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A probabilistic proof of Meir-Moon theorem

Víctor J. Maciá
Washington University in St. Louis
victor.macia@wustl.edu

Abstract: In this work we introduce a probabilistic proof of the Meir-Moon theorem. This theorem gives an asymptotic formula for the coefficients of the solution to Lagrange's equation. Let ψ be an analytic function, with non-negative coefficients, on a disk around $z = 0$ and $f(z) = z\psi(f(z)) = \sum_{n\geq 0} a_n z^n$ Lagrange's equation with data ψ. Under certain conditions over ψ, the coefficients of f satisfy the asymptotic formula

$$a_n \sim \frac{1}{\sqrt{2\pi}} \frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)} \frac{1}{n^{3/2}\tau^n}, \qquad \text{as } n \to \infty,$$

for certain $\tau > 0$. We make no use of saddle point approximation methods: we cast the question in the probabilistic setting of Khinchin families and the local central limit theorem for lattice random variables.

This is based on a joint work with José L. Fernández (Universidad Autónoma de Madrid).

Resumen: En este trabajo presentamos una prueba probabilística del teorema de Meir-Moon. Este teorema da una fórmula asintótica para los coeficientes de la solución de la ecuación de Lagrange. Sea ψ una función analítica en un disco alrededor de $z = 0$, con coeficientes no negativos, y $f(z) = z\psi(f(z)) = \sum_{n\geq 0} a_n z^n$ la ecuación de Lagrange con dato ψ. Bajo ciertas condiciones sobre ψ, los coeficientes de f satisfacen la fórmula asintótica

$$a_n \sim \frac{1}{\sqrt{2\pi}} \frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)} \frac{1}{n^{3/2}\tau^n}, \qquad \text{cuando } n \to \infty,$$

para cierto $\tau > 0$. En esta demostración no utilizamos métodos de punto de silla, los ingredientes son las familias de Khinchin combinadas con cierto teorema local central del límite para variables aleatorias reticulares.

Esto está basado en un trabajo conjunto con José L. Fernández (Universidad Autónoma de Madrid).

Keywords: Khinchin families, probability, Meir-Moon, Lagrange's equation, local central limit theorem, asymptotic analysis, analytic combinatorics.

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1. Introduction and preliminaries

Our objective is to give an asymptotic formula for the coefficients of certain analytic functions. See [5] and [6]. In particular, we want to study generating functions of combinatorial sequences, that is, analytic functions with non-negative coefficients, which are solutions to Lagrange's equation.

In this section, we introduce some definitions and auxiliary results that will be useful later. First, we introduce Khinchin families. Later on, we enumerate some of their properties. We will continue here with Lagrange's equation and its solution: Lagrange inversion formula. Finally, we amalgamate all these tools into a sketch of a proof of Meir-Moon Theorem. For complete details see [1] and [3].

1.1. Khinchin families

We start by defining the class of analytic functions $\mathcal{K}$. This class is convenient for the study of generating functions of combinatorial sequences.

Definition 1. We say that $\psi(z) = \sum_{n\geq 0} b_n z^n$ is in the class $\mathcal{K}$ if ψ is an analytic function with radius of convergence $R > 0$, has non-negative coefficients, $b_0 > 0$ and there exists certain integer $n_0 > 1$ such that $b_{n_0} > 0$. ◂

Now we define the Khinchin family associated to a power series $\psi \in \mathcal{K}$.

Definition 2. Let $\psi \in \mathcal{K}$, an analytic function with power series representation $\psi(z) = \sum_{n\geq 0} b_n z^n$. We define the Khinchin family associated to ψ as the indexed family of discrete random variables $(Y_t)_{[0,R)}$. For each $t \in [0, R)$ we have

$$\mathbb{P}(Y_t = n) = \frac{b_n t^n}{\psi(t)}, \quad \text{for all } n \in \{0, 1, 2, \dots\}.$$ ◂

We will write (Y_t) instead of $(Y_t)_{[0,R)}$ when the radius of convergence is clear from the context. After this line, unless explicitly stated, $R > 0$ will denote the radius of convergence of ψ and (Y_t) the Khinchin family associated to ψ.

The mean and variance functions are

$$\mathbb{E}(Y_t) = m(t) = \frac{t\psi'(t)}{\psi(t)},$$
$$\mathbb{V}(Y_t) = \sigma^2(t) = tm'(t),$$

for all $t \in [0, R)$.

The characteristic function of (Y_t) is

$$\mathbb{E}\left(e^{i\theta Y_t}\right) = \frac{\psi(te^{i\theta})}{\psi(t)}, \quad \text{for all } t \in [0, R) \text{ and } \theta \in \mathbb{R}.$$

In particular, for the normalized Khinchin family $\breve{Y}_t = (Y_t - m(t))/\sigma(t)$ we have

$$\mathbb{E}\left(e^{i\theta \breve{Y}_t}\right) = \mathbb{E}\left(e^{i\theta Y_t/\sigma(t)}\right) e^{-i\theta m(t)/\sigma(t)} = \frac{\psi(te^{i\theta/\sigma(t)})}{\psi(t)} e^{-i\theta m(t)/\sigma(t)}, \quad \text{for all } t \in [0, R) \text{ and } \theta \in \mathbb{R}.$$

The following property will be crucial

Lemma 3. *Suppose $f, g \in \mathcal{K}$, both with radius of convergence at least $R > 0$, and let (X_t) and (W_t) be its Khinchin families, respectively. Denote (Z_t) the Khinchin family associated to $f \cdot g$. Then, for each $t \in [0, R)$, we have*

$$Z_t \stackrel{d}{=} X_t \oplus W_t.$$

Here $\oplus$ denotes sum of independent random variables.

1.2. Lagrange's equation

Let $\psi \in \mathcal{K}$. We will refer to the equation

$$f(z) = z\psi(f(z))$$

as Lagrange equation with data ψ. The coefficients of f, the unique solution to Lagrange's equation with data ψ, are given by the following theorem.

Theorem 4 (Lagrange inversion formula). *Let $f(z)$ and $\psi(z)$ be two analytic functions at certain neighborhood of $z = 0$ such that $\psi(0) \neq 0$ and*

$$f(z) = z\psi(f(z))$$

for $z \in D(0,\delta)$. Then,

$$a_n = COEFF_n\,[f(z)] = \frac{1}{n} COEFF_{n-1}\,[\psi(z)^n]\,.$$

See, for instance, [4].

1.3. A Hayman type formula

Denote $S_t^{(n)} = Y_t^{(1)} \oplus \cdots \oplus Y_t^{(n)}$, where the random variables $Y_t^{(i)}$ are (i.i.d.) copies of Y_t.

Lemma 5. *With the hypotheses above,*

$$a_n = \frac{1}{2\pi}\frac{1}{\sigma_\psi(t)}\frac{\psi(t)^n}{t^{n-1}n^{3/2}}\int_{-\pi\sigma_\psi(t)\sqrt{n}}^{\pi\sigma_\psi(t)\sqrt{n}} \mathbb{E}\left(e^{i\theta \breve{S}_t^{(n)}/\sqrt{n}}\right) e^{i\theta(n-nm_\psi(t))/(\sigma_\psi(t)\sqrt{n})}\, d\theta.$$

Corollary 6. *Suppose there exists $\tau \in (0,R)$ such that $m_\psi(\tau) = 1$, then*

$$a_n = \frac{1}{2\pi}\frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)}\frac{1}{n^{3/2}\tau^n}\int_{-\pi\sigma_\psi(\tau)\sqrt{n}}^{\pi\sigma_\psi(\tau)\sqrt{n}} \mathbb{E}\left(e^{i\theta \breve{S}_\tau^{(n)}/\sqrt{n}}\right) d\theta. \tag{1}$$

See [3].

2. Meir-Moon theorem

Theorem 7 (Meir-Moon). *Let $\psi(z) \in \mathcal{K}$ be a holomorphic function in a disc of radius $R > 0$ around $z = 0$. Let $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$. Suppose that*

- $\gcd\{n \geq 1 : b_n > 0\} = \gcd(\psi) = 1$,
- *there exists $\tau \in (0,R)$ such that*

$$m_\psi(\tau) = \frac{\tau\psi'(\tau)}{\psi(\tau)} = 1.$$

Then, the coefficients of $f(z) = z\psi(f(z)) = \sum_{n=1}^{\infty} a_n z^n$ satisfy the asymptotic formula

$$a_n \sim \frac{1}{\sqrt{2\pi}}\frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)}\frac{1}{n^{3/2}\tau^n}, \qquad \text{as } n \to \infty.$$

Sketch of the proof. Apply the local central limit theorem for lattice random variables to the integral I_n on the right-hand side of formula (1), see, for instance, [2] and [3]. Then we have that $\lim_{n\to\infty} I_n = \sqrt{2\pi}$. ∎

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