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Constructing quadratic 2-step nilpotent Lie algebras

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Abstract: A quadratic Lie algebra is a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form. Among all these algebras, we are going to focus on the nilpotent ones whose nilpotency index is two and, particularly, on those which are reduced. There exist different techniques to construct these algebras. Double extension and T^*-extension are recursive methods that allow us to start from smaller dimensions and grow up. Fixing an appropriate basis and using its definition gives us another approach to these algebras. And finally, we have that their classification is equivalent to the alternating trilinear forms one.

Resumen: Un álgebra de Lie cuadrática es un álgebra de Lie dotada de una forma bilineal invariante simétrica no degenerada. Entre todas las álgebras que cumplen estas condiciones, vamos a centrarnos en aquellas que sean nilpotentes y cuyo índice de nilpotencia sea 2, en particular, aquellas reducidas. Existen diferentes técnicas para construir este tipo de álgebras. La doble extensión y T^*-extensión son métodos clásicos recursivos que nos permiten obtenerlas partiendo de dimensiones pequeñas y aumentando progresivamente. Si fijamos una base apropiada y usamos su definición, junto a alguna propiedades, conseguimos una nueva aproximación. Finalmente, tenemos que su clasificación es equivalente a la de formas trilineales alternadas.

Keywords: nilpotent Lie algebra, quadratic algebra, trilinear form.

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1. Keywords

The main concepts we need in this paper are the following:

Definition 1 (Lie algebra). A Lie algebra is a vector space $\mathfrak{n}$ with an alternating bilinear form $[\cdot,\cdot]: \mathfrak{n}\times\mathfrak{n} \to \mathfrak{n}$ called Lie bracket that satisfies the Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$. ◂

Definition 2 (t-step nilpotent). We say a Lie algebra $\mathfrak{n}$ is t-step nilpotent when $\mathfrak{n}^{t+1} = [\mathfrak{n}^t, \mathfrak{n}] = 0$, but $\mathfrak{n}^t \neq 0$, and where $\mathfrak{n}^1 = \mathfrak{n}$, $[A,B] \coloneqq \operatorname{span}\langle [a,b] : a \in A, b \in B\rangle$. ◂

Definition 3 (quadratic). A quadratic Lie algebra $\mathfrak{n}$ is a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form $f: \mathfrak{n}\times\mathfrak{n} \to \mathbb{K}$, which means that $f([x,y],z) + f(y,[x,z]) = 0$ for every $x,\ y,\ z \in \mathfrak{n}$. ◂

Definition 4 (reduced). An algebra $\mathfrak{n}$ is said to be reduced in case $Z(\mathfrak{n}) \subseteq \mathfrak{n}^2$. ◂

And, as stated in [9, Theorem 6.2]:

Theorem 5. *Any non-reduced and non-abelian quadratic Lie algebra $(\mathfrak{n},\varphi)$ decomposes as an orthogonal direct sum of proper ideals, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{a}$, where $\varphi = \varphi_1 \perp \varphi_2$ and $(\mathfrak{n}_1,\varphi_1)$ is a quadratic reduced Lie algebra and $(\mathfrak{a},\varphi_2)$ is a quadratic abelian algebra.*

Finally, we will note as $\mathfrak{n}_{d,t}$ the free t-step Lie algebra on d generators (see [1] for a formal definition).

2. Constructions

There exist several ways to construct quadratic Lie algebras or equivalent structures. In this section we give an overview of some of them, with focus on the 2-step case.

Unless we specify the contrary, we will work over a generic field $\mathbb{K}$ and (A,f) will be a generic finite-dimensional Lie algebra, while A^* will denote its dual space. Moreover, ad^* will represent the coadjoint representation (i.e., $\operatorname{ad}^*(a)(\alpha)(a') = -\alpha([a,a'])$ for $a, a' \in A$ and $\alpha \in A^*$).

2.1. Double extension

The first way is the classic double extension method (see [7] or [3]). To begin with the extension we need, apart from (A,f) over a field $\mathbb{K}$ of characteristic zero, another finite-dimensional Lie algebra B in the same field and a Lie homomorphism $\phi: B \to \operatorname{Der}_f(A)$ where $\operatorname{Der}_f(A)$ is the space of all f-antisymmetric derivations of A (i.e., $f(d(a),a') + f(a,d(a')) = 0$ for $d \in \operatorname{Der}_f(A)$ and $a, a' \in A$). Let us define $w: A\times A \to B^*$ as $(a,a') \mapsto (b \mapsto f(\phi(b)(a),a'))$ for $b \in B$ and $a, a' \in A$. If we take the vector space $A_B \coloneqq B \oplus A \oplus B^*$, define the following multiplication:

$$[b+a+\beta,\ b'+a'+\beta'] \coloneqq [b,b'] + \phi(b)(a') - \phi(b')(a) + [a,a'] + w(a,a') + \operatorname{ad}^*(b)(\beta') - \operatorname{ad}^*(b')(\beta),$$

and the following symmetric bilinear form f_B on A_B:

$$f_B(b+a+\beta, b'+a'+\beta') \coloneqq \beta(b') + \beta(b) + f(a,a'),$$

for $b, b' \in B$, $a, a' \in A$, $\beta, \beta' \in B^*$. Then, the pair (A_B, f_B) is a metrised Lie algebra over $\mathbb{K}$ and is called the double extension of A by ϕ and B.

And, as we can deduce from [7, Théorème III]:

Corollary 6. *In characteristic zero, every quadratic solvable Lie algebra can be obtained from an abelian Lie algebra extended by successive direct sums and double extensions by one-dimensional algebras.*

In [4, Section 5] we can find examples of indecomposable quadratic t-step nilpotent Lie algebras (arbitary t). The examples include the complete classification up to dimension 7 [4, 5.1. Proposition].

2.2. T^*-extension

The T^*-extension is a one-step method which was introduced in [3]. In contrast to double extension, it can be applied not only to Lie algebras, but to arbitrary nonassociative algebras.

For a Lie algebra B, we consider an arbitrary $w\colon B \times B \to B^*$ bilinear map and define the following multiplication on the vector space $T^*_w B \coloneqq B \oplus B^*$ for $b, b' \in B$ and $\beta, \beta' \in B^*$:

$$[b + \beta,\ b' + \beta'] \coloneqq [b, b'] + w(b, b') + \mathrm{ad}^*(b)(\beta') - \mathrm{ad}^*(b')(\beta).$$

Moreover, we consider the symmetric bilinear form q_B in $B \oplus B^*$ defined as follows:

$$q_B(b + \beta, b' + \beta') \coloneqq \beta(b') + \beta'(b).$$

And, as seen in [3, Lemma 3.1] we know if B, B^*, w and q_B are as above, then the pair $(B \oplus B^*, q_B)$ is a metrised algebra if and only if w is cyclic (i.e., $w(a,b)(c) = w(c,a)(b) = w(b,c)(a)$ for all $a, b, c \in B$).

Finally, we have the following theorem (see [3, Theorem 3.2]), which is really convenient as every quadratic 2-step Lie algebra fulfils every condition.

Theorem 7. *Let (A, f) be a metrised algebra of finite dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then, (A, f) will be isometric to a T^*-extension $(T^*_w B, q_B)$ if and only if n is even and A contains an isotropic ideal I (i.e., $I \subset I^{\perp}$) of dimension $n/2$. In this case: $B \cong A/I$. Note that any isotropic $n/2$-dimensional subspace I of A is an ideal of A if and only if it is abelian, i.e., $I^2 = 0$.*

2.3. Computational approach using Hall Basis

Having a well-defined basis is the first requirement to be able to define algorithmically a construction method. For this purpose, we can use the Hall Basis defined in [6].

The Hall Basis of $\mathfrak{n}_{d,2}$ is $\{x_i \,:\, i = d, \dots, 1\} \cup \{[x_i, x_j] \,:\, i = 1, \dots, d; j = i+1, \dots, d\}$. As we can see, the main advantage of this basis is that the Lie products of every element are already defined, taking into account that every element $[x_i, x_j]$ belongs to the centre as this is a 2-step free nilpotent Lie algebra. And, as stated in [5], any 2-step nilpotent Lie algebra $\mathfrak{n}$ of type d is a homomorphic image of $\mathfrak{n}_{d,2}$ as $\mathfrak{n} \cong \mathfrak{n}_{d,2}/I$, with I an ideal of $\mathfrak{n}_{d,2}$ such that $I \subsetneq \mathfrak{n}^2_{d,2}$.

So we only need to know how the bilinear form works. For this part we can generate a generic symmetric matrix of dimension $\frac{d(d+1)}{2} \times \frac{d(d+1)}{2}$. After that, we just have to reduce the variables in the entries of the matrix by imposing the bilinear form is invariant. The whole process is detailed in [1], where lots of examples are displayed.

Finally, we have to find the kernel of the bilinear form to do the quotient by it, as the bilinear form is non-degenerate. And, every quadratic 2-step nilpotent Lie algebra can be obtained this way as we can see in [1, Proposition 4.1]. This proposition says:

Proposition 8. *Let $(\mathfrak{n}, B)$ be a quadratic 2-step nilpotent Lie algebra of type d and $\varphi\colon \mathfrak{n}_{d,2}/I \to \mathfrak{n}$ be an isomorphism of Lie algebras. If we take the map $\overline{B}\colon \mathfrak{n}_{d,2}/I \times \mathfrak{n}_{d,2}/I \to \mathbb{K}$ defined as $\overline{B}(x + I, y + I) = B(\varphi(x + I), \varphi(y + I))$, then φ is an isometry from $(\mathfrak{n}_{d,2}/I, \overline{B})$ onto $(\mathfrak{n}, B)$.*

It is worthwhile to mention that the kernel of this bilinear form is always of dimension $\frac{d(d+1)}{2} - 2d = \frac{d(d-3)}{2}$ for quadratic 2-step Lie algebras. Indeed, using this property shared by all these algebras, we know that their dimension is always $2d$ and we can simplify the process, as we can see in [2].

2.4. Trivectors

In [8, 3.5 Théorème and 3.6 Corollaire] the relation between quadratic 2-step nilpotent Lie algebras and trivectors appears. They are the following ones:

Theorem 9. *There exists a natural bijection between isomorphism classes of reduced quadratic 2-step nilpotent Lie algebras and dimension $2n$ and the equivalence classes of trilinear forms of rank n.*

Corollary 10. *In an algebraically closed field or $\mathbb{R}$, there exists a finite number of isomorphic classes of reduced quadratic 2-step nilpotent Lie algebras if its dimension is less than 17.*

3. Conclusions

The first clear conclusion we obtain is that having this variety of methods gives us a lot of possibilities. We have several approaches and we can choose the one that fits better for our case.

If we focus on the classic methods (double and T^* extensions), which have been extensively studied, both allow us to incrementally construct all these algebras. The main difference is that:

- Double extension is a more general method but involves several steps.
- T^*-extension is a simpler method, as it is just one step, but it is only valid for some particular Lie algebras. Although it can be used for more general algebras than the Lie ones.

Nevertheless, for the algebras we are interested in, nilpotent 2-step, both methods are perfectly valid for reaching all of them.

On the other hand, the computational approach using Hall Basis is a newer method which can be quite convenient for constructing a lot of examples or checking if some algebra belongs to the class of Lie algebras we are interested in. Moreover, this method can be easily extended to an arbitrary nilpotency index without trouble, and even more, for 2-step Lie algebras we can improve the efficiency using special features of this particular case.

Finally, the fact that trivectors are equivalent allows us to obtain a classification of these algebras, as trivectors have been already classified. Therefore, we can know how many quadratic 2-step Lie algebras are there up to isometrically isomorphisms using less than 9 generators. This data is show in Table 1.

Dimension	6	8	10	12	14	16	≥ 18
Number	1	0	1	2	5	13	∞

Table 1: Non-isometric reduced quadratic 2-step Lie algebras in $\mathbb{C}$ (source [10]).

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