiones de los pares de Lax para dichas ecuaciones. A través del formalismo de las transformaciones binarias de Darboux, se consigue construir soluciones solitónicas racionales para estos tres sistemas.

**Keywords:** integrability, derivative nonlinear Schrödinger equation, Lax pair, rational solitons.

**MSC2010:** 35C08, 35Q55, 37J35.

**Acknowledgements:** The author would like to thank professors P. G. Estévez and J. D. Lejarreta for their useful comments and collaboration. This research has been supported by the predoctoral fellowship FPU17/03246 and MICINN (Grant PID2019-106820RB-C22).

**Reference:** ALBARES, Paz. “Integrability and rational soliton solutions for gauge invariant derivative nonlinear Schrödinger equations”. In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 35-38. ISSN: 2660-6003. URL: <https://temat.es/monograficos/article/view/vol2-p35>.

## 1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most famous integrable equations in soliton theory and mathematical physics. Among the several integrable generalizations of NLS, we are interested in the study of modified NLS systems with derivative-type nonlinearities in  $1 + 1$  dimensions, which are known as derivative nonlinear Schrödinger (DNLS) equations. There exist three celebrated equations of this kind, *i.e.*, the Kaup-Newell (KN) system [4],

$$(1) \quad im_t - m_{xx} - i(|m|^2 m)_x = 0,$$

the Chen-Lee-Liu (CLL) equation [2],

$$(2) \quad im_t - m_{xx} - i|m|^2 m_x = 0,$$

and the Gerdjikov-Ivanov (GI) equation [3]

$$(3) \quad im_t - m_{xx} + im^2 \bar{m}_x - \frac{1}{2}|m|^4 m = 0,$$

where  $m$  is a complex valued function and  $\bar{m}$  denotes the complex conjugate of  $m$ .

It is already known that these three equations are equivalent via a  $U(1)$ -gauge transformation [5]. If  $m(x, t)$  is a solution of the KN system (1), it is easy to find that the new field  $M(x, t)$

$$(4) \quad M(x, t) = m(x, t)e^{\frac{i\gamma}{2}\theta(x,t)}, \quad \text{with } \theta_x = |m|^2, \quad \theta_t = i(m\bar{m}_x - \bar{m}m_x) + \frac{3}{2}|m|^4,$$

satisfies the CLL equation for  $\gamma = 1$ , and the GI equation for  $\gamma = 2$ .

Gauge transformations constitute an useful tool to link integrable evolution equations in soliton theory, since they provide Bäcklund transformations between those equations as well as the relation of their associated linear problems [6]. In this contribution we exploit this gauge invariance property to construct a Lax pair and rational soliton solutions for these three equations. For a detailed analysis and explicit calculations, we refer the reader to [1].

## 2. Integrability and Lax pair

The Painlevé test [7] has been proved to be a powerful criterion for the identification of integrable partial differential equations (PDEs). A PDE is said to possess the Painlevé property, frequently considered as a proof of integrability, when its solutions are single-valued about the movable singularity manifolds. This requires the generalized Laurent expansion for the field  $m(x, t) = \sum_{j=0}^{\infty} a_j(x, t)\phi(x, t)^{j-\mu}$ , where  $\phi(x, t)$  is an arbitrary function called the singular manifold and the index  $\mu \in \mathbb{N}$  is an integer.

The Painlevé test is unable to check the integrability of any DNLS equation since the leading index is not integer,  $\mu = \frac{1}{2}$ . This fact allows us to introduce two new real fields  $\alpha(x, t)$ ,  $\beta(x, t)$

$$(5) \quad m(x, t) = \sqrt{2\alpha_x} e^{\frac{i}{2}\beta(x,t)}, \quad \text{with } \alpha_x = \frac{1}{2}|m|^2, \quad \beta = (2\gamma - 3)\alpha + \int \frac{\alpha_t}{\alpha_x} dx,$$

with  $\gamma = 0$  for the KN system,  $\gamma = 1$  for the CLL equation and  $\gamma = 2$  for the GI equation. This ansatz yields an identical differential equation for  $\alpha$  in each case, expressed in the conservative form

$$(6) \quad [\alpha_x^2 - \alpha_t]_t = \left[ \alpha_{xxx} + \alpha_x^3 - \frac{\alpha_t^2 + \alpha_{xx}^2}{\alpha_x} \right]_x.$$

From expression (4), it can be easily seen that the probability density  $\theta_x = |m|^2 = |M|^2$  is invariant under a  $U(1)$ -gauge transformation, indeed it constitutes the first conservation law for these systems. Due to this symmetry, it is straightforward to see that once we obtain a soliton solution for a particular DNLS equation, it is immediate to derive soliton solutions for any DNLS equation linked by a  $U(1)$ -gauge transformation.

Since  $\alpha_x = \frac{\theta_x}{2}$ , we may conclude that equation (6) is the representative equation for the probability density of any DNLS equation. Equation (6) passes the Painlevé test, but it possesses two branches of expansion. The best method to overcome this inconvenience requires the splitting of the field  $\alpha$  as

$$(7) \quad \alpha = i(u - \bar{u}), \quad \alpha_x^2 - \alpha_t = u_{xx} + \bar{u}_{xx}.$$

The combination of equations in (7) yields two Miura transformations for  $\{u, \bar{u}\}$  and the condition

$$(8) \quad u_{xx} = \frac{1}{2}(\alpha_x^2 - \alpha_t - i\alpha_{xx}), \quad \bar{u}_{xx} = \frac{1}{2}(\alpha_x^2 - \alpha_t + i\alpha_{xx}), \quad iu_t + u_{xx} - i\bar{u}_t + \bar{u}_{xx} + (u_x - \bar{u}_x)^2 = 0,$$

which finally lead to the same nonlocal Boussinesq equation for both  $u(x, t)$  and  $\bar{u}(x, t)$ , of the form

$$(9) \quad \left[ u_{tt} + u_{xxxx} + 2u_{xx}^2 - \frac{u_{xt}^2 + u_{xxx}^2}{u_{xx}} \right]_x = 0.$$

Equation (9) has the Painlevé property with an unique branch of expansion. Hence, this equation is conjectured integrable and it is possible to derive an equivalent linear spectral problem associated to the nonlinear equation (9). This aim may be achieved by means of the singular manifold method (SMM). The SMM [7] focuses on solutions which emerge from the truncated Painlevé series, as auto-Bäcklund transformations of the form  $u^{[1]} = u^{[0]} + \log(\phi)$ . Thus, the singular manifold  $\phi$  is no longer an arbitrary function, since it satisfies the singular manifold equations. The associated linear problem arises from the linearization of these equations, and it can be demonstrated that the Lax pair for  $u$  reads [1]

$$(10) \quad \begin{aligned} \psi_{xx} &= \left( \frac{u_{xxx}^{[0]} - iu_{xt}^{[0]}}{2u_{xx}^{[0]}} - i\lambda \right) \psi_x - u_{xx}^{[0]} \psi, & \psi_t &= i\psi_{xx} - 2\lambda\psi_x + i(2u_{xx}^{[0]} + \lambda^2)\psi, \\ \chi_{xx} &= \left( \frac{u_{xxx}^{[0]} + iu_{xt}^{[0]}}{2u_{xx}^{[0]}} + i\lambda \right) \chi_x - u_{xx}^{[0]} \chi, & \chi_t &= -i\chi_{xx} - 2\lambda\chi_x - i(2u_{xx}^{[0]} + \lambda^2)\chi, \end{aligned}$$

where  $\{\chi, \psi\}$  are two complex conjugated eigenfunctions satisfying  $\frac{\psi_x \chi_x}{\psi \chi} + u_{xx}^{[0]} = 0$  and  $\lambda$  is the spectral parameter. From (10), we may compute the Lax pair for the DNLS equations, obtaining

$$(11) \quad \begin{aligned} \chi_{xx} &= \left[ i\lambda - \frac{i(\gamma - 2)}{2} |m^{[0]}|^2 + \frac{m_x^{[0]}}{m^{[0]}} \right] \chi_x + \frac{1}{2} \left[ im^{[0]} \bar{m}_x^{[0]} - \frac{\gamma - 1}{2} |m^{[0]}|^4 \right] \chi, \\ \chi_t &= i\chi_{xx} - \left[ (\gamma - 2) |m^{[0]}|^2 + \frac{2im_x^{[0]}}{m^{[0]}} \right] \chi_x - i\lambda^2 \chi, \end{aligned}$$

and its complex conjugate, for the corresponding value of  $\gamma$ . It is worthwhile to remark that the condition  $\frac{\psi_x \chi_x}{\psi \chi} - \frac{i}{2} m^{[0]} \bar{m}_x^{[0]} + \frac{\gamma - 1}{4} |m^{[0]}|^4 = 0$  allows us to determine an equivalent Lax pair for those systems.

### 3. Rational soliton solutions

Once the Lax pair have been obtained for a given PDE by means of the SMM, binary Darboux transformations can be constructed in order to obtain iterated solutions for that PDE. We implement the Darboux transformation formalism over the spectral problem (10) so as to provide a general iterative procedure to compute up to the  $n$ th iteration for  $u$ . By virtue of expressions (4), (5) and (7), solutions for the DNLS equations can be forthrightly established. Thus, soliton solutions for DNLS equations may be derived by considering a suitable choice for the seed solution and the eigenfunctions in the Lax pair.

In the following lines we summarize the main results regarding this procedure, oriented to the obtention of rational soliton solutions. Further details and a general rigorous analysis may be found in [1].

We start from a polynomial seed solution  $u^{[0]}$  for (9) and binary exponential eigenfunctions for (10),

$$(12) \quad u^{[0]} = -\frac{j_0^2}{4} \left[ j_0^2 z_0^2 x \left( \frac{x}{2} + j_0^2 (z_0^2 + 1)t \right) i \left( x + j_0^2 \left( z_0^2 + \frac{1}{2} \right) t \right) \right], \quad \chi_\sigma = e^{\frac{i}{2} j_0^2 z_0 \sigma} \left[ x + j_0^2 \left( -\frac{\sigma}{2z_0} (z_0^4 + 7z_0^2 + 1) + 3(z_0^2 + 1) \right) t \right],$$

where  $\psi_\sigma = \bar{\chi}_\sigma$ ,  $j_0$  and  $z_0$  are arbitrary parameters,  $\sigma = \pm 1$  and  $\lambda_\sigma = \frac{j_0^2}{2}(2\sigma z_0 - (z_0^2 + 1))$ . The first and second iterations  $u^{[j]}$ ,  $j = 1, 2$  can be performed and the soliton solution profile may be computed as  $|m^{[j]}|^2 = 2i(u_x^{[j]} - \bar{u}_x^{[j]})$ . The results are displayed in Figure 1.

- The first iteration ( $j = 1$ ) provides a rational soliton-like travelling wave along the  $x - j_0^2(\sigma z_0 - (z_0^2 + 1))t$  direction and constant amplitude, of expression

$$(13) \quad |m_\sigma^{[1]}|^2 = j_0^2 - \frac{4}{j_0^2 z_0 (\sigma - z_0) \left[ (x - j_0^2 (\sigma z_0 - (z_0^2 + 1)) t)^2 + \frac{1}{j_0^4 z_0^2 (\sigma - z_0)^2} \right]}, \quad \sigma = \pm 1.$$

- For the second iteration ( $j = 2$ ), we get the two-soliton solution

$$(14) \quad |m^{[2]}|^2 = j_0^2 + \frac{8 \left[ (x + j_0^2 (z_0^2 + 2) t)^2 + j_0^4 (z_0^2 - 1) t^2 + \frac{1}{j_0^4 (z_0^2 - 1)} \right]}{j_0^2 (z_0^2 - 1) \left[ \left( (x + j_0^2 (z_0^2 + 1) t)^2 - j_0^4 z_0^2 t^2 - \frac{1}{j_0^4 (z_0^2 - 1)} \right)^2 + \frac{4(x + j_0^2 (z_0^2 + 2) t)^2}{j_0^4 (z_0^2 - 1)^2} \right]}.$$

leading to a two asymptotically travelling rational solitons of the form (13) (for  $\sigma = 1$  and  $\sigma = -1$ , respectively) interacting at the origin.

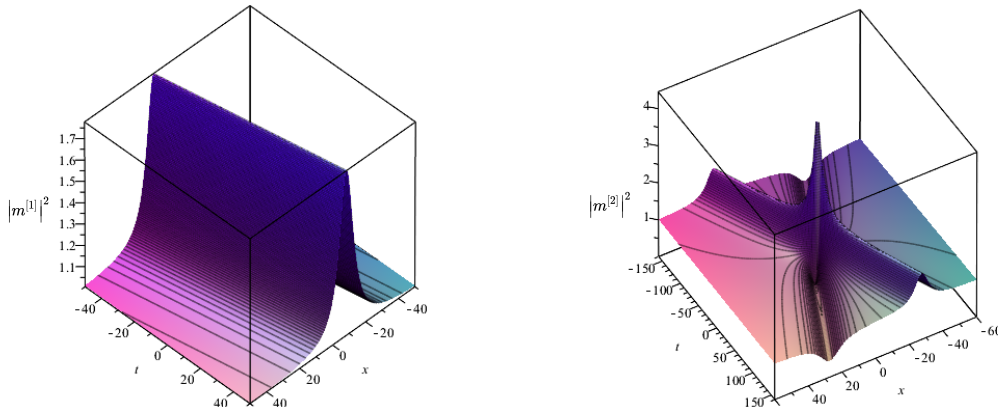


Figure 1: Spatio-temporal plot of  $|m^{[1]}|^2$  and  $|m^{[2]}|^2$  for parameters  $\sigma = -1$ ,  $j_0 = 1$ ,  $z_0 = \frac{1}{6}$ .

## References

- [1] ALBARES, P; GARCÍA ESTÉVEZ, P, and DOMINGO LEJARRETA, J. “Derivative non-linear Schrödinger equation: Singular manifold method and Lie symmetries”. In: *arXiv e-prints* (2020). arXiv: 2010.09411 [nlin.SI].
- [2] CHEN, H. H.; LEE, Y. C., and LIU, C. S. “Integrability of Nonlinear Hamiltonian Systems by Inverse Scattering Method”. In: *Phys. Scr.* 20 (1979), pp. 490–492.
- [3] GERDIKOV, V. and IVANOV, I. “A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures”. In: *Bulg. J. Phys.* 10 (1983), pp. 130–143.
- [4] KAUP, D. J. and NEWELL, A. C. “An exact solution for a derivative nonlinear Schrödinger equation”. In: *J. Math. Phys.* 19 (1978), pp. 798–801.
- [5] KUNDU, A. “Landau-Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger-type equations”. In: *J. Math. Phys.* 25 (1984), pp. 3433–3438.
- [6] WADATI, M. and SOGO, K. “Gauge Transformations in Soliton Theory”. In: *J. Phys. Soc. Jpn.* 52 (1983), pp. 394–398.
- [7] WEISS, J. “The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative”. In: *J. Math. Phys.* 24 (1983), pp. 1405–1413.