3rd *BYMAT Conference* (2020)

Inverse Turán numbers

Ervin Győri
Alfréd Rényi Institute of Mathematics
gyori.ervin@renyi.mta.hu

✉ Nika Salia
Alfréd Rényi Institute of Mathematics
salia.nika@renyi.hu

Casey Tompkins
Discrete Mathematics Group
Institute for Basic Science (IBS)
ctompkins496@gmail.com

Oscar Zamora
Universidad de Costa Rica, San José
oscarz93@yahoo.es

Abstract: For given graphs G and F, the Turán number $\mathrm{ex}(G, F)$ is defined to be the maximum number of edges in an F-free subgraph of G. Foucaud, Krivelevich and Perarnau and later independently Briggs and Cox introduced a dual version of this problem wherein for a given number k, one maximizes the number of edges in a host graph G for which $\mathrm{ex}(G, H) < k$.

Addressing a problem of Briggs and Cox, we determine the asymptotic value of the inverse Turán number of the paths of length 4 and 5 and provide an improved lower bound for all paths of even length. Moreover, we obtain bounds on the inverse Turán number of even cycles giving improved bounds on the leading coefficient in the case of C_4. Finally, we give multiple conjectures concerning the asymptotic value of the inverse Turán number of C_4 and P_ℓ, suggesting that in the latter problem the asymptotic behavior depends heavily on the parity of ℓ.

Resumen: Para dos grafos G y F, el número de Turán $\mathrm{ex}(G, F)$ se define como el número máximo de aristas en un subfrafo F-libre de G. Foucaud, Krivelevich y Perarnau, e independientemente Briggs y Cox, introdujeron una versión dual de este problema en la que, dado un número k, se maximiza el número de aristas en un grafo G tal que $\mathrm{ex}(G, F) < k$.

Abordando un problema de Briggs y Cox, determinamos el valor asintótico del número de Turán inverso de los caminos de longitud 4 y 5, y proporcionamos una cota inferior mejorada para todos los caminos de longitud par. Además, obtenemos cotas para el número de Turán inverso de los ciclos pares, dando cotas mejoradas para el término dominante en el caso de C_4. Por último, planteamos múltiples conjeturas sobre el número de Turán inverso de C_4 y P_ℓ, sugiriendo que en el segundo caso el comportamiento asintótico depende fuertemente de la paridad de ℓ.

Keywords: Turán number, extremal combinatorics, paths, cycles.

MSC2010: 05C35, 05D99.

Acknowledgements: We thank Chris Cox for discussions on the topic of this paper. The research of first, second and forth authors was supported by the National Research, Development and Innovation Office NKFIH, grants K116769, K117879 and K126853. The research of the second author was partially supported by the Shota Rustaveli National Science Foundation of Georgia SRNSFG, grant number FR-18-2499. The research of the third author was supported by the Institute for Basic Science (IBS-R029-C1).

Reference: Győri, Ervin; Salia, Nika; Tompkins, Casey, and Zamora, Oscar. "Inverse Turán numbers". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 79-82. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p79.

1. Introduction

This is an extended abstract of manuscript [12].

Turán's theorem [14] asserts that the maximum number of edges in a subgraph of the complete graph K_n on n vertices with no subgraph isomorphic to the complete graph on r vertices is attained by the complete r-partite graph with parts of size $\lfloor n/r \rfloor$ and $\lceil n/r \rceil$. This graph is referred to as the Turán graph and is denoted by $T(n,r)$.

Since Turán's seminal result, the problem of maximizing the number of edges in an n-vertex graph not containing a fixed graph F as a subgraph has been investigated for a variety of graphs F. A graph G containing no member of $\mathcal{F}$ as a subgraph is said to be $\mathcal{F}$-free, and for $\mathcal{F} = \{F\}$ we say that such a graph is F-free. The Turán number $\mathrm{ex}(n,\mathcal{F})$ is defined to be the maximum number of edges in an $\mathcal{F}$-free subgraph of K_n. The classical Turán problem was settled asymptotically for all finite families of graphs $\mathcal{F}$ of chromatic number at least three by Erdős, Stone and Simonovits [7, 8]. However, for most bipartite graphs F, the Turán problem remains open.

More generally for a given host graph G, the Turán number $\mathrm{ex}(G,\mathcal{F})$ is defined to be the maximum number of edges in an $\mathcal{F}$-free subgraph of G (so $\mathrm{ex}(n,\mathcal{F}) = \mathrm{ex}(K_n,\mathcal{F})$). Common alternative host graphs include the complete bipartite graph $K_{m,n}$ (the so-called Zarankiewicz problem), the hypercube Q_n [4], a random graph [10], as well as the class of n-vertex planar graphs [3].

In this paper, we will be concerned with a dual version of Turán's extremal function introduced by Foucaud, Krivelevich, and Perarnau [9] and later (in a different but equivalent form which we will use) by Briggs and Cox [1]. The number of vertices and edges in a graph G are denoted by $v(G)$ and $e(G)$, respectively. The *inverse Turán number* is defined as follows.

Definition 1. For a given family of graphs $\mathcal{F}$, $\mathrm{ex}^{-1}(k,\mathcal{F}) = \sup\{e(G) : G \text{ is a graph with } \mathrm{ex}(G,\mathcal{F}) < k\}$. For $\mathcal{F} = \{F\}$, we write $\mathrm{ex}^{-1}(k,\{F\}) = \mathrm{ex}^{-1}(k,F)$. ◂

Note that $\mathrm{ex}^{-1}(k,\mathcal{F})$ may be infinite. However, Briggs and Cox [1] observed that $\mathrm{ex}^{-1}(k,F)$ is finite whenever F is not a matching or a star. An equivalent formulation of the problem is that we must find the maximum number of edges in a graph G such that any subgraph of G with k edges contains a copy of some $F \in \mathcal{F}$. Observe that if F_1 is a subgraph of F_2, then $\mathrm{ex}^{-1}(k,F_1) \geq \mathrm{ex}^{-1}(k,F_2)$. Throughout this paper, when discussing inverse Turán numbers, the asymptotic notation O and Ω indicates that k tends to infinity, and constants involving other parameters may be hidden.

Briggs and Cox [1] gave upper and lower bounds on the inverse Turán number of C_4 of the form $\Omega(k^{4/3})$ and $O(k^{3/2})$, respectively. Unknown to Briggs and Cox at the time, this problem was considered earlier in a different form by Foucaud, Krivelevich, and Perarnau [9] where a bound was proved that was sharp up to a logarithmic factor. Even more, according to Perarnau and Reed [13] the problem was already proposed by Bollobás and Erdős at a workshop in 1966 (see [6] for a related problem about union-free families from 1970). More generally a recursive bound on the inverse Turán number of $\mathrm{ex}^{-1}(k,\{C_4,C_6,\ldots,C_{2t}\})$ was also obtained in [9], which is also tight up to a logarithmic factor.

For graphs F with chromatic number at least 3, Foucaud, Krivelevich, and Perarnau [9] and Briggs and Cox [1] determined the inverse Turán number asymptotically. Moreover, Briggs and Cox [1] determined the inverse Turán number of the complete graph precisely as well as the union of a path of length 1 and a path of length 2. They also settled the case of paths of length 3 and proposed a conjecture about the inverse Turán number of a path of length 4.

In Section 2, we will investigate the inverse Turán problem for paths, resolving a conjecture of Briggs and Cox asymptotically and providing a new lower bound for paths of any even length. In Section 3 we will determine the order of magnitude of the inverse Turán number of any complete bipartite graph resolving another conjecture of Briggs and Cox about the order of magnitude of $\mathrm{ex}^{-1}(k,C_4)$. We note however, that this conjecture already follows directly from an unpublished preprint of Conlon, Fox, and Sudakov [2] which preceded the paper of Briggs and Cox [1], but we provide a proof in the formulation introduced by Briggs and Cox for completeness. In the case of C_4, we give improved bounds on the leading coefficient and conjecture that the lower bound is optimal. Additionally, we give some estimates on the inverse Turán number of an arbitrary even cycle. Finally in Section 4, we present some conjectures and directions for future work.

2. Inverse Turán numbers of paths

In this section, we investigate the inverse Turán problem for paths. We begin by recalling a well-known result of Erdős and Gallai.

Theorem 2 (Erdős, Gallai [5]). *For all $n \geq t$, $\mathrm{ex}(n, P_t) \leq (t-1)n/2$, and equality holds if and only if t divides n and G is the disjoint union of cliques of size t.*

Theorem 3 (Briggs, Cox [1]). *For all $t \geq 3$,*

$$\mathrm{ex}^{-1}(k, P_t) \geq \binom{\left\lfloor \frac{2k}{t-1} \right\rfloor - 1}{2}.$$

The bound in Theorem 3 comes from taking a complete graph of the appropriate size and applying Theorem 2. In the case of $t = 3$, Briggs and Cox [1] proved that a complete graph gives the optimal bound for $\mathrm{ex}^{-1}(k, P_3)$. Briggs and Cox also noted that for P_4 one can do better by considering a complete bipartite base graph and using a result of Gyárfás, Rousseau, and Schelp [11] on the extremal number of P_t in such graphs. However, starting with a clique is superior to a complete bipartite graph for P_t, $t \neq 4$. We will improve the lower bound on $\mathrm{ex}^{-1}(k, P_{2t})$ in general by considering balanced complete multipartite graphs. Note that, since the inverse Turán number is non-decreasing when considering supergraphs, it follows that the inverse Turán number of any path of length at least 3 is $\Theta(k^2)$.

Proposition 4. *Among the Turán graphs $T(n, r)$ with $\mathrm{ex}(T(n, r), P_{2t}) < k$, the one with the maximum number of edges is obtained by $r = t$ and $n = \left\lfloor \frac{k-1}{t-1} \right\rfloor + O(k)$. In particular, for $t \geq 2$,*

$$\mathrm{ex}^{-1}(k, P_{2t}) \geq e\left(T\left(\left\lfloor \frac{k-1}{t-1} \right\rfloor, t\right)\right) = \frac{(k-1)^2}{2t(t-1)} + O(k).$$

Theorem 5. *We have $\mathrm{ex}^{-1}(k, P_4) = k^2/4 + O(k^{3/2})$.*

Theorem 6. *We have $\mathrm{ex}^{-1}(k, P_5) = k^2/8 + O(k)$.*

3. Inverse Turán number of complete bipartite graphs and even cycles

While the classical Turán number $\mathrm{ex}(n, K_{s,t})$ is not known, Conlon, Fox, and Sudakov [2] determined the asymptotics of $\mathrm{ex}^{-1}(k, K_{s,t})$.

Theorem 7. *Let s, t be integers with $1 < s \leq t$. Then, $\mathrm{ex}^{-1}(k, K_{s,t}) = \Theta(k^{1+1/s})$.*

In the case of C_4, we give a more precise calculation to prove upper and lower bounds within a factor of $\frac{3\sqrt{3}}{2\sqrt{2}} < 2$.

Theorem 8. $\lfloor \sqrt{2k/3} \rfloor \lfloor 2k/3 - 1 \rfloor \leq \mathrm{ex}^{-1}(k, C_4) \leq k^{3/2} + o(k^{3/2})$.

In the following theorem we offer some bounds for the inverse Turán number of even cycles.

Theorem 9. *Let $t \geq 2$. Then,*

$$\mathrm{ex}^{-1}(k, C_{2t}) = \begin{cases} O\left(k^{2-\frac{2}{3t-3}}\right) & \text{if } t \text{ is odd,} \\ O\left(k^{2-\frac{2}{3t-2}}\right) & \text{if } t \text{ is even,} \end{cases}$$

and

$$\mathrm{ex}^{-1}(k, C_{2t}) = \begin{cases} \Omega\left(k^{2-\frac{2}{t+1}}\right) & \text{if } t \text{ is odd,} \\ \Omega\left(k^{2-\frac{2}{t+2}}\right) & \text{if } t \text{ is even.} \end{cases}$$

4. Remarks and open questions

We pose two conjectures about the inverse Turán number of the path depending on the parity of its length. In agreement with the intuition of Briggs and Cox [1], we believe that the inverse Turán number of a path with odd length is attained by a clique. On the other hand, we believe that a balanced multi-partite graph of t parts is optimal for forcing a path of length $2t$.

Conjecture 10. *The inverse Turán number of a path of length* $2t+1$ *is attained asymptotically by a complete graph. Therefore, for every* t, $\mathrm{ex}^{-1}(k, P_{2t+1}) = \binom{\lfloor \frac{k}{t} \rfloor}{2} + o(k^2)$.

Conjecture 11. *The inverse Turán number of a path of length* $2t$ *is attained asymptotically by a balanced, complete* t*-partite graph. Therefore, for every* t, $\mathrm{ex}^{-1}(k, P_{2t}) = \frac{k^2}{2(t-1)^2}\left(1 - \frac{1}{t}\right) + o\left(k^2\right)$.

We have given upper and lower bounds for the value of $\mathrm{ex}^{-1}(k, C_4)$, and we conjecture that the lower bound is asymptotically sharp.

Conjecture 12. *We have* $\mathrm{ex}^{-1}(k, C_4) = \frac{2\sqrt{2}k^{3/2}}{3\sqrt{3}} + o(k^{3/2})$.

References

[1] Briggs, Joseph and Cox, Christopher. "Inverting the Turán problem". In: *Discrete Mathematics* 342.7 (2019), pp. 1865–1884.

[2] Conlon, David; Fox, Jacob, and Sudakov, Benny. "Large subgraphs without complete bipartite graphs". In: *arXiv e-prints* (2014). arXiv: 1401.6711 [math.CO].

[3] Dowden, Chris. "Extremal C_4-free/C_5-free planar graphs". In: *Journal of Graph Theory* 83.3 (2016), pp. 213–230.

[4] Erdős, Paul. "On some problems in graph theory, combinatorial analysis and combinatorial number theory". In: *Graph Theory and Combinatorics (Cambridge, 1983), Academic Press, London* (1984), pp. 1–17.

[5] Erdős, Paul and Gallai, Tibor. "On maximal paths and circuits of graphs". In: *Acta Mathematica Academiae Scientiarum Hungarica* 10.3-4 (1959), pp. 337–356.

[6] Erdős, Paul and Komlós, János. "On a problem of Moser". In: *Combinat. Theory Appl., Colloquia Math. Soc. Janos Bolyai* 4 (1970), pp. 365–367.

[7] Erdős, Paul and Simonovits, Miklós. "A limit theorem in graph theory". In: *Studia Scientiarum Mathematicarum Hungarica* 1 (1966), pp. 51–57.

[8] Erdős, Paul and Stone, Arthur H. "On the structure of linear graphs". In: *Bulletin of the American Mathematical Society* 52 (1946), pp. 1087–1091.

[9] Foucaud, Florent; Krivelevich, Michael, and Perarnau, Guillem. "Large subgraphs without short cycles". In: *SIAM Journal on Discrete Mathematics* 29.1 (2015), pp. 65–78.

[10] Frankl, Peter and Rödl, Vojtech. "Large triangle-free subgraphs in graphs without K_4". In: *Graphs and Combinatorics* 2.1 (1986), pp. 135–144.

[11] Gyárfás, András; Rousseau, Cecil C, and Schelp, Richard H. "An extremal problem for paths in bipartite graphs". In: *Journal of graph theory* 8.1 (1984), pp. 83–95.

[12] Győri, Ervin; Salia, Nika; Tompkins, Casey, and Zamora, Oscar. "Inverse Turán numbers". In: *arXiv e-prints* (2020). arXiv: 2007.07042 [math.CO].

[13] Perarnau, Guillem and Reed, Bruce. "Existence of spanning $\mathcal{F}$-free subgraphs with large minimum degree". In: *Combinatorics, Probability and Computing* 26.3 (2017), pp. 448–467.

[14] Turán, Paul. "On an external problem in graph theory". In: *Mat. Fiz. Lapok* 48 (1941), pp. 436–452.