3rd *BYMAT Conference* (2020)

Why use topological and analytical methods in aggregation of fuzzy preferences?

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Abstract: In this work, we expose our last results in the aggregation of fuzzy preferences. Comparing these results with the literature, we notice a tendency towards using similar methods than those in classical models and, consequently, also leading to impossibility results. We propose using topological or analytical tools to obtain new possibility results.

Resumen: En este trabajo se exponen nuestros últimos resultados obtenidos sobre la posibilidad de agregar preferencias fuzzy bajo diferentes modelos. Comparando esos resultados con la literatura, observamos una tendencia en usar métodos similares a los propios de los modelos clásicos, y por lo tanto con resultados de imposibilidad. Proponemos el uso de herramientas topológicas o analíticas para obtener nuevos resultados de posibilidad.

Keywords: Arrow's impossibility theorem, topology, social choice, fuzzy set theory.

MSC2010: 91B14, 91-10.

Reference: Campión Arrastia, María Jesús; Induráin Eraso, Esteban, and Raventós Pujol, Armajac. "Why use topological and analytical methods in aggregation of fuzzy preferences?". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 59-62. issn: 2660-6003. url: https://temat.es/monograficos/article/view/vol2-p59.

1. Introduction

Arrow's Impossibility Theorem [1] states that there is no function fusing individual preferences into a social one satisfying certain properties of "common sense". On the contrary, in some of the fuzzy extensions of the Arrovian model, possibility arises [5, 6].

In previous works [7], we developed a technique which has been able to prove new impossibility results in the fuzzy approach. Here, we will explain the grounds of this technique and in which models we can apply it.

This technique is based on controlling the aggregation of fuzzy preferences through some aggregation functions of dichotomic preferences. For each fuzzy aggregation function, we get a family of dichotomic aggregation functions. Studying this family, we obtain information about the initial aggregation function. We will discuss why the fuzzy Arrovian models in which we can apply this technique are, in some sense, "less fuzzy". Moreover, we will expose why we should use topological and analytical methods in the fuzzy models out of the scope of our technique.

2. Classic Arrovian model and the theorem of impossibility

Let X be the set including all alternatives involved in a decision. They can be ordered by using binary relations satisfying certain properties. Particularly, in the Arrovian model, these binary relations are total preorders (reflexive, transitive and complete binary relations). Moreover, to give a total preorder on X is equivalent to give a ranking with ties on X.

Every binary relation $\succsim$ factorizes into the relations $\succ$ and $\sim$ defined as: $x \succ y \Leftrightarrow x \succsim y \wedge \neg(y \succsim x)$ and $x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x$. These binary relations are the strict preference (or asymmetric part) and the indifference (or symmetric part) of $\succsim$. If $x \succsim y$ we say that x is at least as good as y, if $x \succ y$ that x is better than y, and if $x \sim y$ that x and y are equally preferred.

Arrow [1] proved that, given a finite set of agents $N = \{1, \dots, n\}$, each one expressing their preferences over a set of alternatives X with total preorders, there is no "fair" rule which aggregates all individual preferences obtaining a social one. Formally, if the set of all total preorders on X is denoted by $\mathcal{O}_X$:

Theorem 1 (Arrow's Impossibility Theorem). *There is no function $f : {\mathcal{O}_X}^n \to \mathcal{O}_X$ on a set of alternatives with $|X| \geq 3$ satisfying, for every $x, y \in X$ and profiles $\succsim, \succsim' \in {\mathcal{O}_X}^n$, the following conditions:*

(i) *Paretian:* $\forall i \in N\ x \succ_i y \Rightarrow x \succ_{f(\succsim)} y$.

(ii) *Independence of irrelevant alternatives (IIA):*

$$\left[\forall i \in N \ \ \succsim_{i|\{x,y\}} = \succsim'_{i|\{x,y\}}\right] \Rightarrow f(\succsim)_{|\{x,y\}} = f(\succsim')_{|\{x,y\}}.$$

(iii) *Non dictatoriship:* $\nexists\, k \in N\, [x \succ_k y \Rightarrow x \succ_{f(\succsim)} y]$.

Given this result, many researchers looked for alternative ways to aggregate preferences. We will focus on using fuzzy sets to find new aggregation methods.

3. Extending the Arrovian model to the fuzzy setting

Studying the Arrovian model in the fuzzy framework consists in generalizing the objects and the properties from the previous section, and checking if the aggregation of preferences is possible in the new framework. All these properties can be generalized in different manners. So, a huge number of fuzzy Arrovian models is obtained.

In the fuzzy setting, a preference is a fuzzy binary relation $R : X \times X \to [0, 1]$. There are many generalizations of the crisp strict preference $\succ$ (of $\succsim$) to the fuzzy strict preference P_R (of R). For every fuzzy Arrovian model, we have to set a method of factorization to obtain P_R.

The properties of preferences $\precsim$ can be generalized to the fuzzy setting in different ways. For instance, the transitivity can be extended saying that R is T-transitive (with T a t-norm) if, $\forall x, y, z \in X$, $R(x, z) \geq T(R(x, y), R(y, z))$. However, it also may be generalized to the weak transitivity defined as $R(x, y) \geq R(y, x) \wedge R(y, z) \geq R(z, y) \Rightarrow R(x, z) \geq R(z, x)$. The completeness can be generalized to being S-connected (with S a t-conorm) as $\forall x, y \in X\ S(R(x, y), R(y, x)) = 1$.

Let $\mathcal{FP}$ be a set of fuzzy preferences on X. An aggregation fuzzy rule is a function $f : \mathcal{FP}^n \to \mathcal{FP}$. Arrow axioms can also be generalized in various ways. For instance, the Paretian property may be generalized to the weakly (resp. strongly) Paretian property as $\forall x, y \in X\ P_{R_i}(x, y) > 0 \Rightarrow P_{f(\mathbf{R})}(x, y) > 0$ (resp. $P_{f(\mathbf{R})}(x, y) \geq \min_{i \in N} P_{R_i}(x, y)$). The dictatorship may be extended to the weak (resp. strong) dictatorship as $\exists k \in N\ P_{R_k}(x, y) > 0 \Rightarrow P_{f(\mathbf{R})}(x, y) > 0$ (resp. $\forall t \in [0, 1]\ P_{R_k}(x, y) > t \Rightarrow P_{f(\mathbf{R})}(x, y) > t$. And the IIA may be generalized to $\forall x, y \in X\ [\forall i \in N\ R_i \approx_{\{x,y\}} R'_i \Rightarrow f(\mathbf{R}) \approx_{\{x,y\}} f(\mathbf{R}')]$, where $\approx_{\{x,y\}}$ can be defined as, $R \approx^1_{\{x,y\}} R' \Leftrightarrow R_{]\{x,y\}} = R'_{]\{x,y\}}$, $R \approx^2_{\{x,y\}} R' \Leftrightarrow \operatorname{supp}(R_{]\{x,y\}}) = \operatorname{supp}(R'_{]\{x,y\}})$ or $R \approx^3_{\{x,y\}} R' \Leftrightarrow R \approx^2_{\{x,y\}} R' \wedge [\forall \overline{z}, \overline{z}' \in \{x, y\}^2\ R(\overline{z}) > R(\overline{z}') \Leftrightarrow R'(\overline{z}) > R(\overline{z}')]$, among others (see [8]).

4. Studying fuzzy aggregation using crisp preferences

In this section, we draft a strategy to study fuzzy aggregation functions using the Arrovian theorem and other combinatorial techniques from the crisp model.

Consider a set of fuzzy preferences $\mathcal{FP}$ were all its preferences are reflexive and satisfy one type of fuzzy transitivity and one type of fuzzy connectedness. Then, we define a projection p from $\mathcal{FP}$ to a set of crisp preferences $\mathcal{B}$ on X. These projections are interpreted as collapsing the fuzzy preferences into its qualitative factor (a crisp binary relation). Some examples of projections are:

(i) If R is a weak transitive and S-connected preferences, $\succsim^1_R$ defined as $x \succsim^1_R y \Leftrightarrow R(x, y) \geq R(y, x)$ is a total preorder.

(ii) If R is a T-transitive and max-connected preference, $\succsim^2_R$ defined as $x \succsim^2_R y \Leftrightarrow R(x, y) = 1$ is a total preorder.

(iii) If R is a min-transitive and S-complete preference, $\succsim^3_R$ defined as $x \succsim^3_R y \Leftrightarrow R(x, y) \geq R(y, x)$ is a quasi-transitive binary relation.

The second step is finding the same but applied to aggregation functions. Here, given a fuzzy aggregation function f and n embeddings $\iota_i : \mathcal{B} \to \mathcal{FP}$, we define $f_\iota := p \circ f \circ (\iota_i \times \cdots \times \iota_n)$. We have to choose the right embeddings in order for f_ι to be an Arrovian aggregation function. Then, each f_ι is dictatorial. However, they may have different dictators. When all of them have the same dictator, and the image of all embeddings covers $\mathcal{FP}$, we can ensure that f is dictatorial.

Let $\mathcal{P}$ be the set of weak transitive and S-connected fuzzy preferences on X. Using the strategy above, we proved in [7] the following theorem:

Theorem 2. *Let $f : \mathcal{P}^n \to \mathcal{P}$ be a fuzzy aggregation function satisfying IIA defined by $\{\approx^3_{\{x,y\}}\}_{x,y \in X}$ and weakly Paretian. Then, f is dictatorial.*

The theorem above is an example illustrating that we can reduce the study of a fuzzy model to the study of a family of crisp functions from the Arrovian model (and we obtain an impossibility result), then the fuzziness of the model is an illusion.

In the next section, we will see the relation of some aggregation functions with the projections exposed in the beginning of the present section.

5. Aggregation functions using ordinal expressions

These *illusory fuzziness* arises when we study the fuzzy Arrovian aggregation functions in the literature. We can consider some of these expressions. In [5] there is an aggregation function defined as $f(\mathbf{R})(x, y) = 1$ if $\forall i \in N\ R_i(x, y) > R_i(y, x)$, and $f(\mathbf{R})(x, y) = 0.5$ otherwise. In [6] we find an aggregation function defined as

$f(\mathbf{R})(x,y) = \frac{1}{n}\sum_{i\in N} R_i(x,y)$. Finally, in [4] we find $f(\mathbf{R})(x,y) = \text{median}\{\min_i\{R_i(x,y)\}, h, \max_i\{R_i(x,y)\}\}$, where $T(h,h) = 0$ for any $h \in (0,1)$.

Notice that in the first and the third functions, the same expression we used in $\succsim_R^1$ and $\succsim_R^3$ is employed, and the second is the well-known arithmetic mean. These three examples represent the present situation in the existing literature. All functions are built using the reasoning based on crisp binary relations or testing pre-existing well-known algebraic expressions as means.

If we look for functions capturing the vagueness, we should think out of the box of crisp binary relations. Moreover, testing the functions with an algebraic expression we know does not seem a suitable method. For these reasons, we stand up for the methods explained in the next section.

6. Conclusions and future research

In order to get more satisfactory results and classify the fuzzy Arrovian models, we cannot rely on functions built as algebraic expressions or close to binary relations. We need a richer framework able to express the vagueness, and it cannot be constrained by human dichotomic thinking.

We propose using topological or analytical tools to build this general framework. Using the fact that the degrees of a preference are in $[0,1]$, we can interpret a preference as a point in the cube $[0,1]^{X^2}$, the spaces of preferences as topological subspaces of $[0,1]^{X^2}$, and the aggregation functions as continuous functions (see [2] for an extended discussion). Using this framework, we expect to find suitable aggregation functions with no need to write them explicitly. For example, using differential equations.

It is important to remark that our approach is different from the topological models proposed by Chichilnisky [3]. We depart from a model with no topological structure, whereas Chichilnisky built her models using a topological background.

Considering our conclusions, we are working on finding a general framework to create suitable binary relation form fuzzy preferences and use them to study fuzzy aggregation functions. Furthermore, we will continue the study initiated in [2] about how fuzzy Arrovian models can be translated to differential equations.

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