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On Grünbaum type inequalities

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Abstract: Given a compact set $K \subset \mathbb{R}^n$ of positive volume, and fixing a hyperplane H passing through its centroid, we find a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, depending on the concavity nature of the function that gives the volumes of cross-sections (parallel to H) of K, where K^- denotes the intersection of K with a halfspace bounded by H. When K is convex, this inequality recovers a classical result by Grünbaum. To this respect, we also show that the log-concave case is the limit concavity assumption for such a generalization of Grünbaum's inequality.

Resumen: Dado un conjunto compacto $K \subset \mathbb{R}^n$ y un hiperplano H pasando por su centroide, encontramos una cota inferior óptima para el cociente $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, dependiendo de la concavidad de la función que nos da el volumen de las secciones (paralelas a H) de K, donde K^- denota la intersección de K con el semiespacio delimitado por H. Cuando K es convexo, esta desigualdad recupera un resultado clásico de Grünbaum. Además, veremos que el caso log-cóncavo es la mínima concavidad exigible para este tipo de generalización de la desigualdad de Grünbaum.

Keywords: centroid, convex body, Grünbaum, inequality.

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1. Introduction

Let $K \subset \mathbb{R}^n$ be a compact set with positive volume $\mathrm{vol}(K)$ (i.e., with positive n-dimensional Lebesgue measure). The centroid of K is the affine-covariant point

$$\mathrm{g}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x \,\mathrm{d}x.$$

Furthermore, if we write $[\cdot]_1$ for the first coordinate of a vector with respect to the basis, by Fubini's theorem, we get

$$[\mathrm{g}(K)]_1 = \frac{1}{\mathrm{vol}(K)} \int_a^b t f(t) \,\mathrm{d}t. \tag{1}$$

The classical Grünbaum inequality, originally proven in [2], states that if $K \subset \mathbb{R}^n$ is a convex body with centroid at the origin, then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n, \tag{2}$$

where $K^- = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 0\}$ and $K^+ = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \geq 0\}$ represent the parts of K which are split by the hyperplane $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$, for any given $u \in \mathbb{S}^{n-1}$. Equality holds, for a fixed $u \in \mathbb{S}^{n-1}$, if and only if K is a cone in the direction u, i.e., the convex hull of $\{x\} \cup (K \cap (y + H))$, for some $x, y \in \mathbb{R}^n$.

The underlying key fact in the original proof of (2) (see [2]) is the following classical result (see, e.g., [1, Section 1.2.1] and also [4, Theorem 12.2.1]).

Theorem 1 (Brunn's concavity principle). *Let $K \subset \mathbb{R}^n$ be a non-empty compact and convex set and let H be a hyperplane. Then, the function $f : H^\perp \to \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is $(1/(n-1))$-concave.*

In other words, for any given hyperplane H, the cross-sections volume function f to the power $1/(n-1)$ is concave on its support, which is equivalent (due to the convexity of K) to the well-known *Brunn-Minkowski inequality*.

Although this property cannot be in general enhanced, one can easily find compact convex sets for which f satisfies a stronger concavity, for a suitable hyperplane H. Thus, on the one hand, it is natural to wonder about a possible enhanced version of Grünbaum's inequality (2) for the family of those compact convex sets K such that (there exists a hyperplane H for which) f is p-concave, i.e., f to the power p is concave, with $1/(n-1) < p$. On the other hand, one could expect to extend this inequality to compact sets K, not necessarily convex, for which f is p-concave (for some hyperplane H), with $p < 1/(n-1)$.

Observing that the equality case in Grünbaum's inequality (2) is characterized by cones, that is, those sets for which f is $(1/(n-1))$-affine (i.e., such that $f^{1/(n-1)}$ is an affine function), the following sets of revolution, associated to p-affine functions, arise as natural candidates to be the extremal sets, in some sense, of these inequalities.

Definition 2. Let $p \in \mathbb{R}$ and let $c, \gamma, \delta > 0$ be fixed. Then:

(i) If $p \neq 0$, let $g_p : I \to \mathbb{R}_{\geq 0}$ be the non-negative function given by $g_p(t) = c(t + \gamma)^{1/p}$, where $I = [-\gamma, \delta]$ if $p > 0$ and $I = (-\gamma, \delta]$ if $p < 0$.

(ii) If $p = 0$, let $g_0 : (-\infty, \delta] \to \mathbb{R}_{\geq 0}$ be the non-negative function defined by $g_0(t) = c\mathrm{e}^{\gamma t}$.

Let $u \in \mathbb{S}^{n-1}$ be fixed. By C_p we denote the set of revolution whose section by the hyperplane $\{x \in \mathbb{R}^n : \langle x, u\rangle = t\}$ is an $(n-1)$-dimensional ball of radius $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ with axis parallel to u. (We warn the reader that, in the following, we will use the word "radius" to refer to such a generating function $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ of the set C_p, for short.) ◂

In this short paper we discuss the above-mentioned problem and show that it has a positive answer in the full range of $p \in [0, \infty]$ (in the following, $\sigma_{H^\perp}$ denotes the *Schwarz symmetrization* with respect to $H^\perp$).

2. Main results

As mentioned in the introduction, the sets C_p associated to (cross-sections volume) functions that are p-affine (see Definition 2) seem to be possible extremal sets of such expected inequalities. So, we start by showing the precise value of the ratio $\operatorname{vol}(\cdot^-)/\operatorname{vol}(\cdot)$ for the sets C_p.

Lemma 3 ([3]). *Let $p \in (-\infty,-1)\cup[0,\infty)$ and let H be a hyperplane with unit normal vector $u \in \mathbb{S}^{n-1}$. Let g_p and D_p, with axis parallel to u, be as in Definition 2, for any fixed $c,\gamma,\delta > 0$. If C_p has centroid at the origin, then*

$$\frac{\operatorname{vol}(C_p^-)}{\operatorname{vol}(C_p)} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{3}$$

where, if $p = 0$, the above identity must be understood as

$$\frac{\operatorname{vol}(C_0^-)}{\operatorname{vol}(C_0)} = \lim_{p\to 0^+}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p} = \mathrm{e}^{-1}. \tag{4}$$

Before showing the general case, we have that if the cross-sections volume function f associated to a compact set K is increasing in the direction of the normal vector of H, then the minimum of the ratios $\operatorname{vol}(K^-)/\operatorname{vol}(K)$ and $\operatorname{vol}(K^+)/\operatorname{vol}(K)$ is attained at $\operatorname{vol}(K^-)/\operatorname{vol}(K)$, independently of the concavity nature of f.

Proposition 4 ([3]). *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane, with unit normal vector $u \in \mathbb{S}^{n-1}$, such that the function $f : H^\perp \to \mathbb{R}_{\geq 0}$ given by $f(x) = \operatorname{vol}_{n-1}(K \cap (x+H))$ is quasi-concave with $f(bu) = \max_{x\in H^\perp} f(x)$, where $[au, bu] = K|H^\perp$. Then,*

$$\frac{\operatorname{vol}(K^+)}{\operatorname{vol}(K)} \geq \frac{1}{2}.$$

Our main result reads as follows:

Theorem 5 ([3]). *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane such that the function $f : H^\perp \to \mathbb{R}_{\geq 0}$ given by $f(x) = \operatorname{vol}_{n-1}(K\cap(x+H))$ is p-concave, for some $p \in [0,\infty)$. If $p > 0$, then*

$$\frac{\operatorname{vol}(K^-)}{\operatorname{vol}(K)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{5}$$

with equality if and only if $\sigma_{H^\perp}(K) = C_p$. If $p = 0$, then

$$\frac{\operatorname{vol}(K^-)}{\operatorname{vol}(K)} \geq \mathrm{e}^{-1}. \tag{6}$$

The inequality is sharp, that is, the quotient $\operatorname{vol}(K^-)/\operatorname{vol}(K)$ comes arbitrarily close to e^{-1}.

Note that the "limit case" $p = \infty$ in Theorem 5 is also trivially fulfilled. Indeed, if f is ∞-concave, then f is constant on $[a,b]$, and thus $0 = [\mathrm{g}(K)]_1 = b + a$ (see (1)), which yields that $a = -b$ and, hence,

$$\frac{\operatorname{vol}(K^-)}{\operatorname{vol}(K)} = \frac{1}{2} = \lim_{p\to\infty}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

Finally, we show that Theorem 5 cannot be extended to the range of $p \in (-\infty,-1)$. In fact, we have a more general result:

Proposition 6 ([3]). *Let $p \in (-\infty,-1)$. There exists no positive constant β_p such that*

$$\min\left\{\frac{\operatorname{vol}(K^-)}{\operatorname{vol}(K)}, \frac{\operatorname{vol}(K^+)}{\operatorname{vol}(K)}\right\} \geq \beta_p$$

for all compact sets $K \subset \mathbb{R}^n$ with non-empty interior and with centroid at the origin, for which there exists H such that $f(x) = \operatorname{vol}_{n-1}(K\cap(x+H))$, $x \in H^\perp$, is p-concave.

We conclude this work by discussing that the statement of Theorem 5 cannot be extended to the range of $p \in (-1/2, 0)$ either. Therefore, this fact (jointly with the case in which $p \in (-\infty, -1)$, collected in Proposition 6) gives that $[0, \infty]$ is the largest subset of the real line (with respect to set inclusion) for which C_p provides us with the infimum value for the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$, among all compact sets with (centroid at the origin and) p-concave cross-sections volume function.

Note 7. The results presented in this contribution were originally proven in [3]. ◂

References

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