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Model theory and metric approximate subgroups

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Abstract: In combinatorics, approximate subgroups are objects similar to subgroups up to a constant error. In 2011, using model theory, a connexion was found between finite approximate subgroups and Lie groups. This result, known as the Lie model theorem, was the starting point used to finally give a complete classification of finite approximate subgroups.

This short essay is a partial summary of a joint paper with Ehud Hrushovski (in current development) in which we prove a generalization of the Lie model theorem to the case of metric groups.

Resumen: En combinatoria, los subgrupos aproximados son objetos semejantes a subgrupos salvo un error constante. En 2011, usando teoría de modelos, se encontró una relación entre subgrupos aproximados finitos y grupos de Lie. Este resultado, conocido como el teorema de modelos de Lie, fue el punto de partida para establecer finalmente una clasificación completa de los subgrupos aproximados finitos.

Este breve ensayo es un resumen parcial de un artículo conjunto con Ehud Hrushovski (en desarrollo) en el que obtenemos una generalización del teorema de modelos de Lie al caso de grupos métricos.

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1. The Lie model theorem

Approximate subgroups are basic combinatorial structures modeling objects similar to subgroups up to a constant error. Although first definitions where given in the abelian setting by Freiman in 1973 and Ruzsa in 1994, the current definition was definitely established by Tao [4].

Notation 1. Here, for subsets of a group $X, Y \subseteq G$, we write $XY \coloneqq \{xy : x \in X, y \in Y\}$ and $X^2 \coloneqq XX$. ◂

Definition 2. A *k-approximate subgroup* of a group G is a symmetric subset $X \subseteq G$ containing the identity such that $X^2 \subseteq \Delta X$ for some $\Delta \subseteq G$ with $|\Delta| < k$.

We say that two subsets $X, Y \subseteq G$ are *k-commensurable* if there is $\Delta \subseteq G$ such that $X \subseteq \Delta Y$ and $Y \subseteq \Delta X$ with $|\Delta| \leq k$. ◂

Example 3 (geometric progressions). Let G be abelian and $u_1, \dots, u_m \in G$. The set of words $w(\bar{u})$ in G with at most N_i occurrences of u_i is a 2^m-approximate subgroup. ◂

Example 4 (nilprogressions). Let G be nilpotent of nilpotent length s and $u_1, \dots, u_m \in G$. The set of words $w(\bar{u})$ with at most N_i occurrences of u_i is an $k(s, m)$-approximate subgroup. ◂

Using model theory, Hrushovski [2] found a connexion between approximate subgroups and Lie groups. This result, known as the Lie model theorem, was the starting point used to finally give a complete classification of finite approximate subgroups by Breuillard, Green and Tao.

To explain Hrushovski's result, we need to introduce the model theoretic notion of ultraproduct. The idea is to construct models by taking "limits". Formally, we consider a sequence $(\mathfrak{M}_m)_{m\in\mathbb{N}}$ of structures (e.g., groups, graphs, fields, linear orders) and an non-atomic measure $\mathrm{u} : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$. The *ultraproduct* $\widehat{\mathfrak{M}} = \prod \mathfrak{M}_m/\mathrm{u}$ is the set of sequences $x \in \prod \mathfrak{M}_m$ modulo the equivalence relation $x = x'$ almost surely.

A fundamental theorem by Łoś says that the ultraproduct satisfies all first-order properties which are almost surely satisfied by the factors. More generally, Łoś's theorem says us that, for any first-order formula $\varphi(x)$ in $\widehat{\mathfrak{M}}$ (possibly with parameters), the definable subset $\widehat{A} = \varphi(\widehat{\mathfrak{M}}) \coloneqq \{a : a \text{ satisfies } \varphi \text{ in } \widehat{\mathfrak{M}}\}$ of $\widehat{\mathfrak{M}}$ can be written as the ultraproduct $\widehat{A} = \prod A_m/\mathrm{u}$ where $A_m = \varphi(\mathfrak{M}_m)$. The topology generated by taking as clopen subsets the definable subsets is called the logic topology.

Example 5 (non-standard analysis). When $\mathfrak{M}_m = \mathbb{R}$, we get a model of the hyperreal numbers $\widehat{\mathbb{R}}$. ◂

Morally, the Lie model theorem says that every finite approximate subgroup is "in the limit" commensurable to a compact neighbourhood of the identity of some Lie group.

Theorem 6 (Hrushovski's Lie model theorem). *Let $\widehat{G}$ be an ultraproduct of groups and $\widehat{X} \subseteq \widehat{G}$ an ultraproduct of respective finite k-approximate subgroups. Then, there exists a Lie model of $\widehat{X}$, i.e., a surjective group homomorphism $\pi : H \leq G \to L$ where*

(i) *L is a connected Lie group,*

(ii) *$X^8 \cap H$ is an approximate subgroup, generates H and is commensurable to X,*

(iii) *$K \coloneqq \ker \pi \subseteq X^4$, and*

(iv) *π is continuous and closed from the logic topology (with enough parameters).*

Using this result, Breuillard, Green, and Tao [1] concluded that every finite approximate subgroup is commensurable to a nilprogression modulo some normal subgroup.

Theorem 7 (Breuillard-Green-Tao Classification Theorem). *In the Lie model theorem, L is nilpotent and K could be made definable taking enough structure. Hence, we get the following result:*

Let G be a group and X a finite k-approximate subgroup. Then, there are $H \leq G$ and $K \trianglelefteq H$ such that

(i) *$H \cap X^8$ is $C(k)$-commensurable to X and generates H,*

(ii) *$K \subseteq X^4$, and*

(iii) *H/K is a nilpotent group of $s(k)$ nilpotent length.*

2. The metric Lie model theorem

Approximate subgroups could be further generalized to the context of metric groups. By a *metric group* we understand a group G together with a metric d invariant under translations.

Notation 8. Write $\mathbb{D}_r(X) = \{y : \text{there is } x \in X\ d(x,y) < r\}$. Note that $\mathbb{D}_r(X) = X\mathbb{D}_r(1) = \mathbb{D}_r(1)X$. ◂

Remark 9. We assume invariance under two-side translations to simplify the statements. In fact, it is possible to assume only that d is invariant under left (alternatively right) translations and some local Lipschitz condition for the right (alternatively left) translations. ◂

Definition 10. A δ-metric k-approximate subgroup $X \subseteq G$ is a symmetric subset $1 \in X^{-1} = X$ such that $X^2 \subseteq \Delta\mathbb{D}_\delta(X)$ with $|\Delta| \leq k$.

We say that two subsets $X, Y \subseteq G$ are *δ-metrically k-commensurable* if there is $\Delta \subseteq G$ such that $X \subseteq \Delta\mathbb{D}_\delta(Y)$ and $Y \subseteq \Delta\mathbb{D}_\delta(X)$ with $|\Delta| \leq k$. ◂

We do no longer assume finiteness, instead we assume that using the metric we can find nice discretizations. An *r-entropy discretization* of a set X is an r-separated finite subset $Z \subseteq X$ of maximal size. Write $N^{\mathrm{ent}}_r(X) = \sup\{|Z| : Z \subseteq X\ r\text{-separated}\}$. If there are arbitrary large r-separated finite sets, write $N^{\mathrm{ent}}_r(X) = \infty$.

Remark 11. N^{ent} is subadditive and decreasing on r. ◂

Our aim is to generalize Hrushovski's Lie model theorem to the case of metric approximate subgroups. In our case, the ultraproduct of metric groups is then a *non-standard metric group*, i.e., a group $\widehat{G}$ together with a function $\hat{d} : \widehat{G} \times \widehat{G} \to \widehat{\mathbb{R}}$ into the hyperreal numbers that satisfies the usual properties of a metric.

For a sequence $r = (r_n)_{n\in\mathbb{N}}$ of non-standard positive numbers in $\widehat{\mathbb{R}}$ with $2r_{n+1} < r_n$, we define the *r-infinitesimal thickening* of X by

$$o_r(X) := \bigcap_{n=0}^{\infty} \mathbb{D}_{r_n}(X) = \{g \in \widehat{G} : \forall n \in \mathbb{N}\ \exists x \in X\ d(g,x) < r_n\}.$$

It follows that $o_r(1_G) \trianglelefteq \widehat{G}$. Also, $o_r(X) = X o_r(1_G) = o_r(1_G)X$.

Now, we quotient out by $o_r(1)$ and check that the original arguments done by Hrushovski with $\widehat{X}$ could be adapted to $\widehat{X}/o_r(1)$. This requires to generalize various model theoretic techniques to the context of piecewise hyperdefinable sets as it was done in [3].

Theorem 12 (metric Lie model [2]). *Let (G_m, X_m, r^m) be a sequence such that*

1. *G_m is a metric group,*
2. *X_m is a symmetric subset containing the identity,*
3. *$r^m = (r^m_1, \dots, r^m_m)$ satisfies $r^m_i \geq 2r^m_{i+1}$ and*

$$N^{\mathrm{ent}}_{r^m_i/2}(X^9_m) \leq C \cdot N^{\mathrm{ent}}_{9r^m_i/2}(X_m) \in \mathbb{R} \text{ for each } i.$$

Let $\widehat{G} = \prod_{m\in\mathbb{N}} G_m/\mathcal{U}$, $\widehat{X} = \prod_{m\in\mathbb{N}} X_m/\mathcal{U}$ and $r = r^m/\mathcal{U}$ be ultraproducts. Then, there is a Lie model of $o_r(\widehat{X})$, i.e., a surjective group homomorphism $\pi : H \leq G \to L$ where

(i) *L is a connected Lie group,*

(ii) *$H \cap o_r(\widehat{X}^{12})$ is a C-approximate subgroup, generates H and is commensurable to $o_r(\widehat{X})$,*

(iii) *$K = \ker \pi \subseteq o_r(\widehat{X}^8)$ and $o_r(1_G) \leq K$,*

(iv) *π is continuous and closed from the logic topology (with enough parameters).*

Hence, as a corollary of the metric Lie model theorem we get the following result for metric approximate subgroups.

Corollary 13. *Fix constants $k \in \mathbb{R}_{>0}$, $C \in \mathbb{R}_{>1}$, $\delta \in \mathbb{R}_{>0}$ and $N, p, q \in \mathbb{N}$. Take $\alpha \geq 144^2$. There are $e := e(k, C)$ and $m := m(k, C, N, p)$ such that the following holds.*

Let G be a metric group and X a δ-metric k-approximate subgroup such that

$$N^{\mathrm{ent}}_{\delta}(X) \leq C^m N^{\mathrm{ent}}_{\alpha^m \delta}(X) \in \mathbb{R}_{>0}.$$

Then, there is a sequence $X_N \subseteq \cdots \subseteq X_1 \subseteq X^8$ satisfying the following properties:

(i) X^2 *and* X_1 *are* $2^{-p}\alpha^m$ δ*-metrically* e*-commensurable.*

(ii) $X_{n+1}X_{n+1} \subseteq \mathbb{D}_{2^{-p}\alpha^m\delta}(X_n)$.

(iii) X_n *is covered by* e *right cosets of* $\mathbb{D}_{2^{-p}\alpha^m\delta}(X_{n+1})$.

(iv) $X_{n+1}^{X_1} \subseteq \mathbb{D}_{2^{-p}\alpha^m\delta}(X_n)$.

(v) $[X_{n_1}, X_{n_2}] \subseteq \mathbb{D}_{2^{-p}\alpha^m\delta}(X_n)$ *whenever* $n < n_1 + n_1$.

(vi) $\{x \in X_1 : x^2, x^4 \in X_1 \text{ and } x^8 \in X_n\} \subseteq X_{n+1}$.

(vii) *If* $x, y \in X_1$ with $x^2 = y^2$, *then* $y^{-1}x \in \mathbb{D}_{2^{-p}\alpha^m\delta}(X_N)$.

References

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