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Higher derivations with Lie structure of associative rings

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Abstract: In this paper, we suppose R is a prime ring with the centre $Z(R)$, $D = (d_i \neq 0)_{i \in \mathbb{N}}$ is a higher derivation of R and L is a Lie ideal of R, this gives under certain conditions R has a weak zero-divisor or a weakly semiprime ideal.

Resumen: En este artículo, suponemos que R es un anillo principal con centro $Z(R)$, $D = (d_i \neq 0)_{i \in \mathbb{N}}$ es una derivación de R y L es un ideal de Lie de R. Bajo ciertas condiciones, resulta que R tiene un divisor de cero débil o un ideal semiprimo débil.

Keywords: weakly semiprime ideal, weak zero-divisor, Lie ideal, derivation, prime ring.

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1. Introduction

One of the earliest results on Lie derivations of associative rings is dur to Martindale [10], who proved that every Lie derivation on a primitive ring can be written as a sum of derivation and an additive mapping of ring to its center that maps commutators into zero, *i.e.*, Lie derivation has the standard form. In 1993, Brešar [5] gave a characterization of Lie derivations of prime rings. This result together with other results initiated the theory of functional identities on rings. Actually, study behaviour of a derivation on the whole ring with many of the results achieved by extending the other ones proven previously. For a full account on the theory of functional identities and zero Lie product we refer the reader to the paper of Brešar [6]. Lie derivations, as well as other Lie maps, have been active research subjects for a long time (see, e.g., [1], Benkovič [3] and Brešar [6]). Also, Cheung [8] gave a characterization of linear Lie derivations on triangular algebras. Qi and Hou [11] discussed additive ξ-Lie derivations on nest algebras. The most interesting result on additive Lie derivations of prime rings was obtained by Brešar [6].

Throughout the article, R will represent an a commutative ring with identity $1 \neq 0$. The center of R is denoted by $Z(R)$. The symbols $[x, y]$ stand for the commutator $xy - yx$, and $x \circ y$ stands for the anticommutator $xy + yx$, for any $x, y \in R$. A ring R is called a prime if $xRy = 0$ implies either $x = 0$ or $y = 0$. Suppose L is an additive subgroup of R, L is said to be a Lie ideal of R if for every $u \in L$, $r \in R$ then the commutator $[u, r] = ur - ru \in L$. Any ordinary, two-sided ideal of R is automatically a Lie ideal of R. Let $n > 1$ be an integer; then, a ring R is said to be n-torsion free, in case $nx = 0$ implies that $x = 0$ for any $x \in R$.

The idea of a weakly semiprime ideal is due to Badawi [2]. He introduced that the ideal L is a weakly semiprime ideal of R such that R is a commutative ring with identity $1 \neq 0$ and L is a proper ideal of R. If $a \in R$ and $0 \neq a^2 \in L$ then $a \in L$. While the concept of a weak zero-divisor of a ring R introduced by Burgess, Lashgari, and Mojiri [7], where the authors defined an element $a \in R$ is called a weak zero-divisor. If there is $r, s \in R$ with $ras = 0$ and $rs \neq 0$. A derivation d is an additive mapping $d : R \to R$ satisfies the Leibniz's formula $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Moreover, D is said to be a higher derivation of U into R if for every $n \in \mathbb{N}$, we conclude that $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in L$ and $D = (d_i \neq 0)$ for all $i \in \mathbb{N}$ is the family of additive mappings of R such that $d_0 = id_R$ and $\mathbb{N}$ is set of a positive integers, where L is a Lie ideal of R.

By the above facts, it is fascinating to study weakly semiprime ideals and weak zero-divisors on a Lie ideal of a prime ring R via a higher derivation $D = (d_i \neq 0)_{i \in \mathbb{N}}$. This is our main motivation for this paper. The following lemmas are also going to be applied:

Lemma 1 (Bergen, Herstein, and Kerr [4], Lemma 4). *Suppose R is a prime ring with characteristic not two and $a, b \in R$. If L is a non-central Lie ideal of R such that $aUb = 0$, then either $a = 0$ or $b = 0$.*

Lemma 2 (Herstein [9], Lemma 1.8). *Let R be a semiprime ring, and $a \in R$ be a centralizer of all commutators $[x, y]$, $x, y \in R$. Then, $a \in Z(R)$.*

2. The main results

Theorem 3. *Let R be a prime ring with the centre $Z(R)$ and L be a Lie ideal of R. Suppose that $D = (d_i \neq 0)_{i \in \mathbb{N}}$ is a higher derivation of U into R. If d satisfy one of the following relations*

(i) $[a, d_i(u)] \in Z(R)$ *for all* $u \in L$, $a \in R$, *then* L *is a weakly semiprime ideal.*

(ii) $[d_i(L), d_i(L)] \subseteq Z(R)$, *then either* L *is a weakly semiprime ideal of* R *or* $d_n(L)$ *is a weak zero-divisor of* R.

(iii) $[a, d_i(u)] \in Z(R)$ *and* $d_i(Z(R)) \neq 0$ *for all* $u \in L, a \in R$, *then* $[a, [L, R]] \subseteq Z(R)$.

Based on Theorem 3, we can easily prove the following theorem.

Theorem 4. *Let R be a prime ring with the centre $Z(R)$ and L be a Lie ideal of R. Suppose that $D = (d_i \neq 0)_{i\in\mathbb{N}}$ is a higher derivation of U into R. If d satisfy one of the following relations*

(i) $[d_i(L), d_i(L)] \subseteq Z(R)$ *and* $d_i(Z(R)) \neq 0$, *then R has a weakly semiprime ideal.*

(ii) $d_i^2(L) \subseteq Z(R), d_i(Z(R)) \neq 0$ *and* $d_i d_j(L) \subseteq Z(R)$, $i, j \in \mathbb{N}$, *then either $d_n(L)$ is a weak zero-divisor of R or R has a weakly semiprime ideal, where R is 2-torsion free.*

(iii) $ad_i(L) \subseteq Z(R)$ *and* $d_i(Z(R)) \neq 0$, *then either a is a weak zero-divisor of R or R has a weakly semiprime ideal, where $a \in R$.*

In the following theorem, R not to be a commutative ring with identity $1 \neq 0$.

Theorem 5. *For any fixed integers $n, q > 1$, let R be prime ring with the centre $Z(R)$ and D be a derivation on R. If D satisfy one of the following relations*

(i) $D^n(xoy) \mp [x, y] \in Z(R)$;

(ii) $D^n(x \circ y) \pm D^q(x \circ y) \mp [x, y] \in Z(R)$ *and R is 2-torsion free;*

(iii) $D^n([x, y]) \pm D^q([x, y]) \mp (x \circ y) \in Z(R)$ *and R is 2-torsion free*

for all $x, y \in R$, then R has a weak zero-divisors.

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