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Convergence of manifolds with totally bounded curvature

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Abstract: When we study the topological consequences of the curvature, one of the most successful tools is the Gromov-Hausdorff distance. It allows us to study the convergence of manifolds under some metric constrictions. In this paper, we will focus on the convergence with totally bounded sectional curvature. We will explain some of the most important results in the area (Cheeger-Gromov, Fukaya, Naber & Tian). We will use a result due to S. Roos explaining the collapse with totally bounded curvature with codimension 1, to show our current work: we are trying to generalize it to every codimension using the Uryson k-widths instead of the injectivity radius.

Resumen: Al intentar estudiar las consecuencias topológicas de la curvatura seccional en las variedades riemannianas, la distancia Gromov-Hausdorff es una herramienta muy útil. Gracias a ella podemos estudiar la convergencia de variedades con ciertas restricciones métricas. En este artículo, nos centraremos en las variedades con curvatura seccional totalmente acotada. Explicaremos algunos de los resultados más significativos (Cheeger-Gromov, Fukaya, Naber y Tian). A raíz de un resultado de S. Roos sobre el colapso con codimensión 1 y curvatura totalmente acotada, mostraremos nuestra línea de trabajo actual en pos de generalizar dicho resultado para cualquier codimensión y usando los Uryson k-widths en vez del radio de inyectividad.

Keywords: Riemannian geometry, collapse.

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The main purpose of this study is to understand the geometry of the limit space X, in the Gromov-Hausdorff sense, of a sequence of n-Riemannian manifolds $\{M_i^n\}$ with $|\sec_{M_i}| \leq 1$ and $\operatorname{diam}(M_i) \leq D$, for all $i \in \mathbb{N}$. This has certain implications in some areas where the sequence is formed by Kähler or Einstein manifolds.

We are trying to generalize, for every dimension, a result due to S. Roos. We will be using the Hausdorff dimension on the limit space because, on the majority of the situations, it will be a metric space.

Theorem 1 ([13]). *Let $\{(M_i^n, g_i)\}$ be a sequence of n-Riemannian manifolds such that $\operatorname{diam}(M_i) \leq D$ and $|\sec_{M_i}| \leq 1$, for all i. We have that $M_i \xrightarrow{GH} X$, where X is a metric space. Then, these two are equivalent:*

- $\dim_H X \geq (n-1)$.
- *For every $r > 0$, there exist $C(n, r, X) > 0$ such that*

$$C \leq \frac{\operatorname{vol}(B_r^{M_i}(x))}{\operatorname{inj}_{M_i}(x)}, \text{ for every } x \in M_i \text{ and } i \in \mathbb{N}.$$

We are interested in generalizing this theorem to every dimension on the limit X. In this situation we will work with the Uryson k-widths instead of the injectivity radius. This approach allows us to obtain Roos's result as a corollary. We will also obtain results of Gromov [10] and Perelman [12] as corollaries.

For that purpose we are working with a commuting diagram developed by Fukaya [6, 7],

$$\begin{array}{ccc} FM_i & \xrightarrow{\tilde{\eta}_i} & \tilde{X} \\ \pi_i \downarrow & & \downarrow \pi \\ M_i & \xrightarrow{\eta_i} & X \end{array}$$

which relates a sequence of n-Riemannian manifolds $\{(M_i, g_i)\}$ with totally bounded curvature and bounded diameter, its Gromov-Hausdorff limit X, the frame bundles FM_i of each manifold of the sequence and the $C^{1,\alpha}$ limit $\tilde{X}$ of those frame bundles. We will like to relate the widths of the fibres of these maps to the widths of the manifolds and, with that, generalize Roos's result.

1. Gromov-Hausdorff Distance

The Gromov-Hausdorff distance allows us to define a convergence for sequences of metric spaces. In general, the limit space does not need to conserve any regularity properties if the items of the sequence are manifolds. There can appear topological and metric singularities.

Gromov extended the notion of Hausdorff distance involving two different metric spaces. It is known as Gromov-Hausdorff distance:

Definition 2. Let X, Y be metric spaces. We define the *Gromov-Hausdorff distance* between X and Y as

$$\mathrm{d}_{GH}(X, Y) = \inf_Z \{\mathrm{d}_H(f(X), g(Y)\},$$

where the infimum is taken between all the isometric embeddings $f : X \to Z$ and $g : Y \to Z$ in the same ambient metric space Z. ◂

To be precise, d_{GH} is a distance in the set of compact metric spaces after identifying isometric pairs.

2. Collapse of Riemannian manifolds with $|\sec_{M_i}| \leq 1$

We are going to show some of the most relevant results of the convergence of Riemannian manifolds with totally bounded curvature. We will use the Hausdorff dimension in the limit space and the usual one on the manifolds of the sequence.

The first case is when the limit space has the same dimension as the manifolds of the sequence. Cheeger showed that we can extract a subsequence of manifolds which converges to a manifold:

Theorem 3 ([2, 3]). *Let $\{(M_i^n, g_i)\}$ be a sequence of compact Riemannian manifolds such that $|\sec_{M_i}| \leq 1$, $\operatorname{diam}(M_i) \leq D$ and $\operatorname{vol}(M_i) > v$, for all i. Then, there exists a subsequence $\{M_j\} \subset \{M_i\}$ such that $M_j \xrightarrow{GH} N$, where N is a $C^{1,\alpha}$ Riemannian manifold with $0 < \alpha < 1$.*

Now, our aim is working with limit spaces with less dimension than the manifolds of the sequence.

Definition 4 (collapse). Let $\{M_i^n\}$ be a sequence of Riemannian manifolds such that $M_i \xrightarrow{GH} X$, where X is a metric space. We say that there exists *collapse* if $\dim_H X < \dim M_i = n$. ◂

One of the most important results on the field is the *almost flat theorem* of Gromov.

Definition 5. An *infranil manifold* N/Γ, is a quotient manifold where N is a simply connect nilpotent Lie group and Γ is a discrete cocompact subgroup of $\operatorname{Aut}(N) \ltimes N$. ◂

Definition 6 (almost flat manifold). We say that M^n is an almost flat Riemannian manifold if there exists a set of metrics g_ϵ such that $|\sec_{M_\epsilon}| \leq 1$, $\operatorname{diam}(M_\epsilon) \leq \epsilon$, for all $\epsilon > 0$.

For example, every flat manifold is almost flat. ◂

Theorem 7 ([8]). *A Riemannian manifold M^n is almost flat if and only if it is infranil. In other words, if $M_\epsilon \xrightarrow{GH} \{pt\}$, we have that M_ϵ is diffeomorphic to one which is infranil.*

Later on, Cheeger and Gromov worked on *F-structures* [4]. They defined them as actions of torus sheaves on normal coverings of the manifolds on the sequence. This action gives orbits on the manifolds which are going to collapse to points. They proved that if a manifold admits such structure, we can construct a family of metrics g_δ such that the manifold collapses when $\delta \to 0$ while the curvature is totally bounded. In [5], they constructed the converse of the above result.

At present, the most up-to-date results are due to Naber and Tian [11]. In that paper, they try to understand all the geometry besides Fukaya's diagram. They built two fibre bundles $V^T, V^{ad} \to \widetilde{X}$ which show the unwrapped limit geometry above the limit space X.

3. k-dimensional Uryson width

We begin with the definition of the k-width:

Definition 8 (Uryson k-width [12]). The *k-dimensional Uryson width* $w_k(X)$ of a metric space X is defined as the exact lower bound of those $\delta > 0$ for which there exists a k-dimensional space P and a continuous map $f : X \to P$ all of whose inverse images have diameters at most δ. ◂

Remark 9. Let M^n be an n-Riemannian manifold. Then,

- $w_0(M) = \operatorname{diam}(M)$.
- $w_i(M) = 0$, for all $i \geq n$. ◂

Using these metric invariants, Gromov and later Perelman proved some inequalities involving the volume of a Riemannian manifold and the product of all k-widths:

Theorem 10 ([10, 12]). *Let M be an almost flat n-Riemannian manifold. Then, there exists $c > 0$ such that*

$$c^{-1} \cdot \operatorname{vol}(M) \leq \prod_{i=0}^{n-1} w_i(M) \leq c \cdot \operatorname{vol}(M). \tag{1}$$

Let M be a closed n-Riemannian manifold nonnegatively curved. Then, there exists $c > 0$ such that (1) holds.

Due to this result, we can conjecture the following taking into account that the fibres of our collapse are infranil manifolds:

Conjecture 11. *Let $\{(M_i^n, g_i)\}$ be a sequence of n-Riemannian manifolds such that $\operatorname{diam}(M_i) \leq D$ and $|\sec_{M_i}| \leq 1$, for all i. We have that $M_i \xrightarrow{GH} X$, where X is a metric space. Then, these two are equivalent:*

- $\dim_H X \geq (n-k)$.
- *For every $r > 0$, there exists $C(n,r,X) > 0$ such that*

$$C \leq \frac{\operatorname{vol}(B_r^{M_i}(x))}{\Pi_{j=0}^{k-1} w_{n-k+j}(M_i)}, \text{ for every } x \in M_i \text{ and } i \in \mathbb{N}.$$

Remark 12. If $k = 1$,

$$w_{n-1}(M_i) = w_0(F_i^p) = \operatorname{diam}(F_i^p) = 2\operatorname{inj}(M_i),$$

where F_i^p is de fibre in the Fukaya's map $f : M_i \to X$. Therefore, our conjecture implies Theorem 1. ◂

Remark 13. Suppose $M_i \xrightarrow{GH} \{pt\}$ with totally bounded curvature (Theorem 7). Then, our conjecture implies Theorem 10, because it is our desired result with that kind of collapse. ◂

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