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Efimov K-theory of diamonds

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Abstract: Motivated by Scholze and Fargues's geometrization of the Local Langlands Correspondence using perfectoid diamonds and Clausen and Scholze's work on the K-theory of adic spaces using condensed mathematics, we introduce the Efimov K-theory of diamonds. We propose a large stable $(\infty, 1)$-category of diamonds $\mathcal{D}^{\diamond}$, a diamond spectra and chromatic tower, and a localization sequence for diamond spectra. Commensurate with the localization sequence, we detail three potential applications of the Efimov K-theory of $\mathcal{D}^{\diamond}$: to quantum gravity and reconstructing the holographic principle using diamonds and Scholze's six operations in the étale cohomology of diamonds; to post-quantum diamond cryptography in the form of programming AI with Efimov K-theory of $\mathcal{D}^{\diamond}$; and to nonlocality in perfectoid quantum physics.

Resumen: Motivados por la geometrización de Scholze y Fargues de la Correspondencia Local de Langlands usando diamantes perfectoides y el trabajo de Clausen y Scholze con la K-teoría de espacios ádicos usando matemáticas condensadas, nosotros introducimos la K-teoría de diamantes de Efimov. Proponemos una $(\infty, 1)$-categoría de diamantes $\mathcal{D}^{\diamond}$; un espectro de diamantes y una torre cromática, y una secuencia de localización del espectro de un diamante. Acorde con esta secuencia de localización, detallamos tres potenciales aplicaciones de la K-teoría de $\mathcal{D}^{\diamond}$ de Efimov: a gravedad cuántica y la reconstrucción del principio holográfico usando diamantes y las seis operaciones de Scholze en la cohomología étale de diamantes; a criptografía de diamante postcuántica, en forma de programación de IA con K-teoría de $\mathcal{D}^{\diamond}$ de Efimov, y a no localidad en física perfectoide cuántica.

Keywords: perfectoid spaces, Efimov K-theory, diamonds, Fargues-Fontaine curve, geometric Langlands, $(\infty, 1)$-topoi.

MSC2010: 11F77, 11S70, 19D06, 19E08, 19E20.

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1. Introduction

K-theory is defined on the category of small stable ∞-categories which are idempotent, complete, and where morphisms are exact functors. A certain category of large compactly generated stable ∞-categories is equivalent to this small category. In Efimov K-theory the idea is to weaken to 'dualizable' the condition of being compactly generated so that K-theory is still defined. A category $\mathcal{C}$ being dualizable implies that $\mathcal{C}$ fits into a localization sequence $\mathcal{C} \to \mathcal{S} \to \mathcal{X}$ with $\mathcal{S}$ and $\mathcal{X}$ compactly generated. The Efimov K-theory should be the fiber of the K-theory in the localization sequence. As Clausen and Scholze propose [6]:

Proposition 1. *Let Nuc_R be a full subcategory of solid modules [1]. Nuc_R is a presentable stable ∞-category closed under all colimits and tensor products. If R is a Huber ring, then Nuc_R is dualizable, making its K-theory well-defined. Nuc_R embeds into Mod_R. Let $\mathcal{X}$ be a Noetherina formal scheme and X the torsion perfect complexes of modules over R. Then,*

Theorem 2. *$(R, R^+) \to Nuc_R$ satisfies descent over $Spa(R)$ and so does its Efimov K-theory. There exists a localization sequence $K(X) \to K^{Efimov}(\mathcal{X}) \to K^{Efimov}(\mathcal{X}^{rig})$ [2].*

We introduce the Efimov K-theory of diamonds.

2. Main conjectures

Conjecture 3. *There exists a large, stable, presentable $(\infty, 1)$-category of diamonds $\mathcal{D}^\diamond$ with spatial descent datum. $\mathcal{D}^\diamond$ is dualizable. Therefore, the Efimov K-theory is well defined.*

Conjecture 4. *Let S be a perfectoid space, $\mathcal{D}^\diamond$ a stable dualizable presentable category, and R a sheaf of E_1-ring spectra on S. Let $\mathcal{T}$ be a stable compactly generated $(\infty, 1)$-category and $F\colon \mathrm{Cat}_{St}^{idem} \to \mathcal{T}$ a localizing invariant that preserves filtered colimits. Then, $F_{cont}(Shv(\mathbb{S}^n, \mathcal{D}^\diamond)) \simeq \Omega^n F_{cont}(\mathcal{D}^\diamond)$.*

Conjecture 5. *Let $\mathcal{D}_\diamond$ be the complex of v-stacks of locally spatial diamonds. Let $\mathcal{Y}_{(R,R^+),E} = Spa(R, R^+)$ x_{SpaF_q} $SpaF_q[[t]]$ be the relative Fargues-Fontaine curve. Let $(\mathcal{Y}^\diamond_{S,E})$ be the diamond relative Fargues-Fontaine curve. There exists a localization sequence $K(\mathcal{D}_\diamond) \to K^{Efimov}(\mathcal{Y}^\diamond_{S,E}) \to K^{Efimov}(\mathcal{Y}_{(R,R^+),E})$.*

Conjecture 6. *$\mathcal{D}^\diamond$ admits a topological localization, in the sense of Grothendieck-Rezk-Lurie $(\infty, 1)$-topoi.*

Conjecture 7. *There exists a diamond chromatic tower $\mathcal{D}^\diamond \to \ldots \to L_n\mathcal{D}^\diamond \to L_{n-1}\mathcal{D}^\diamond \to \ldots \to L_0\mathcal{D}^\diamond$ for L_n a topological localization for $K\mathcal{D}^\diamond$ the K-theory spectrum that represents the étale cohomology of diamonds.*

Conjecture 8. *The $(\infty, 1)$-category of perfectoid diamonds is an $(\infty, 1)$-topos.*

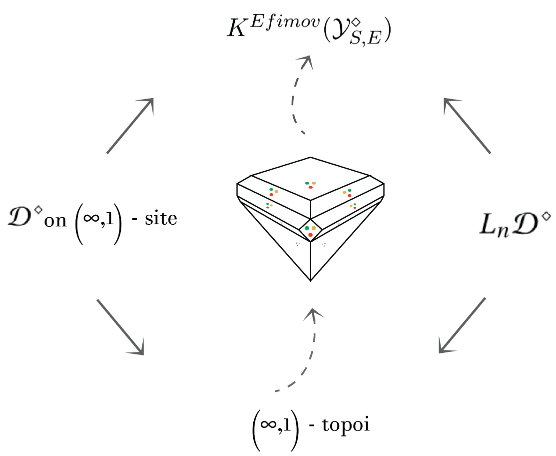

Figure 1: Efimov K-theory and Diamond Chromatic Tower.

3. Efimov K-theory of diamonds

Our terminology and exposition of Efimov K-theory follows Hoyois [4]

∞-categories are called categories. Let $\mathcal{P}r$ denote the category of presentable categories and colimit-preserving functors. Let $\mathcal{P}r^{dual} \subset \mathcal{P}r$ denote the subcategory of dualizable objects and right-adjointable morphisms (with respect to the symmetric monoidal and 2-categorical structures of $\mathcal{P}r$). Let $\mathcal{P}r^{cg} \subset \mathcal{P}r$ be the subcategory of compactly generated categories and compact functors. Compact functors are functors whose right adjoints preserve filtered colimits. Let $\mathcal{P}r^{\star}_{St}$ denote the corresponding full subcategories consisting of stable categories.

Definition 9. A functor $F\colon \mathcal{P}r^{dual}_{St} \to \mathcal{T}$ is called a localizing invariant if it preserves final objects and sends localization sequences to fiber sequences. ◂

Definition 10. Let $C \in \mathcal{P}r$ be stable and dualizable. The continuous K-theory of $\mathcal{C}$ is the space $\mathrm{K}_{cont}(\mathcal{C}) = \Omega$ $\mathrm{K}(Calk(C)^{\omega})$. ◂

Lemma 11. *If $\mathcal{C}$ is compactly generated, then $K_{cont}(\mathcal{C}) = K(\mathcal{C}^{w})$. Proof. The localization sequence is Ind of the sequence $\mathcal{C}^{w} \hookrightarrow \mathcal{C} \to Calk(\mathcal{C})^{w}$. Since $K(\mathcal{C}) = 0$, the result follows from the localization theorem.*

Definition 12. A functor $F\colon \mathcal{P}r^{dual}_{St} \to \mathcal{T}$ is called a localizing invariant if it preserves final objects and sends localization sequences to fiber sequences. ◂

Theorem 13 (Efimov). *Let $\mathcal{T}$ be a category. The functor $Fun(\mathcal{P}r^{dual}_{St}, \mathcal{T}) \longrightarrow Fun(Cat^{idem}_{St}, \mathcal{T})$, $F \mapsto F \circ Ind$, restricts to an isomorphism between the full subcategories of localizing invariants, with inverse $F \mapsto F_{cont}$. In particular, if $\mathcal{C} \in \mathcal{P}r^{cg}_{St}$, then $F_{cont}(\mathcal{C}) = F(C^{\omega})$. Proof. See* [4, Theorem 10].

Theorem 14 (Efimov*). *Let X be a locally compact Hausdorff topological space, $\mathcal{C}$ a stable dualizable presentable category, and R a sheaf of E_1-ring spectra on X. Suppose that $Shv(X)$ is hypercomplete (i.e., X is a topological manifold). Let $\mathcal{T}$ be a stable compactly generated category and $F\colon \mathrm{Cat}^{idem}_{St} \to \mathcal{T}$ a localizing invariant that preserves filtered colimits. Then, $F_{cont}(Mod_R(Shv(X, \mathcal{C}) \simeq \Gamma_c(X, F_{cont}(Mod_R(\mathcal{C})))$. Specifically, $F_{cont}(Shv(\mathbb{R}^n, \mathcal{C})) \simeq \Omega^n F_{cont}(\mathcal{C})$. Proof. See* [4, Theorem 15].

Remark 15. The main goal is to develop a Waldhausen S-construction to obtain the K-theory spectrum $K\mathcal{D}^{\diamond}$ on the $(\infty,1)$-category of diamonds $\mathcal{D}^{\diamond}$. In parallel, to construct a topology on the $(\infty,1)$-category of diamonds, we must first construct the $(\infty,1)$-site on the $(\infty,1)$-category of diamonds. Recall, the definition of an $(\infty,1)$-site. ◂

Definition 16. The $(\infty,1)$-site on an $(\infty,1)$-category $\mathcal{C}$ is the data encoding an $(\infty,1)$-category of $(\infty,1)$-sheaves $Sh(\mathcal{C}) \hookrightarrow PSh(\mathcal{C})$ inside the $(\infty,1)$-category of $(\infty,1)$-presheaves on $\mathcal{C}$ [5]. ◂

Remark 17. $\mathcal{D}^{\diamond}$ admits a topological localization. Recall equivalence classes of topological localizations are in bijection with Grothendieck topologies on $(\infty,1)$-categories C. Topological localizations are appropos because in passing to the full reflective sub-$(\infty,1)$-category, objects and morphisms have reflections in the category, just as geometric points have reflections in the profinitely many copies of $Spa(\mathcal{C})$. ◂

Remark 18. Recall, the category of sheaves on a (small) site is a Grothendieck topos. Lurie discusses the structure needed for our construction. Recall the following [5]. ◂

Definition 19. An $(\infty,1)$-category of $(\infty,1)$-sheaves is a reflective sub-$(\infty,1)$-category $Sh(C) \overset{\overset{L}{\leftarrow}}{\hookrightarrow} PSh(C)$ of an $(\infty,1)$-category of $(\infty,1)$-presheaves such that the following equivalent conditions hold:

(i) L is a topological localization.

(ii) There is the structure of an $(\infty,1)$-site on C such that the objects of $Sh(C)$ are precisely those $(\infty,1)$-presheaves A that are local objects with respect to the covering monomorphisms $p\colon U \to j(c)$ in $PSh(C)$ in that $A(c) \simeq PSh(j(c), A) \xrightarrow{PSh(p,A)} PSh(U, A)$ is an $(\infty,1)$-equivalence in ∞Grpd.

(iii) The $(\infty,1)$-equivalence is the descent condition and the presheaves satisfying it are the $(\infty,1)$-sheaves. ◂

4. Diamonds

The construction of diamonds imitates that of algebraic spaces in taking the quotient of a scheme by an étale equivalence relation. Our terminology and exposition follows [6].

Definition 20. Let *Perfd* be the category of perfectoid spaces and *Perf* be the subcategory of perfectoid spaces of characteristic p. A *diamond* is a pro-étale sheaf $\mathcal{D}$ on *Perf* which can be written as the quotient X/R of a perfectoid space X by a pro-'etale equivalence relation $R \subset X \times X$. ◂

The diamond quotient lives in a category of sheaves on the site of perfectoid spaces with pro-étale covers.

Examples of diamonds are the following:

$SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^{\times}}$. $SpdQ_p$ is the coequalizer of $\underline{Z_p^{\times}} \times Spa(Q_p^{cycl})^{\flat} \rightrightarrows Spa(Q_p^{cycl})^{\flat}$. $SpdQ_p$ attaches to any perfectoid space S of characteristic p the set of all untilts $S\#$ over Q_p. The moduli space of shtukas for $(\mathcal{G}, b, \{\mu_1, ..., \mu_m\})$ fibered over $SpaQ_p \times SpaQ_p ... \times_m SpaQ_p$; the diamond relative Fargues-Fontaine Curve: $\mathcal{Y}_{S,E}^{\diamond} = S \times (Spa\mathcal{O}_E)^{\diamond}$; any $\diamond$ product: $SpdQ_p \times_{\diamond} SpdQ_p$.

Definition 21. Let C be an algebraically closed affinoid field and $\mathcal{D}$ a diamond. A *geometric point* $Spa(C) \to \mathcal{D}$ is "visible" by pulling it back through a quasi-pro-étale cover $X \to \mathcal{D}$, resulting in profinitely many copies of $Spa(C)$. The geometric point $Spa(C) \to \mathcal{D}$ is a mathematical minerological impurity. ◂

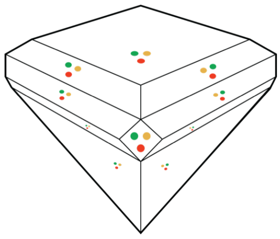

Figure 2: Diamond $SpdQ_p = Spd(Q_p^{cycl})/\underline{Z_p^{x}}$ with geometric point $Spa(C) \to \mathcal{D}$.

Remark 22. For a detailed discussion of the author's applications of the six operations and diamonds to quantum gravity, post-quantum diamond cryptography, and nonlocality of perfectoid quantum physics, see [3]. Recall, a perfectoid space is an adic space covered by affinoid adic spaces of the form $Spa(R, R^+)$ where R is a perfectoid ring. Any completion of an arithmetically profinite extension is perfectoid. A nice source of APF extensions is p-divisible formal group laws [6]. ◂

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