3rd *BYMAT Conference* (2020)

On the prime graph associated with class sizes of a finite group

Víctor Sotomayor
University Centre EDEM
vsotomayor@edem.es

Abstract: The aim of this paper is to present some current results that investigate the relation between the structure of a finite group G and graph-theoretical properties of the prime graph associated with its conjugacy class sizes.

Resumen: En este trabajo presentamos algunos resultados recientes que estudian la relación existente entre la estructura de un grupo finito G y las propiedades del grafo primo asociado a los tamaños de sus clases de conjugación.

Keywords: finite groups, conjugacy classes, prime graph.

MSC2010: 20E45.

Acknowledgements: This research is supported by Proyecto PGC2018-096872-B-I00, Ministerio de Ciencia, Innovación y Universidades (Spain), and by Proyecto AICO/2020/298, Generalitat Valenciana (Spain).

Reference: Sotomayor, Víctor. "On the prime graph associated with class sizes of a finite group". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 43-46. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p43.

1. Introduction

Hereafter, only finite groups will be considered. A well-established area of research within finite group theory is the study of the connection between the structure of a group G and the arithmetical properties of certain sets of positive integers associated to it. In particular, the set $cs(G) = \{|G : \mathbf{C}_G(x)| \ : \ x \in G\}$ of conjugacy class sizes of G has been thoroughly analysed. For instance, a classical result within this research line states that a group G has a central Sylow p-subgroup, for a given prime p, if and only if p does not divide any number in $cs(G)$. Indeed, non-divisibility properties have been studied since many decades ago, as the next theorem due to N. Itô in 1953 (see the paper [5]): if p and q are two distinct prime numbers that divide two distinct class sizes of a group G, but pq does not divide any number in $cs(G)$, then G has a normal p-complement and abelian Sylow p-subgroups.

A useful tool that is gaining an increasing interest for studying the arithmetical structure of the set $cs(G)$ is the (complement) prime graph. In general, if X is a set of positive integers, then the *prime graph* $\Delta(X)$ is defined as the simple undirected graph whose vertex set $V(X)$ consists of the prime divisors of the numbers in X, and whose edge set $E(X)$ contains $\{p, q\} \subseteq V(X)$ whenever pq divides some element in X. Further, the *complement prime graph* $\overline{\Delta(X)}$ is the graph with the same vertex set $V(X)$, and two primes p and q are adjacent in $\overline{\Delta(X)}$ if and only if they are not adjacent in $\Delta(X)$. In this paper, we consider the prime graph $\Delta(G)$ built on the set $cs(G)$, with vertex and edge set $(V(G), E(G))$, respectively; and in particular, we will present some current results in collaboration with S. Dolfi, E. Pacifici, and L. Sanus.

Two natural questions that arise in this context are the following ones:

- What can be said about the structure of G if some properties of $\Delta(G)$ are known?
- What graphs can occur as $\Delta(G)$ for some finite group G?

2. Features of $\Delta(G)$

Both classical results stated in the first paragraph can be framed within the first question, since they have the next transcription in terms of $\Delta(G)$.

Lemma 1. *Let G be a group, and let $p, q \in V(G)$ with $p \neq q$. Then we have:*

(i) *$p \notin V(G)$ if and only if $P \leqslant \mathbf{Z}(G)$, for some Sylow p-subgroup P of G.*

(ii) *If $\{p, q\} \notin E(G)$, then G has a normal p-complement and abelian Sylow p-subgroups.*

In the context of the second question above, those graphs that possess "few" edges cannot occur as $\Delta(G)$ for a group G. This is due to the following result of S. Dolfi, which we call the "Three-Vertex Theorem".

Theorem 2. [2, Theorem A] *Let G be a group. Then for every choice of three vertices in $\Delta(G)$, there exists at least an edge that joins two of them.*

Indeed, this result is an improvement of [1, Theorem 16], where Dolfi proved the soluble version of the Three-Vertex Theorem. As a direct consequence, we obtain the next result, which actually was known to be true even before the existence of the Three-Vertex Theorem (see the paper [1]).

Corollary 3. *Let G be a group. Then we have:*

(i) *If $\Delta(G)$ is connected, then its diameter is at most* 3.

(ii) *If $\Delta(G)$ is non-connected, then it is the union of two complete subgraphs.*

In fact, a group G has non-connected prime graph $\Delta(G)$ if and only if $G = AB$ with A and B abelian Hall subgroups of coprime order, and $G/\mathbf{Z}(G)$ is a Frobenius group with kernel $A\mathbf{Z}(G)/\mathbf{Z}(G)$ (see [1, Theorem 4]). Further, the set of prime divisors of $|A\mathbf{Z}(G)/\mathbf{Z}(G)|$ and the one of $|B\mathbf{Z}(G)/\mathbf{Z}(G)|$ are two *cliques* of $\Delta(G)$, so they form the two complete connected components of $\Delta(G)$. In general, a subset of vertices of a graph Δ is called a *clique* if their induced subgraph in Δ is complete.

3. The complement prime graph

Recall that the complement prime graph $\overline{\Delta(G)}$ has the same vertex set $V(G)$, and there is and edge between two primes p and q whenever they are not adjacent in $\Delta(G)$. Observe that the Three-Vertex Theorem can be expressed in terms of the complement prime graph as follows: for every finite group G, the graph $\overline{\Delta(G)}$ does not contain any cycle of length 3. This fact suggests the study of the (non-)existence of cycles within $\overline{\Delta(G)}$ of length larger than 3.

Example 4. Let $N = C_{31} \times C_{61} = C_{1891}$ and $H = C_3 \times C_5 = C_{15}$. Then H acts on N fixed-point-freely, and we can consider the semidirect product $G = N \rtimes H$, which is a Frobenius group. It follows that $cs(G) = \{1, 15, 1891\}$ and that $\Delta(G)$ is the union of two complete connected components, which are the prime divisors of N and H, respectively. So $\overline{\Delta(G)}$ is a cycle of length 4. ◂

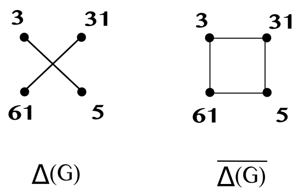

Figure 1: an illustration of $\Delta(G)$ and $\overline{\Delta(G)}$ where $G = C_{1891} \rtimes C_{15}$.

In view of the above example, the next natural step would be to study the case of a cycle of length 5 in $\overline{\Delta(G)}$. Nevertheless, this is also impossible. Indeed, this fact is more general, as the main theorem of the paper [3] shows.

Theorem 5. [3, Theorem A] *Let G be a group. Then the graph $\overline{\Delta(G)}$ does not contain any cycle of odd length.*

In other words, this means that $\overline{\Delta(G)}$ is a bipartite graph, i.e., a graph where the vertex sex can be partitioned into two disjoint subsets A and B such that every edge connects a vertex in A to another one in B. As an immediate consequence, we obtain the next result.

Corollary 6. [3, Corollary B] *Let G be a group. Then the vertex set of $\Delta(G)$ can be partitioned into two subsets of pairwise adjacent vertices.*

We have previously commented that if $\Delta(G)$ is disconnected for some group G, then $V(G)$ is the union of two cliques. But from the above corollary, this property turns out to hold in full generality.

Therefore, at least half of the vertices of $\Delta(G)$ are pairwise adjacent, for any group G. So denoting by $w(G)$ the clique number (i.e., the maximum size of a clique) of $\Delta(G)$, we obtain what follows.

Corollary 7. [3, Corollary C] *Let G be a group. Then, the inequality $|V(G)| \leq 2w(G)$ holds.*

It is not difficult to see that this bound is best possible, as the group in Example 4 shows. We close this section with another illustrating example.

Example 8. Let $G = A\Gamma(11^3) = ((C_{11} \times C_{11} \times C_{11}) \rtimes C_{11^3-1}) \rtimes C_3$ be an affine semilinear group. Then $V(G) = \{2, 3, 5, 7, 11, 19\}$, and it follows that $\Delta(G)$ is the union of the clique $V(G) \setminus \{3\}$ and the edge $\{3, 11\}$ (see Figure 2). ◂

4. Cut vertices

The last example has the following interesting property: if we remove the vertex 11 from the graph and all the edges adjacent to 11 from $\Delta(G)$, then the resulting graph is disconnected. Let us define this "almost non-connectedness" feature of $\Delta(G)$ in general: $r \in V(G)$ is called a *cut vertex* of $\Delta(G)$ if the subgraph

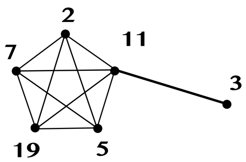

Figure 2: the prime graph $\Delta(G)$ for $G = A\Gamma(11^3)$.

induced by $V(G) \setminus \{r\}$ in $\Delta(G)$ (i.e., the graph $\Delta(G) - r$ obtained by removing the vertex p and all edges incident to r from $\Delta(G)$) has more connected components than $\Delta(G)$.

Example 9. There is an easy way of obtaining groups G such that $\Delta(G)$ has a cut vertex r. It is enough to consider $G = R \times (A \rtimes B)$ where $A \rtimes B$ is a Frobenius group with A and B abelian, and R is a non-abelian r-group such that r does not divide the order of $A \rtimes B$. ◂

The next result states, among other facts, that $\Delta(G)$ can have at most two cut vertices.

Theorem 10. [4, Theorem A] *Let G be a group such that $\Delta(G)$ has a cut vertex r. Then, the following conclusions hold.*

(i) *G is soluble with Fitting height at most 3, and its Sylow p-subgroups are abelian for all primes $p \neq r$.*

(ii) *$\Delta(G) - r$ is a graph with two connected components, that are both complete graphs.*

(iii) *If r is a complete vertex of $\Delta(G)$, then it is the unique complete vertex and the unique cut vertex of $\Delta(G)$. If r is non-complete, then $\Delta(G)$ is a graph of diameter 3, and it can have at most two cut vertices; moreover, G is metabelian with abelian Sylow subgroups.*

Example 11. Let $R = C_{31}$, $A = C_{11} \times C_{61}$, $B_0 = C_2 \times C_3$, and $B_1 = C_5$, and consider $G = (A \times R) \rtimes (B_0 \times B_1)$, where there is a Frobenius action of $B_0 \times B_1$ on R, another Frobenius action of B_1 on A, and B_0 acts trivially on A. It is not difficult to show that $\Delta(G)$ is the union of the two cliques $\{11, 31, 61\}$ and $\{2, 3, 5\}$ together with the edge $\{5, 31\}$, so 5 and 31 are cut vertices of $\Delta(G)$. ◂

Moreover, Theorem 3.3 and Theorem C of [4] completely characterise the structure of G (and the corresponding one of $\Delta(G)$) in both the cases when $\Delta(G)$ has either one or two cut vertices, respectively. In particular, these results yield a classification of those groups G such that $\Delta(G)$ is acyclic, i.e., that it has no cycle as an induced subgraph (see [4, Corollary 3.4]).

In addition, there is a necessary and sufficient condition for a graph that possesses a cut vertex to occur as $\Delta(G)$ for a suitable group G.

Theorem 12. [4, Theorem D] *Let Δ be a graph having a cut vertex. Then there exists a finite group G such that $\Delta = \Delta(G)$ if and only if Δ is connected and the vertex set of Δ can be partitioned in two subsets of pairwise adjacent vertices.*

References

[1] Dolfi, Silvio. “Arithmetical conditions on the length of the conjugacy classes of a finite group”. In: *J. Algebra* 174 (1995), pp. 753–771.

[2] Dolfi, Silvio. “On independent sets in the class graph of a finite group”. In: *J. Algebra* 303 (2006), pp. 216–224.

[3] Dolfi, Silvio; Pacifici, Emanuele; Sanus, Lucía, and Sotomayor, Víctor. “The prime graph on class sizes of a finite group has a bipartite complement”. In: *J. Algebra* 542 (2020), pp. 35–42.

[4] Dolfi, Silvio; Pacifici, Emanuele; Sanus, Lucía, and Sotomayor, Víctor. “Groups whose prime graph on class sizes has a cut vertex”. In: *Israel J. Math.* (to appear).

[5] Itô, Noboru. “On finite groups with given conjugate types. I”. In: *Nagoya Math. J.* 6 (1953), pp. 17–28.