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Principles of least action in geometric mechanics

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Abstract: A path c is said to be a solution of Hamilton's least action principle if it is a critical point of the action functional. Here, the action is the integral of a Lagrangian function L along c. This principle describes many physical theories, and has applications in other fields (optimal control, Riemannian geometry). Its solutions have a nice geometric characterization: they are integral curves of a Hamiltonian vector field on a symplectic manifold.

We introduce a generalization of this principle: the so-called *Herglotz's principle*. Here the Lagrangian not only depends on the positions and velocities, but also on the action itself. Hence, the action is no longer the integral of the Lagrangian, but it is the solution of a non-autonomous ODE. Herglotz's principle allows us to model new problems, such as some dissipative systems in mechanics (where energy is lost), thermodynamics, and some modified optimal control systems. This principle is also related to Hamiltonian systems, but switching symplectic by contact geometry. We will compare both principles, their applicability and the geometric properties of their solutions.

Resumen: Un camino c es una solución del principio de mínima acción de Hamilton si es un punto crítico del funcional de acción. En este caso, la acción es la integral de una función lagrangiana L a lo largo de c. Este principio describe numerosas teorías físicas y tiene aplicaciones en otros campos (control óptimo, geometría riemmaniana). Sus soluciones tienen una interesante caracterización geométrica: son las curvas integrales de un campo Hamiltoniano en una variedad simpléctica.

Proponemos una generalización the este principio: el *principio de Herglotz*. Ahora, el lagrangiano depende de la propia acción, además de las posiciones y velocidades. Aquí, la acción ya no es la intregral del lagrangiano, sino la solución a una EDO no autónoma. El principio de Herglotz nos permite modelizar nuevos problemas, como algunos sistemas disipativos en mecánica (con pérdidas de energía), termodinámica y algunos problemas de contol óptimo. Este principio también está relacionado con los sistemas Hamiltonianos, pero cambiando la geometría simpléctica por geometría de contacto. Compararemos ambos principios, sus aplicaciones y las propiedades geométricas de sus soluciones.

Keywords: variational principles, Herglotz principle, contact Hamiltonian systems, Lagrangian mechanics.

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1. Principles of least action

In the 17th century, Fermat formulated the laws of geometric optics in the following way: "light travels between two given points along the path of shortest time". This is known as *the principle of least time* or *Fermat's principle*. Knowing the velocity of light at every point of space, one can use this principle to compute the trajectories of the light rays, obtaining the laws of refraction and reflection.

Many principles such as this one were introduced in mechanics, by Maupertuis, Euler, Lagrange and Hamilton. Although they have different physical interpretation, all these principles (including Fermat's) fit on the same mathematical framework. Given a Lagrangian function $L : TQ \times \mathbb{R} \to \mathbb{R}$[1], let Ω be the space[2] of curves $c : [0, T] \to Q$ with fixed endpoints (say, $c(0) = q_0$, $c(T) = q_1$). We define the action $\mathcal{A} : \Omega \to \mathbb{R}$ of any curve c as

$$\mathcal{A}(c) = \int_{t_0}^{t_1} L(c(t), \dot{c}(t))\, \mathrm{d}t.$$

The principle of least action states that a path c will be followed by the system if and only if c is a critical point of $\mathcal{A}$ among all paths in Ω. The solutions of this principle are precisely the paths that satisfy the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}(c(t), \dot{c}(t))\right) - \frac{\partial L}{\partial q^i}(c(t), \dot{c}(t)) = 0.$$

Picking as a Lagrangian the inverse of the velocity of light in the media, we retrieve Fermat's principle. If we instead pick as the Lagrangian the kinetic minus the potential energy, $L = T - V$, we obtain *Hamilton's principle* for conservative mechanical systems (where energy remains constant), whose solutions satisfy Newton's Second Law[3].

1.1. Why variational principles?

There are many mathematical and physical[4] reasons to study variational principles. In physics, it has been found that the least action principle (sometimes with extensions) can model a wide range of phenomena, including field theory and general relativity. Furthermore, developments on this principle lead to quantum field theory (through Feynman path integral). Outside of physics, least action principles appear in control theory (optimal control problems) and characterize geodesics in Riemannian and Finsler geometry. If we are working with a second order ODE that is the Euler-Lagrange equation of some Lagrangian also provides access to useful mathematical tools.

- The problem is framed in "*generalized coordinates*", i.e., the Euler-Lagrange equation is the same on every coordinate system[5]. This does not hold with Newton's equation, where new terms appear when we work in non-cartesian coordinates or in non-inertial frames.
- Presence of *symplectic geometry* [1, 3]. A (regular) Lagrangian provides a symplectic form $\omega_L = \mathrm{d}q^i \wedge \mathrm{d}(\partial L/\partial \dot{q}^i)$ which is preserved by the evolution of the system. Knowledge on the topology and geometry of symplectic manifolds provides a better understanding on the dynamics of the system.
- It allows to prove *Noether theorems* relating symmetries and conserved quantities.
- It can be used to construct *variational integrators* [12], that preserve the geometry of the system and have better long term behavior than methods for more general ODEs, such as Runge-Kutta.

[1] Here, TQ is the tangent bundle of the configuration manifold TQ, i.e., the space of positions and velocities, with coordinates $(q^i, \dot{q}^i)$.

[2] This space is an infine-dimensional manifold locally modeled on a space of functions $[0, 1] \to \mathbb{R}^n$. We recommend the interested reader [1] and the references therein.

[3] In cartesian coordinates x^i, if $L = \frac{1}{2} m \sum_i (\dot{x}^i)^2 - V(q^i)$, then the equation of motion is $m\ddot{x}^i = -\frac{\partial V}{\partial x^i} = F_i$, where F_i is the force.

[4] Some natural philosophers, such as Maupertuis, were interested in these principles on metaphysical grounds, since they express that nature "acts by the simplest means"[13]. These arguments would now probably be considered unscientific.

[5] Now, within the framework of geometric mechanics, modern differential geometric language is used and the dynamics can be described without the use of coordinates.

2. Herglot's variational principle

There are, however, many interesting systems that cannot be modeled with Hamilton's principle. For example, all mechanical systems that do not preserve the energy, such as the damped harmonic oscillator:

$$\ddot{q}^2 + q = -\gamma \dot{q}. \tag{1}$$

A simple extension is to allow that the Lagrangian depends explicitly on time, however this is not enough on many situations. In 1930, Herglotz [11] proposed a more general formulation, the so-called *Herglotz's variational principle*. Here the Lagrangian not only depends on the positions and velocities, but also on the action itself. Hence, the action is no longer the integral of the Lagrangian, but it is the solution of a non-autonomous ODE. This will allow us to model a wider class of systems.

2.1. Herglotz's principle and Herglotz's equations

Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be the Lagrangian function, where the last coordinate will be denoted by z[6]. In order to formalize the idea of an "action dependent Lagrangian", we will define the action through a non-autonomous ODE, instead of an integral. First we fix the *initial action* $z_0 \in \mathbb{R}$, and we define the Herglotz action $\mathcal{A} : \Omega \to \mathbb{R}$ as follows. Given $c \in \Omega$, we solve the Cauchy problem $\dot{z}_c = L(c, \dot{c}, z_c)$ with initial condition $z_c(0) = z_0$. Now we define the Herglotz action[7] $\mathcal{A}$ as

$$\mathcal{A}(c) = z_c(T) - z_0 = \int_0^T L(c(t), \dot{c}(t), z_c(t))\, \mathrm{d}t.$$

In this case, c is a critical point of $\mathcal{A} : \Omega \to \mathbb{R}$ if and only if $(c, \dot{c}, z_c)$ satisfies Herglotz's equations [6]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}.$$

We note that the energy $E_L = L - \dot{q}^i \frac{\partial L}{\partial \dot{q}_i}$ is dissipated along the solutions χ of Herglotz equations at a rate $\partial L/\partial z$. Indeed, if we pick $L = \frac{1}{2}(\dot{q})^2 - q - \gamma z$, the Herglotz equation is the equation of motion of the damped harmonic oscillator (1), and we have $\mathrm{d}E_L/\,\mathrm{d}t = -\gamma E_L$.

3. Further topics

In the recent years a considerable amount of new results related to the Herglotz principle and Lagrangian contact mechanics have been published. We list some of the topics on which there is active research.

- *Contact geometry* is to Herglotz's principle [7] as symplectic geometry is to Hamilton's principle. A contact form $\eta_L = \mathrm{d}z - \partial L/\partial \dot{q}^i \mathrm{d}q^i$ is preserved by the flow of the system.
- *Noether theorems* [8] also exist in this context. However, symmetries do not correspond to conserved, but to *dissipated quantities*, that is, quantities that decay at the same rate as the energy.
- *Variational integrators* can be constructed through the Herglotz principle [15, 16].
- Herglotz's principle and some related variational principles can be applied to the description of *thermodynamic processes* [14] and mechanical systems with dissipation [2], among others.

[6] We can also use Hamilton's principle for explicitly time dependent Lagrangians $L : TQ \times \mathbb{R} \to \mathbb{R}$, where we think of the $\mathbb{R}$ coordinate as "time" t. The corresponding Euler-Lagrange equations have the same form as in Hamilton's principle. These should not be confused with contact Hamiltonian systems and Herglotz's variational principle, where the extra coordinate represents the "action".

[7] We remark that this action coincides with the Euler-Lagrange action when L does not depend on z. It is also important to note that the action functional does not only depend on the Lagrangian, like in Hamilton's principle, but also on the initial action z_0.

- *Higher order systems* can be considered [4]. Lagrangians depend not only on positions and velocities, but also on higher order derivatives.
- Constraints can be added to the motion of these systems. They can be either *vakonomic*, that is, implemented on the variations, or *nonholonomic* [5], on the infinitesimal variations. The first ones are useful for optimal control theory [9], while the second ones appear on mechanical systems.
- We can also study *the inverse problem.* Given a second order ODE, does there exist a Lagrangian such that the ODE is its Euler-Lagrange/Herglotz equation?
- Contact Lagrangian mechanics can be extended to *noncorservative field theories* [10].

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