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Neumann p-Laplacian problems with a reaction term on metric spaces

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Abstract: We use a variational approach to study existence and regularity of solutions for a Neumann p-Laplacian problem with a reaction term on metric spaces equipped with a doubling measure and supporting a Poincaré inequality. Trace theorems for functions with bounded variation are applied in the definition of the variational functional and minimizers are shown to satisfy De Giorgi type conditions.

Resumen: Utilizamos un enfoque variacional para estudiar la existencia y regularidad de soluciones para un problema de Neumann p-Laplaciano con un término de reacción en espacios métricos dotados de una medida de duplicación y que permiten una desigualdad de Poincaré. Se aplican teoremas de traza para funciones con variación acotada en la definición del funcional variacional y se demuestra que los minimizadores satisfacen condiciones de tipo De Giorgi.

Keywords: p-Laplacian operator, measure metric spaces, minimal p-weak upper gradient, minimizer.

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1. Introduction

We extend existence and regularity results for a Neumann boundary value problem valid on the Euclidean setting and, more generally, in Riemannian manifolds (see Nastasi [8]) to the general setting of metric spaces. Applying variational methods such as those based on De Giorgi classes [2], we study a Neumann boundary value problem as in Lahti, Malý, and Shanmugalingam [5] and Malý and Shanmugalingam [6], but the new feature is that we include a reaction term (see Nastasi [9]). Under appropriate conditions on the reaction term, we prove existence and boundedness properties of solutions with a reaction term in a metric space equipped with a doubling measure and supporting a Poincaré inequality and thus extending the corresponding results in Kinnunen and Shanmugalingam [3] and Malý and Shanmugalingam [6].

2. Mathematical background

Let (X, d, μ) be a metric measure space, where μ is a Borel regular measure. Let $B(x, \rho) \subset X$ be a ball with center $x \in X$ and radius $\rho > 0$.

Definition 1 ([1, Section 3.1]). A measure μ on X is said to be doubling if there exists a constant K, called the doubling constant, such that $0 < \mu(B(x, 2\rho)) \leq K\mu(B(x, \rho)) < +\infty$ for all $x \in X$ and $\rho > 0$. ◂

The following notion of upper gradient has been introduced in order to satisfy the lack of a differentiable structure.

Definition 2 ([1, Definition 1.13]). A non negative Borel measurable function g is said to be an upper gradient of function $u : X \to [-\infty, +\infty]$ if, for all compact rectifiable arc lenght parametrized paths γ connecting x and y, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s \tag{1}$$

whenever $u(x)$ and $u(y)$ are both finite and $\int_\gamma g \, \mathrm{d}s = +\infty$ otherwise. ◂

We note that, if g is an upper gradient of function u and ϕ is a non negative Borel measurable function, then $g + \phi$ is still an upper gradient of u. In order to overcome this aspect, we use the following notions that will lead to the definition of the minimal p-weak upper gradient of u.

Definition 3 ([1, Definition 1.33]). Let $p \in [1, +\infty[$. Let Γ be a family of paths in X. We say that $\inf_\phi \int_X \phi^p \, \mathrm{d}\mu$ is the p-modulus of Γ, where the infimum is taken among all non negative Borel measurable functions ϕ satisfying $\int_\gamma \phi \, \mathrm{d}s \geq 1$, for all rectifiable paths $\gamma \in \Gamma$. ◂

Definition 4 ([1, Definition 1.32]). If (1) is satisfied for p-almost all paths γ in X, that is, the set of non constant paths that do not satisfy (1) is of zero p-modulus, then g is said a p-weak upper gradient of u. ◂

The family of weak upper gradients satisfy a result concerning the existence of a minimal element g_u, that is called the minimal p-weak upper gradient of u.

Definition 5 ([1, Definition 4.1]). Let $p \in [1, +\infty[$. A metric measure space X supports a $(1, p)$-Poincaré inequality if there exist $K > 0$ and $\lambda \geq 1$ such that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq Kr \left(\frac{1}{\mu(B(x,\lambda r))} \int_{B(x,\lambda r)} g_u^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

for all balls $B(x, r) \subset X$ and for all $u \in L^1_{loc}(X)$, where $u_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, \mathrm{d}\mu$. ◂

Let X be a complete metric space equipped with a doubling measure supporting a $(1, p)$-Poincaré inequality. We recall the concept of Newtonian space, which is based on the notion of minimal p-weak upper gradient.

Definition 6. Let X be a complete metric space equipped with a doubling measure supporting a $(1,p)$-Poincaré inequality, $p \in [1,+\infty]$. The Newtonian space $N^{1,p}(X)$ is defined by $N^{1,p}(X) = V^{1,p}(X) \cap L^p(X)$, where $V^{1,p}(X) = \{u : u \text{ is measurable and } g_u \in L^p(X)\}$. We consider $N^{1,p}(X)$ equipped with the norm

$$\|u\|_{N^{1,p}(X)} = \|g_u\|_{L^p(X)} + \|u\|_{L^p(X)}.$$

We denote with $N_*^{1,p}(X) = \{u \in N^{1,p}(X) : \int_X u \,\mathrm{d}x = 0\}$. ◂

The Newtonian space $N^{1,p}(X)$ defined above is a complete normed vector space, which generalizes the Sobolev space $W^{1,p}(\Omega)$ to a metric setting.

Definition 7 (see [7]). A Borel set $E \subset X$ is said to be of finite perimeter if there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $N^{1,1}(X)$ such that $u_n \to \chi_E$ in $L^1(X)$ and $\liminf_{n\to+\infty} \int_X g_{u_n} \,\mathrm{d}\mu < \infty$. The perimeter $P_E(X)$ of E is the infimum of the above limit among all sequences $\{u_n\}$ as above. For an open set $U \subset X$, the perimeter of E in U is

$$P_E(U) = \inf\left\{\liminf_{n\to+\infty} \int_X g_{u_n} \,\mathrm{d}\mu : \{u_n\}_{n\in\mathbb{N}} \subset N^{1,1}(U), u_n \to \chi_{E\cap U} \text{ in } L^1(U)\right\}.$$ ◂

From now on, we consider a bounded domain (non empty, connected open set) Ω in X with $X \setminus \Omega$ of positive measure such that Ω is of finite perimeter with perimeter measure P_Ω. Let $f : \partial\Omega \to \mathbb{R}$ be a bounded P_Ω-measurable function with $\int_{\partial\Omega} f \,\mathrm{d}P_\Omega = 0$. We make the following assumptions on Ω:

(H_1) There exists a constant $K \geq 1$ such that, for all $y \in \Omega$ and $0 < \rho \leq \operatorname{diam}(\Omega)$, we have

$$\mu(B(y,\rho) \cap \Omega) \geq \frac{1}{K}\mu(B(y,\rho)).$$

(H_2) (Ahlfors codimension 1 regularity of P_Ω) For all $y \in \partial\Omega$ we have that

$$\frac{1}{K\rho}\mu(B(y,\rho)) \leq P_\Omega(B(y,\rho)) \leq \frac{K}{\rho}\mu(B(y,\rho)),$$

where K and ρ are as in (H_1).

(H_3) $(\Omega, d_{|\Omega}, \mu_{|\Omega})$ admits a $(1,p)$-Poincaré inequality with $\lambda = 1$, where $p \in]1,+\infty[$.

Definition 8 ([4, Definition 4.1]). Let $\Omega \subset X$ be an open set and let u be a μ-measurable function on Ω. A function $Tu : \partial\Omega \to \mathbb{R}$ is the trace of u if for $\mathcal{H}$-almost every $y \in \partial\Omega$ we have

$$\lim_{\rho\to0^+} \frac{1}{\mu(\Omega \cap B(y,\rho))} \int_{\Omega\cap B(y,\rho)} |u - Tu(y)| \,\mathrm{d}\mu = 0.$$ ◂

For the existence theorem of the trace operator see Malý and Shanmugalingam [6] and references therein.

Given a Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G, we associate the following functional

$$J(u) = \int_\Omega g_u^p \,\mathrm{d}\mu - \int_\Omega G(u) \,\mathrm{d}\mu + \int_{\partial\Omega} Tuf \,\mathrm{d}P_\Omega \quad \text{for all } u \in N^{1,p}(\Omega).$$

Definition 9. A function $u_0 \in N_*^{1,p}(\Omega)$ is a p-harmonic solution to the Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G if

$$\begin{aligned} J(u_0) &= \int_\Omega g_{u_0}^p \,\mathrm{d}\mu - \int_\Omega G(u_0) \,\mathrm{d}\mu + \int_{\partial\Omega} Tu_0 f \,\mathrm{d}P_\Omega \\ &\leq \int_\Omega g_v^p \,\mathrm{d}\mu - \int_\Omega G(v) \,\mathrm{d}\mu + \int_{\partial\Omega} Tvf \,\mathrm{d}P_\Omega = J(v) \end{aligned}$$

for every $v \in N_*^{1,p}(\Omega)$, where g_{u_0}, g_v are the minimal p-weak upper gradients of u_0 and v in Ω, respectively, and Tu_0 and Tv are the traces of u_0 and v on $\partial\Omega$, respectively. ◂

Later on, in considering the trace Tu of u we will omit T and just write u.

Here, we assume that $G\colon\, \Omega \to \mathbb{R}$ is defined as $G(u) = c - |u|^\gamma$ for all $u \in N^{1,p}(\Omega)$, for some $c > 0$ and $1 < \gamma < p^* = \frac{ps}{s-p}$ if $p < s$ and $1 < \gamma < +\infty$ otherwise.

In the metric setting, we will look for a minimizer of J in the Newtonian space $N^{1,p}_*(\Omega)$.

3. Existence of a solution and a weaker uniqueness result

The existence of a nontrivial solution to the Neumann boundary value problem with non zero boundary data f and reaction term G is an immediate consequence of the following theorem which shows that J has a minimizer.

Theorem 10. *J has a minimizer in $N^{1,p}_*(\Omega)$. If $u_1, u_2 \in N^{1,p}_*(\Omega)$ are two minimizers of J, then $g_{u_1} = g_{u_2}$ a.e. in Ω.*

4. Boundedness property

We show that minimizers are locally bounded near the boundary under appropriate hypothesis on the boundary data f. In order to do so, the following De Giorgi type inequality plays a key role.

Lemma 11. *Let $u \in N^{1,p}_*(\Omega)$ be a minimizer of J and $f \in L^\infty(\partial\Omega)$. If $y \in \partial\Omega$, $0 < \rho < R < \frac{\operatorname{diam}(\Omega)}{10}$ and $\alpha \in \mathbb{R}$, then there is $K \geq 1$ such that the following De Giorgi type inequality*

$$\int_{\Omega\cap B(y,\rho)} g^p_{(u-\alpha)_+}\, \mathrm{d}\mu \leq \frac{K}{(R-\rho)^p} \int_{\Omega\cap B(y,R)} (u-\alpha)^p_+\, \mathrm{d}\mu + K \int_{\partial\Omega\cap B(y,R)} |f|(u-\alpha)^p_+\, \mathrm{d}P_\Omega$$

is satisfied.

Theorem 12. *Let $0 < R < \frac{\operatorname{diam}(\Omega)}{4}$ and $\Omega_R = \left\{y \in \Omega\, :\, d(y, \partial\Omega) < \frac{R}{2}\right\}$. If $u \in N^{1,p}_*(\Omega)$ is a minimizer of J and $f \in L^\infty(\partial\Omega)$, then $u \in L^\infty(\Omega_R)$ and $Tu \in L^\infty(\partial\Omega_R)$.*

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