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Rooted structures in graphs

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Abstract: A *transversal* of a partition is a set which contains exactly one element from each member of the partition and nothing else. A *colouring* of a graph is a partition of its vertex set into anticliques, that is, sets of pairwise nonadjacent vertices. We study the following problem: Given a transversal T of a proper colouring $\mathcal{C}$ of some graph G, is there a partition $\mathfrak{H}$ of a subset of $V(G)$ into connected sets such that T is a transversal of $\mathfrak{H}$ and any two distinct sets of $\mathfrak{H}$ are adjacent?

It has been conjectured by Matthias Kriesell [9] that for any transversal T of a colouring $\mathcal{C}$ of order k of some graph G such that any pair of colour classes induces a connected subgraph, there exists such a partition $\mathfrak{H}$ with pairwise adjacent sets. This would prove Hadwiger's conjecture for the class of uniquely optimally colourable graphs; however it is open for each $k \geq 5$.

This paper will provide an overview about the stated conjecture. It extracts associated results from my PhD thesis and the related papers [2, 10, 11], summarises their relevence to the stated problem, and discusses some unsuccessful attempts.

Resumen: Una *transversal* de una partición es un conjunto que contiene exactamente un elemento de cada miembro de la partición y nada más. Una *coloración* de un grafo es una partición de sus vértices en conjuntos independientes, es decir, conjuntos de vértices no adyacentes entre sí. Nosotros estudiamos el siguiente problema: dada una transversal T de una coloración $\mathcal{C}$ de un grafo G, ¿existe alguna partición $\mathfrak{H}$ de un subconjunto de $V(G)$ en conjuntos conexos tal que T sea una transversal de $\mathfrak{H}$ y cualesquiera dos conjuntos distintos de $\mathfrak{H}$ sean adyacentes?

Matthias Kriesell [9] conjeturó que, para cualquier transversal T de orden k de una k-coloración $\mathcal{C}$ de algún grafo G tal que cualquier par de clases de colores inducen un subgrafo conexo, existe tal partición $\mathfrak{H}$ con conjuntos adyacentes dos a dos. Esto demostraría la conjetura de Hadwiger para la clase de grafos óptimamente coloreables de forma única; sin embargo, el problema sigue abierto para todo $k \geq 5$.

Este artículo presenta una visión general sobre esta conjetura. Expone resultados de mi tesis doctoral y los artículos relacionados [2, 10, 11], resume su relevancia con el problema planteado y discute algunos intentos fallidos.

Keywords: rooted minor, Hadwiger's conjecture.

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1. Hadwiger's conjecture

Hadwiger's conjecture states that the order of a largest clique minor in a graph G is at least its chromatic number $\chi(G)$ [8]. It is known to be true for graphs with chromatic number at most 6, with $\chi(G) = 5$ and $\chi(G) = 6$ being merely equivalent to the Four-Colour-Theorem [14]. Even for subclasses of graphs, Hadwiger's conjecture seems to be challenging. Though it is solved for line graphs [13], only partial results exist for claw-free graphs [7]. Paul Erdős [3] stated that it is "one of the deepest unsolved problems in graph theory", thus reinforcing the extreme nature and difficulty of this conjecture.

Instead of restricting to subclasses of graphs, one could also uniformly bound the order of the colour classes. But even when forbidding anticliques of order 3, Hadwiger's conjecture is widely open. This variant is stated in a conjecture of Seymour (see [1]).

Matthias Kriesell suggested in [9] to bound the number of colourings and, in particular, consider uniquely optimally colourable graphs. We will be interested in a rooted version of Hadwiger's conjecture that imposes additional assumptions on the colourings.

2. Kempe colourings

All graphs in the present paper are assumed to be finite, undirected, and simple. For terminology not defined here we refer to contemporary textbooks such as [4] or [6]. By K_S we denote the complete graph on a finite set S. A (minimal) *transversal* of a set $\mathfrak{C}$ of disjoint sets is a set T containing exactly one member of every $A \in \mathfrak{C}$ and nothing else; we also say that $\mathfrak{C}$ is *traversed* by T. A *colouring* of a graph G is a partition $\mathcal{C}$ of its vertex set $V(G)$ into *anticliques*, that is, sets of pairwise non-adjacent vertices. The *order* of a colouring $\mathcal{C}$ is the number of anticliques in $\mathcal{C}$ and an *optimal* colouring is a colouring of smallest order. The *chromatic number* $\chi(G)$ is the order of an optimal colouring of G. A *Kempe chain* is a connected component of $G[A \cup B]$ for some $A \neq B$ from $\mathcal{C}$.

We call a graph G *uniquely k-colourable* if $\chi(G) = k$ and for any two optimal colourings $\mathcal{C}$ and $\mathcal{C}'$ of G, we have $\mathcal{C} = \mathcal{C}'$. Such graphs have the property that the union of any two distinct colour classes induces a connected graph [5]. To see this, assume to the contrary that there is a graph G with a unique optimal colouring $\mathcal{C}$ and there are $A, B \in \mathcal{C}$, $A \neq B$, such that $G[A \cup B]$ has at least two components. Let H be one of the components and consider the colouring $\mathcal{C}'$ with

$$\mathcal{C}' = (\mathcal{C} \setminus \{A, B\}) \cup \{(A \setminus V(H)) \cup (B \cap V(H))\} \cup \{(B \setminus V(H)) \cup (A \cap V(H))\}.$$

Then $\mathcal{C}'$ is another optimal colouring of G and distinct from $\mathcal{C}$, a contradiction.

However restricting to uniquely colourable graphs seems excessive provided that in most situations we only make use of the above property that any two colour classes induce a connected graph. This leads us to the following definition.

Definition 1. A colouring $\mathcal{C}$ of a graph G is a *Kempe colouring* if any two vertices from distinct colour classes belong to the same Kempe chain or, in other words, the union of any two colour classes induces a connected subgraph in G. ◂

If G is graph and $\mathcal{C}$ is a Kempe colouring of G for which the vertices $x_1, \dots, x_k$ are given different colours, then it is easy to see that there exists a system of edge-disjoint x_i,x_j-paths ($i \neq j$ from $\{1, \dots, k\}$), a so-called *(weak) clique immersion* of order k at $x_1, \dots, x_k$. The natural question to ask is whether there exists a *clique minor* of the same order such that $x_1, \dots, x_k$ are in different bags.

A graph H is a *minor* of a graph G if there exists a family $c = (V_t)_{t \in V(H)}$ of pairwise disjoint subsets of $V(G)$, called *bags*, such that V_t is nonempty and $G[V_t]$ is connected for all $t \in V(H)$ and there is an edge connecting V_t and V_s for all $st \in E(H)$. Any such c is called an *H-certificate* in G, and a *rooted H-certificate* if, moreover, $V(H) \subseteq V(G)$ and $t \in V_t$ for all $t \in V(H)$. If there exists a rooted H-certificate, then H is a *rooted minor* of G.

A positive answer to the question above was conjecture by Matthias Kriesell.

Conjecture 2 (Kriesell [9]). *Let G be a graph, $\mathcal{C}$ be a Kempe colouring of size k and T a transversal of $\mathcal{C}$, then G contains a K_k-minor rooted at T.*

Using a result by Fabila-Monroy and Wood [15], a confirmation of Conjecture 2 for $k \leq 4$ follows immediately. In [11], it is proved for line graphs.

Theorem 3 (Kriesell, Mohr [11]). *For every transversal T of every Kempe colouring of the line graph $L(G)$ of any graph G there exists a complete minor in $L(G)$ traversed by T.*

It should be mentioned that a Kempe colouring can have significantly more colours than an optimal colouring. However a positive answer to Conjecture 2 will prove Hadwiger's conjecture for uniquely colourable graphs.

3. Two-coloured paths

In the previous section, we have seen that for each transversal T of any Kempe colouring of order k, there exists a clique immersion of order k at T. It is natural to ask whether the requirement of a Kempe colouring can be weakened to only demanding that two distinct vertices x, y of a transversal T belong to the same connected component of $G[A \cup B]$, where $A, B \in \mathcal{C}$, $\mathcal{C}$ is the colouring of the graph G, and $x \in A, y \in B$.

Conjecture 4. *Let G be a graph and $\mathcal{C}$ be one of its k-colourings (k not necessary optimal). Furthermore, let T be an arbitrary transversal of $\mathcal{C}$.*

Assume that for each pair of distinct vertices $x, y \in T$ there is a Kempe chain containing both vertices x and y. Then G contains a K_k-minor rooted at T.

Conjecture 2 would follow if Conjecture 4 held for all graphs G, colourings $\mathcal{C}$, and transversals T. However Conjecture 4 turned out to be too restrictive to be true: There exists a graph with a 7-colouring that does not contain a rooted K_7-minor.

Theorem 5 (Kriesell, Mohr [10]).

(i) *Let G be a graph and $\mathcal{C}$ be one of its k-colourings (k not necessary optimal). Furthermore, let T be an arbitrary transversal of $\mathcal{C}$. Assume that for each pair of distinct vertices $x, y \in T$ there is a Kempe chain containing both vertices x and y.*

If $k \leq 4$, or $k = 5$ and $G[T]$ is connected, then G contains a K_k-minor rooted at T.

(ii) *There is a graph with a 7-colouring $\mathcal{C}$ and a transversal T of $\mathcal{C}$ such that each pair of distinct vertices $x, y \in T$ belongs to the same Kempe chain, and this graph does not contain a K_7-minor rooted at T.*

We have seen that the setting of Conjecture 4 is insufficient to guarantee a rooted K_7-minor (and any K_k-minor with $k \geq 7$). This troublesome graph K_7 is known to be the smallest 6-connected graph and one may ask whether it is possible to find a 6-connected minor instead.

To address this problem, we move away from colourings and ask the following question [12]:
Given an integer k, does there exist an integer $\ell(k)$ such that for each graph G and $X \subseteq V(G)$ for which there is no separator S in G with $|S| < \ell(k)$ separating vertices of X, G has a k-connected minor (or topological minor) that "contains X"?

This questions demands a local connectedness of the vertices from X. Clearly, $\ell(k)$ must be at least k since X might be equal to $V(G)$. If each separator S in a graph G with $|S| < \ell$ splits the graph into components such that only one contains vertices from X ($|X| \geq \ell + 1$), we say that X is *ℓ-connected in G*. Moreover, the definition of a rooted minor given in Section 2 is not suitable in our setting since the k-connected minor can have significantly more vertices than $|X|$. We adapt the definition and say that a graph H is an *X-minor* of a graph G with $X \subseteq V(G)$ if $X \subseteq V(H)$ and there exists an H-certificate $c = (V_t)_{t \in V(H)}$ in G such that $t \in V_t$ for all $t \in X$. Armed with this refined definition, we prove the following:

Theorem 6 (Böhme, Harant, Kriesell, Mohr, Schmidt [2]). *Let $k \in \{1, 2, 3, 4\}$, G be a graph, and $X \subseteq V(G)$ be a k-connected set in G. Then:*

(i) *G has a k-connected X-minor.*

(ii) *If $1 \leq k \leq 3$, then G has a k-connected topological X-minor.*

Moreover, the theorem is best possible in the sense that there exist graphs G_1 and G_2 with $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_2)$ such that X_1 (X_2) is 5-connected (4-connected) in G_1 (G_2) and neither G_1 nor G_2 contain a 5-connected X_1-minor and a 4-connected topological X_2-minor, respectively.

In summary, Theorems 5 and 6 lead us to the following insights. First, to confirm Conjecture 2 and in turn prove the rooted version of Hadwiger's conjecture for uniquely colourable graphs, it is not possible to restrict to clique immersions. It would seem that a certain connectedness property, which is provided by Kempe colourings, is necessary. Second, lifting the problem away from the colouring doesn't feel like a promising approach either, since a high connectedness gives no guarantee on any highly connected minor.

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