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Character varieties of torus knots

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Abstract: Attached to any topological space X we find its character variety. This is an algebraic variety parametrizing isomorphism classes of representations $\pi_1(X) \to G$ of the fundamental group of X into an algebraic reductive group G. These spaces are particularly useful in classical knot theory, since they provide very subtle invariants of knot $K \subset \mathbb{R}^3$ by taking $X = \mathbb{R}^3 - K$. However, even in the simplest cases a full understanding of these character varieties is an open problem. In this paper, we compute the motif of the irreducible character variety of representations of the fundamental group of the complement of an arbitrary torus knot into $G = \mathrm{SL}_4(k)$. For that purpose, we introduce a stratification of the variety in terms of the type of a canonical filtration attached to any representation. This allows us to reduce the computation of the virtual class to a purely combinatorial problem.

Resumen: Asociado a cada espacio topológico X tenemos su variedad de caracteres. Esta es una variedad algebraica que parametriza las clases de isomorfismo de representaciones $\pi_1(X) \to G$ del grupo fundamental de X en un grupo algebraico reductivo G. Estos espacios resultan especialmente útiles en teoría de nudos clasica, pues proveen de invariates muy sutiles de nudos $K \subset \mathbb{R}^3$ al tomar $X = \mathbb{R}^3 - K$. A pesar de esta importancia, incluso en los casos más simples el entendimiento completo de estas variedades de caracteres es un problema abierto. En este artículo, calculamos el motivo de la variedad de caracteres irreducible de representaciones del grupo fundamental de un nudo toroidal arbitrario en $G = \mathrm{SL}_4(k)$. Para este fin, introducimos una estratificación de la variedad en términos del tipo de una filtración canónica asociada a cada representación. Esto permite reducir el cálculo de la clase virtual a un problema puramente combinatorio.

Keywords: torus knot, character varieties, representations.

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1. Motivic theory of character varieties

Let Γ be a finitely generated group and let G be a reductive linear algebraic group over an algebraically closed field k. The space $R(\Gamma, G)$ of representations $\rho : \Gamma \to G$ forms an algebraic variety known as the *G-representation variety*. Additionally, consider the open subset $R^{\mathrm{irr}}(\Gamma, G) \subseteq R(\Gamma, G)$ of irreducible representations. By Schur's lemma the adjoint action of G by conjugation on $R^{\mathrm{irr}}(\Gamma, G)$ is closed and its stabilizer at any point is the center of G. Therefore, the orbit space

$$\mathfrak{M}^{\mathrm{irr}}(\Gamma, G) = R^{\mathrm{irr}}(\Gamma, G)/G$$

is an algebraic variety known as the *irreducible G-character variety*. These varieties play a prominent role in the topology of 3-manifolds, starting with the foundational work of Culler and Shalen [1] on the study of hyperbolic geometry via $\mathrm{SL}_2(\mathbb{C})$-character varieties. Due to its importance, the algebro-geometric properties of character varieties have been widely studied, particularly regarding their motivic class.

Definition 1. The *Grothendieck ring of algebraic varieties $K\mathcal{V}ar_k$* is the ring generated by isomorphism classes of algebraic varieties $[X]$, called virtual classes or motives in this context, with the relations $[X_1 \sqcup X_2] = [X_1] + [X_2]$ and $[X_1 \times X_2] = [X_1] \cdot [X_2]$ for any algebraic varieties X_1 and X_2. ◂

Remark 2. Great efforts have been made to compute the virtual classes $[\mathfrak{M}^{\mathrm{irr}}(\Gamma, G)] \in K\mathcal{V}ar_k$. Three approaches are proposed in the literature: the arithmetic viewpoint [4], the geometric perspective [6] and through Topological Quantum Field Theories [3]. ◂

An useful tool for studying the geometry of the character variety is the so- called *semi-simple filtration*. This is the analogue of the composition series or the Harder-Narasimhan filtration in the representation theoretic framework. Working similarly to the Jordan-Hölder theorem, we get the following result.

Proposition 3. *Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation. There exists an unique filtration of Γ-modules*

$$0 = V_0 \subset V_1 \subset \ldots \subset V_i \subset \ldots \subset V_s = V,$$

such that $\mathrm{Gr}_i(V_\bullet) = V_i/V_{i-1}$ is a maximally semi-simple subrepresentation of V/V_{i-1}.

By restriction, the semi-simple filtration also exists for representations onto any linear group G. Thanks to this filtration, we can decompose the graded pieces of a representation into its isotypic components $\mathrm{Gr}_i(V_\bullet) \cong \bigoplus_{j=1}^{s_i} W_{i,j}^{m_{i,j}}$, with $W_{i,1}, \ldots, W_{i,s_i}$ non-isomorphic representations. From this information, we define the *shape* of the representation as the tuple collecting of dimensions and multiplicities of this decomposition $\xi = (\{(\dim W_{i,j}, m_{i,j})\}_j)_i$.

Moreover, we can add spectral information to the shape. For each $\gamma \in \Gamma$, denote by $\sigma_{i,j}(\gamma)$ the collection of eigenvalues of $\rho(\gamma) \in \mathrm{End}(W_{i,j})$ and set $\sigma = (\sigma_{i,j}(\gamma))$. The pair $\tau = (\xi, \sigma)$ is called the *type* of the representation and it is invariant under the adjoint action. Writing $\mathcal{T}(\Gamma, G)$ for the space of possible types arising in representations $\Gamma \to G$, we get a natural map

$$\Phi : R(\Gamma, G) \to \mathcal{T}(\Gamma, G)$$

assigning each representation to its underlying type. Also set $\mathcal{T}^{\mathrm{irr}}(\Gamma, G)$ for the types of irreducible representations, all of which have the same shape. The map Φ restricts to $\Phi : R^{\mathrm{irr}}(\Gamma, G) \to \mathcal{T}^{\mathrm{irr}}(\Gamma, G)$. Notice that if $\mathcal{T}^{\mathrm{irr}}(\Gamma, G)$ is finite, the morphism Φ induces a stratification of $R^{\mathrm{irr}}(\Gamma, G)$.

2. Character varieties of torus knots

Given a knot $K \subset \mathbb{R}^3$, it natural to study the fundamental group of its complement $\pi_1(\mathbb{R}^3 - K)$. An important case arises when $K = K_{n,m}$ is the (n, m)-torus knot ($\gcd(n, m) = 1$) whose fundamental group of the complement is $\Gamma_{n,m} = \pi_1(\mathbb{R}^3 - K_{n,m}) = \langle x, y \mid x^n = y^m \rangle$. Using the image of the generators x, y to identify a representation, we get that the representation variety is

$$R(\Gamma_{n,m}, G) = \{(A, B) \in G \mid A^n = B^m\}.$$

The G-character varieties of torus knots have been studied for $G = \mathrm{SL}_2(\mathbb{C})$ [5, 8], $G = \mathrm{SL}_3(\mathbb{C})$ [9] and $G = \mathrm{SU}(2)$ [7], among others. However, very little is known in the higher rank case $G = \mathrm{SL}_r(k)$ for $r \geq 4$. A key observation towards this aim is the following.

Proposition 4. *Any irreducible representation $\rho : \Gamma_{n,m} \to G$ lifts, up to rescalling, to a representation $\tilde{\rho} : \mathbb{Z}_n \star \mathbb{Z}_m \to G$.*

Proof. Set $P = A^n = B^m$. Trivially $PA = AP$ and $PB = BP$, so $P^{-1}\rho P = \rho$. Thus, P is a Γ-equivariant automorphism of ρ which, by Schur's lemma, implies that $P = \varpi \mathrm{Id}$ for some $\varpi \in k^*$. ∎

Corollary 5. *$\mathcal{T}^{\mathrm{irr}}(\Gamma_{n,m}, \mathrm{SL}_r(k))$ is finite.*

Proof. In this case the scalling factor $\varpi \in k^*$ of Proposition 4 must satisfy $\varpi^r = 1$, so there are only finitely many posibilities. Thus, it is enough to show that $\mathcal{T}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m, \mathrm{SL}_r(k))$ is finite. If $(A, B) \in R^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m, \mathrm{SL}_r(k))$, then $A \in R(\mathbb{Z}_n, \mathrm{SL}_r(k))$ and $B \in R(\mathbb{Z}_m, \mathrm{SL}_r(k))$ so they are diagonalizable in so far as representations of finite abelian groups. Moreover, $A^n = B^m = \mathrm{Id}$ so the eigenvalues of A and B must be roots of unit. These are finitely many for fixed n, m, implying that $\mathcal{T}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m, \mathrm{SL}_r(k))$ is finite. ∎

From now on, we focus on $G = \mathrm{SL}_r(k)$, $r \geq 1$, as target group so we will omit it from the notation. Fixed a spectrum $\kappa = (\sigma_A, \sigma_B)$ for the matrices of a representation $(A, B) \in R(\Gamma_{n,m})$, let us denote by $\mathcal{T}_\kappa$ the set of types $\tau = (\xi, \sigma) \in \mathcal{T}(\Gamma_{n,m})$ whose spectral data σ are drawn from κ. Set $\mathcal{T}_\kappa^{\mathrm{irr}} = \mathcal{T}_\kappa \cap \mathcal{T}^{\mathrm{irr}}(\Gamma_{n,m})$, $\mathcal{T}_\kappa^{\mathrm{red}} = \mathcal{T}_\kappa - \mathcal{T}_\kappa^{\mathrm{irr}}$, $R_\kappa(\mathbb{Z}_n \star \mathbb{Z}_m) = \Phi^{-1}(\mathcal{T}_\kappa)$ and $R_\kappa^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) = \Phi^{-1}(\mathcal{T}_\kappa^{\mathrm{irr}})$. Then, we have that

$$R^{\mathrm{irr}}(\Gamma_{n,m}) \cong \bigsqcup_\kappa R_\kappa^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) = \bigsqcup_\kappa \Big(R_\kappa(\mathbb{Z}_n \star \mathbb{Z}_m) - \bigsqcup_{\tau \in \mathcal{T}_\kappa^{\mathrm{red}}} X(\tau) \Big), \tag{1}$$

where $X(\tau) = \Phi^{-1}(\tau)$ is the set of (reducible) representations of type τ. The virtual class $[R_\kappa(\mathbb{Z}_n \star \mathbb{Z}_m)] \in K\mathcal{V}ar_k$ can be easily computed as the product of the adjoint orbits of two diagonal matrices. Hence, Equation (1) shows that, to compute the virtual class of $R^{\mathrm{irr}}(\Gamma_{n,m})$, it is enough to compute the virtual classes of $X(\tau)$ for all κ and $\tau \in \mathcal{T}_\kappa^{\mathrm{red}}$. This amounts to a combinatorial problem and the knowledge of $[R^{\mathrm{irr}}(\Gamma_{n,m}, \mathrm{SL}_s(k))]$ for $s < r$, so the computation can be performed recursively. For further details, check [2, Section 3].

2.1. Counting components

Consider partitions $\pi = \{1^{e_1}, 2^{e_2}, \dots, r^{e_r}\}$ and $\pi' = \left\{1^{e'_1}, 2^{e'_2}, \dots, r^{e'_r}\right\}$ of r with $r = \sum_i ie_i = \sum_i ie'_i$. Denote by $M_{n,m,r}^{\pi,\pi'}$ the collection of (unordered) spectra $\kappa = (\sigma_A, \sigma_B)$ where σ_A (resp. σ_B) has e_i (resp. e'_i) collections of i equal eigenvalues for $i = 1, \dots, r$. Notice that, for any $\kappa, \kappa' \in M_{n,m,r}^{\pi,\pi'}$ we have that $\left[\Phi^{-1}(\mathcal{T}_\kappa^{\mathrm{red}})\right] = \left[\Phi^{-1}(\mathcal{T}_{\kappa'}^{\mathrm{red}})\right]$. Hence, we can collect the summands in (1) that contribute equaly to get

$$\left[R^{\mathrm{irr}}(\Gamma_{n,m})\right] = \sum_{\pi,\pi'} |M_{n,m,r}^{\pi,\pi'}| \Big(\left[R_{\kappa(\pi,\pi')}(\mathbb{Z}_n \star \mathbb{Z}_m)\right] - \sum_{\tau \in \mathcal{T}_{\kappa(\pi,\pi')}^{\mathrm{red}}} [X(\tau)] \Big). \tag{2}$$

Here, we have fixed an element $\kappa(\pi, \pi') \in M_{n,m,r}^{\pi,\pi'}$ for every permutations π, π'. The first step towards the calculation of all the terms involved this sum is provided in the following result.

Theorem 6 ([2, Section 6 and Theorem 6.8]). *If $\gcd(n, r) = \gcd(m, r) = 1$ or $r \leq 4$, then we have*

$$|M_{n,m,r}^{\pi,\pi'}| = \frac{r}{nm} \binom{n}{e_1, e_2, \dots, e_r} \binom{m}{e'_1, e'_2, \dots, e'_r} = \frac{r}{nm} \frac{n!}{e_1! \cdots e_r!(n - e_1 - \dots - e_r)!} \frac{m!}{e'_1! \cdots e'_r!(n - e'_1 - \dots - e'_r)!}.$$

Remark 7. It is an open problem whether this formula also holds true for $r \geq 5$ without the awkward hypothesis $\gcd(n, r) = \gcd(m, r) = 1$. ◂

2.2. Counting representations of fixed type

Fix a type τ, let $m_{i,j}$ be the multiplicity of the isotypic piece $W_{i,j}$ of the semi-simple filtration and set $\kappa_{i,j} = (\sigma_{i,j}(x), \sigma_{i,j}(y))$ for the corresponding eigenvalues of these pieces. Then we consider

$$\mathcal{I}(\tau) = \prod_{i=1}^{s} \prod_{j=1}^{s_i} \mathrm{Sym}^{m_{i,j}} \left(R_{\kappa_{i,j}}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) \right), \quad \hat{\mathcal{I}}(\tau) = \prod_{i=1}^{s} \prod_{j=1}^{s_i} \left(R_{\kappa_{i,j}}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) \right)^{m_{i,j}}.$$

There is a map $\mathrm{Gr}_\bullet : X(\tau) \to I(\tau)$ that assigns any representation to its graded complex (its "semi-simplification"). Pulling-back $\mathrm{Gr}_\bullet$ through the quotient map $\hat{\mathcal{I}}(\tau) \to \mathcal{I}(\tau)$, we obtain a morphism $\hat{\mathrm{Gr}}_\bullet : X(\tau) \times_{\mathcal{I}(\tau)} \hat{\mathcal{I}}(\tau) \to \hat{\mathcal{I}}(\tau)$. It is a Zariski locally trivial fibration whose fiber F_ϱ at $\varrho \in \hat{\mathcal{I}}(\tau)$ is the set of ways we can complete the block- diagonal semi-simple representation induced by ϱ with off-diagonal elements.

These calculations of the virtual classes of the fibers F_ϱ can be carried out using Schubert calculus (see [2, Sections 4 and 5], where the calculations for rank $r \leq 4$ are performed). Moreover, if for every $m_{i,j} > 1$ we have that $\dim W_{i,j} = 1$ (i.e. if all the repeated irreducible representations are 1-dimensional) then we have that $\hat{\mathcal{I}}(\tau) = \mathcal{I}(\tau)$ so $[X(\tau)] = [F_\varrho][\mathcal{I}(\tau)]$. These conditions hold for $r \leq 4$ [2, Corollary 4.7 and Proposition 8.1]. Thus performing these calculations for all the possible combinations of permutations and types, we can compute the virtual class of $R^{\mathrm{irr}}(\Gamma_{n,m}, \mathrm{SL}_r(k))$ by means of (2) for $r \leq 4$.

In the case $r = 4$, there are 10 posible partitions such that $\mathcal{T}^{\mathrm{irr}}_{\kappa(\pi,\pi')} \neq \varnothing$ and more than 350 types must be analyzed for these partitions. Carrying out the calculations with a symbolic algebra system, we finally obtain the following result (see [2, Section 8] for further details).

Theorem 8. *The virtual class of the irreducible* $\mathrm{SL}_4(k)$*-character variety of the* (n, m)*-torus knot is*

$$
\begin{aligned}
[\mathfrak{M}^{\mathrm{irr}}(\Gamma, \mathrm{SL}_4(k)] = {} & \tfrac{4}{nm}\tbinom{n}{4}\tbinom{m}{4}(q^9 + 6q^8 + 20q^7 + 17q^6 - 98q^5 - 26q^4 + 38q^3 + 126q^2 - 144) \\
& + \tfrac{4}{nm}\tbinom{n}{2,1}\tbinom{m}{2,1}(q^5 + 2q^4 - 10q^3 + 7q^2 + 11q - 17) + \tfrac{4}{nm}\left(\tbinom{n}{4}\tbinom{m}{2} + \tbinom{n}{2}\tbinom{m}{4}\right)(q^5 + 4q^4 - 11q^3 + q^2 + 18q - 18) \\
& + \tfrac{4}{nm}\left(\tbinom{n}{4}\tbinom{m}{1,1} + \tbinom{n}{1,1}\tbinom{m}{4}\right)(q^3 - 4q^2 + 6q - 4) + \tfrac{4}{nm}\left(\tbinom{n}{2,1}\tbinom{m}{2} + \tbinom{n}{2}\tbinom{m}{2,1}\right)(q^3 - 3q^2 + 5q - 4) \\
& + \tfrac{4}{nm}\left(\tbinom{n}{4}\tbinom{m}{2,1} + \tbinom{n}{2,1}\tbinom{m}{4}\right)(q^7 + 5q^6 + 7q^5 - 34q^4 + 34q^2 + 18q - 48),
\end{aligned}
$$

where $q = [\mathbb{A}^1_k] \in K\mathcal{V}ar_k$ *is the virtual class of the affine line.*

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