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Bilbao, 5-10 de marzo de 2018

IX *Escuela-Taller de Análisis Funcional*

en memoria del profesor Bernardo Cascales

Bilbao, 4-8 de marzo de 2019

Asociación Nacional de Estudiantes de Matemáticas

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TEMat persigue el doble objetivo de dar visibilidad a la calidad y diversidad de los trabajos realizados por estudiantes de matemáticas a la vez que permite a los estudiantes publicar sus primeros artículos. Se contemplan para su publicación artículos escritos en castellano de todas las áreas de las matemáticas, puras y aplicadas, así como aplicaciones científicas o tecnológicas en las que las matemáticas jueguen un papel central.

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Sobre este volumen / About this volume

Este primer volumen de *TEMat monográficos* contiene artículos sobre la mayoría de los cursos impartidos en las VIII y IX Escuela-Taller de Análisis Funcional, una escuela de análisis organizada por la Red de Análisis Funcional y Aplicaciones en la que los participantes estudian un tema de análisis concreto junto con un profesor. Los artículos presentados aquí los han escrito los estudiantes que participaron en cada uno de los cursos, con la supervisión de su respectivo profesor.

This first volume of *TEMat monográficos* contains papers about most of the courses taught during the 8th and 9th editions of the Escuela-Taller de Análisis Funcional, an analysis school organised by the Red de Análisis Funcional y Aplicaciones where participants study a given topic in analysis together with a lecturer. The papers presented here have been written by the students that took part in each of the courses, with the supervision of their respective instructor.

VIII *Escuela-Taller de Análisis Funcional* (2018)
IX *Escuela-Taller de Análisis Funcional* (2019)

Artículo invitado

La Escuela-Taller de Análisis Funcional «Bernardo Cascales»

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Resumen: Desde el año 2011, la Red de Análisis Funcional y Aplicaciones, apoyada por más de treinta y cinco proyectos de investigación nacionales, organiza anualmente la *Escuela-Taller de Análisis Funcional*. Más de doscientos estudiantes de últimos años de licenciatura, grado y másteres en Matemáticas de veinte universidades nacionales han participado en esta iniciativa. En la edición de 2019, se decidió dedicarla a la memoria de Bernardo Cascales, profesor de la Universidad de Murcia y promotor de la Escuela-Taller.

Abstract: Since 2011, the Spanish Functional Analysis Network, supported by over thirty five national research projects, organises the *Escuela-Taller de Análisis Funcional* yearly. More than two hundred undergraduate or master Mathematics students from twenty Spanish universities have taken part in this initiative. The 2019 edition of the school was dedicated to the memory of Bernardo Cascales, professor at the University of Murcia and promoter of the Escuela-Taller.

Palabras clave: Bernardo Cascales, Escuela-Taller de Análisis Funcional.

MSC2010: 01A70, 01A72, 01A74.

Agradecimientos: Los autores agradecemos los comentarios, sugerencias y observaciones de los editores de *TEMat* que, con su desinteresada labor, han contribuido notablemente a la versión final de este artículo. Todos los miembros y amigos de la Red de Análisis Funcional y Aplicaciones han contribuido con su trabajo y esfuerzo por crear, mantener y mejorar la *Escuela-Taller de Análisis Funcional «Bernardo Cascales»*.

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1. Origen de la Escuela-Taller: la Red de Análisis Funcional y Aplicaciones

En 2004, a iniciativa de José Luis González Llavona, de la Universidad Complutense de Madrid, se puso en marcha la red temática en Análisis Funcional y Aplicaciones, que fue financiada por el Ministerio de Educación y Ciencia mediante dos acciones especiales (referencias BFM2002-11782-E y MTM2004-22129-E). Desde sus comienzos, la Red buscó aglutinar a todos los grupos nacionales cuya investigación se centrara en el análisis funcional. En estos quince años, uno de los objetivos fundamentales de la Red ha sido fomentar la investigación interdisciplinar y la interacción entre los distintos grupos que la integran, poniendo especial atención en la formación de jóvenes estudiantes o doctores.

Durante el periodo 2004-2006, las principales actividades de la Red fueron la creación de una página web (http://www.mat.ucm.es/~nfaas, ya no operativa[1]) y la organización de dos encuentros en la residencia La Cristalera, dependiente de la Universidad Autónoma de Madrid, en Miraflores de la Sierra (Madrid).

En el primero de ellos, celebrado del 29 al 30 de octubre de 2004, dieciocho investigadores de doce universidades españolas tuvieron la oportunidad de informar del estado actual de su trabajo y sus líneas de investigación prioritarias. El segundo, celebrado del 30 de marzo al 1 de abril de 2006, estuvo dedicado a jóvenes investigadores en análisis funcional. Doce estudiantes de doctorado expusieron los principales resultados de sus tesis, corriendo a cargo de Manuel Valdivia la ponencia inaugural.

Una vez celebradas estas dos reuniones, quedó clara la filosofía que debía marcar el devenir de la Red de Análisis Funcional y Aplicaciones: por un lado, dar la oportunidad de exponer sus trabajos a jóvenes investigadores en un ambiente menos rígido que el que predomina en los congresos tradicionales y contando entre el público a reconocidos investigadores; por otro, fomentar la interacción entre grupos de investigación que trabajan en temas muy diferentes. Convivir durante los días del congreso permitía realizar investigaciones conjuntas, compartir información sobre gestión y temas científicos, y surgir profundas amistades.

En diciembre de 2006, la Red pasó a ser coordinada por Miguel Martín y Rafael Payá, de la Universidad de Granada. Cada año la Red iba creciendo y tomando fuerza, uniéndose nuevos grupos de investigación y aumentando el número de participantes en los encuentros.

En el periodo 2007-2010 se organizaron cuatro encuentros de la Red: el primero de ellos, otra vez en la residencia La Cristalera (del 20 al 23 de junio de 2007), y los otros tres en Salobreña, en Granada (del 2 al 5 de abril del 2008, del 22 al 25 de abril del 2009 y del 14 al 17 de abril del 2010, respectivamente). Se impartieron una media de diecisiete conferencias y participaron más de sesenta congresistas cada año.

En este tiempo, la Red fue financiada mediante las acciones complementarias de referencias MTM2007-29355-E y MTM2008-05274-E. Estas acciones complementarias y actividades de la Red fueron avaladas por numerosos proyectos del Plan Nacional y de los programas financiadores de las comunidades autónomas. Datos concretos de estos encuentros pueden consultarse a través de la web https://www.uv.es/functanalys/conferencias-encuentros.html.

En el último de los encuentros en Salobreña, en 2010, se propuso que Bernardo Cascales, de la Universidad de Murcia, fuera el nuevo coordinador de la Red, quien al asumir la dirección aportó nuevas iniciativas.

2. Bernardo Cascales, promotor de la Escuela-Taller

Nuestro amigo Bernardo Cascales Salinas, investigador muy relevante en el área de análisis funcional y ciclista empedernido, nació en Alcantarilla (Murcia) el 10 de enero de 1958. Su interés por las matemáticas le llevó a estudiar, formando parte de la primera promoción, la licenciatura en Matemáticas en la Universidad de Murcia (UM) desde el curso 1975-1976 al 1979-1980. En 1981, como becario de investigación, comenzó su actividad docente en la UM. Realizó la tesis doctoral bajo la dirección del profesor Manuel Valdivia, leyéndola en 1985 y obteniendo por ella el premio extraordinario de doctorado. En 1987 realizó su primera estancia en el extranjero en la Universität Oldenburg, trabajando con Klaus Floret. Allí dejó un gran

[1] Disponible en http://web.archive.org/web/20110306075008/http://www.mat.ucm.es/~nfaas gracias a la Internet Wayback Machine.

Bernardo Cascales. XIV *Encuentro de Análisis Funcional Murcia-Valencia*, 24-26 de septiembre de 2015.

recuerdo por la profundidad de sus ideas y por sobrevivir dos meses a base de bocadillos. A la vuelta, obtuvo su plaza de profesor titular de universidad. De 1990 a 1996 fue director del Departamento de Matemáticas de la UM.

En agosto de 1998 comenzó una estancia científica de un año en la University of Missouri-Columbia (Estados Unidos), donde trabajó con el extraordinario matemático Nigel Kalton. En 2004 obtuvo su plaza de catedrático de universidad en la UM, y en el curso académico 2009-2010 realizó otra estancia de año sabático, en este caso en la Kent State University (Estados Unidos), invitado por Richard M. Aron. Desde 2014 hasta 2017 fue vicerrector de Coordinación e Internacionalización de la Universidad de Murcia, donde desarrolló una actividad extraordinaria.

Fue editor de varias revistas de investigación, entre ellas la *Journal of Mathematical Analysis and Applications* (Elsevier). También fue miembro del comité asesor de la Real Sociedad Matemática Española (RSME) para publicaciones conjuntas con la American Mathematical Society (AMS).

Fue investigador principal en numerosos proyectos competitivos de investigación financiados por el Ministerio de Investigación y Ciencia de España y por la Comunidad Europea. Dirigió ocho tesis doctorales. Impartió un número apreciable de conferencias invitadas en eventos científicos de su campo en universidades españolas y extranjeras.

Sus intereses científicos principales fueron aspectos geométricos de la teoría de espacios de Banach, métodos topológicos de dichos espacios y la teoría de la medida. Sus importantes contribuciones están recogidas en cincuenta y nueve artículos de investigación, prácticamente todos en revistas de reconocido prestigio internacional como *Advances in Mathematics*, *Mathematische Annalen*, *Journal of Functional Analysis*, *Journal of Mathematical Analysis and Applications*, etc. Pero, además de científico encerrado en su mundo, desde siempre tuvo una visión clara de crear a su alrededor un ambiente científico y de colaboración con los mejores matemáticos de su área y que atrajera a jóvenes talentos. Consecuente con esa visión, Bernardo creó en 1990 el Seminario del Departamento de Matemáticas de la UM, consiguiendo un continuo flujo de conferenciantes de muy alto nivel nacional e internacional que continúa hasta hoy en día.

Fue también la *alma mater* detrás de los *Encuentros de Análisis Funcional Murcia-Valencia*, creados en 2007, y de los cuales en vida de Bernardo se celebraron quince ediciones. La decimosexta fue en su memoria en diciembre de 2018. Los concibió como lugar de encuentro informal pero productivo de los investigadores de análisis funcional de las universidades de Murcia, Politécnica de Valencia y de Valencia, y también como escenario natural para que jóvenes pre- y posdoctorales tuvieran su primera audiencia sénior donde exponer sus resultados. El Comité Científico de las primeras ediciones de estos encuentros estuvo formado por José Bonet (Universitat Politècnica de València), Bernardo Cascales (UM) y Manuel Maestre (Universitat de València).

Bernardo fue miembro fundador, y esencial en la consolidación, de la Red de Análisis Funcional y Aplicaciones española, de la que fue coordinador durante tres años, desde el 2011 hasta el 2013.

En 2010 tuvo una idea fundamental y que cambio la Red radicalmente: la creación de la Escuela-Taller. Esta idea fue acogida en medio de grandes dudas de todos los miembros de la Red. Pero Bernardo, con su empuje y bonhomía, consiguió llevarla adelante, siendo su primer organizador (junto con Jesús Bastero), en 2011, en Jaca. El resultado de esta primera edición fue tan extraordinario que la Escuela-Taller quedó como la principal actividad de la Red de Análisis Funcional y Aplicaciones.

Un pilar básico y fundamental en su vida fue su mujer, Cecilia, que, junto con sus hijos Juan Pedro, Bernardo y Miguel, era lo más importante de su vida. Bernardo era un hombre de gran energía, de humor e ironía fina, y con una gran capacidad de trabajo. Gran comunicador de hablar pausado y con gran gusto matemático. Tuvo la virtud de hacer muchos y grandes amigos que ahora le añoramos.

3. Objetivos, estructura y desarrollo de la Escuela-Taller

A mediados de 2010, la Red de Análisis Funcional y Aplicaciones estaba bien afianzada en el panorama matemático nacional y los Encuentros anuales eran una cita obligada y de prestigio para los investigadores de esta área. Por iniciativa de Bernardo, se consideró que sería de gran interés ampliar estas actividades, organizando una escuela-taller destinada a alumnos de los últimos cursos de licenciatura o máster con especial interés en el análisis funcional.

La Escuela-Taller contaba con elementos novedosos en su concepción. Estaba pensada para y por los alumnos. La organización de la Escuela elige unos temas centrales dentro del análisis funcional y unos profesores responsables de los mismos. Los profesores elegidos no tenían que tener especial relación con el tema escogido, prefiriendo que no estuviera en su campo de investigación. Son los responsables de los grupos de investigación que forman la Red de Análisis Funcional y Aplicaciones quienes animan y deciden qué estudiantes de su centro son seleccionados para participar en la Escuela-Taller.

Los estudiantes llegan el domingo por la tarde a la localidad donde se realiza la Escuela-Taller. El lunes por la mañana, los profesores presentan los temas. Asesorados por los docentes, los grupos de alumnos estudian en sesiones intensivas de mañana y tarde los temas, que finalmente exponen a sus compañeros en la última sesión. Inicialmente, se decidió organizar la Escuela-Taller de domingo a miércoles y, a continuación, de jueves a sábado, los Encuentros de la Red, invitando a los alumnos participantes a la Escuela-Taller a asistir también al Encuentro.

Cada Escuela-Taller requiere una financiación notable al necesitar cubrir el desplazamiento, alojamiento y manutención de alrededor de veinticinco estudiantes durante un mínimo de cinco días. Para ello, era imprescindible tener financiación nacional del Ministerio, así como otras ayudas de institutos o centros de investigación locales.

En las nueve ediciones realizadas, han participado doscientos veinte alumnos de veinte universidades nacionales y una extranjera. Se han conseguido unas ayudas de 69 500 euros de parte del Ministerio para la realización de la Escuela-Taller. Para valorar el éxito de esta iniciativa, hay que señalar que muchos de estos alumnos han realizado o están realizando en la actualidad su tesis doctoral en temas relacionados con el análisis funcional y que, de hecho, algunos de ellos ya han participado en los encuentros de la Red, exponiendo el trabajo de investigación desarrollado en el apartado de jóvenes investigadores.

Jornada de trabajo del taller «Teorema de Brouwer y aplicación a equilibrios de Nash». II *Escuela-Taller de Análisis Funcional*, La Manga del Mar Menor, 16-18 de abril de 2012.

4. Las primeras Escuelas-Taller: Jaca (2011) y La Manga del Mar Menor (2012)

Varios problemas logísticos y administrativos se tuvieron que salvar para poder organizar la primera *Escuela-Taller de Análisis Funcional*. Jesús Bastero y Pedro J. Miana, de la Universidad de Zaragoza, ofrecieron la Residencia Universitaria de Jaca, sede de numerosas reuniones científicas desde 1929.

Así, la I *Escuela-Taller de Análisis Funcional* tuvo lugar del 4 al 6 de abril de 2011 en Jaca, organizada en colaboración con el Instituto Universitario de Matemáticas y Aplicaciones (IUMA) de la Universidad de Zaragoza. En tres días completos de actividad se desarrollaron los siguientes talleres:

Taller 1. «El teorema de Baire y el análisis funcional»;

Taller 2. «La transformada de Fourier y el análisis funcional».

Los profesores responsables para esta edición fueron

- José Bonet (Universitat Politècnica de València),
- María Jesús Carro (Universitat de Barcelona),
- Tomás Domínguez (Universidad de Sevilla),
- Rafael Payá (Universidad de Granada) y
- Javier Soria (Universitat de Barcelona).

Participaron veinticinco alumnos, estudiantes de once universidades en últimos cursos de licenciatura o de másteres. A continuación se celebró el VII *Encuentro de la Red de Análisis Funcional y Aplicaciones*, entre el 6 y el 9 de abril de 2011, en la propia Residencia Universitaria de Jaca. La cena del Encuentro se recordará por varias intervenciones, principalmente la imitación del humorista Eugenio y la extensa intervención de Bernardo que, micrófono en mano, agradecía a todos y cada uno de los asistentes su presencia y trabajo.

A mediados de julio de 2011, Bernardo organizó una reunión de coordinación de la Red, en la Universidad de Murcia, para analizar la escuela organizada en Jaca, modernizar el eterno asunto de la página web,

Participantes de la I *Escuela-Taller de Análisis Funcional* y del VII *Encuentro de la Red de Análisis Funcional y Aplicaciones*, Jaca, 4-9 de abril de 2011.

actualizar el listado de tesis y proyectos y planear las futuras actividades de la Red. El grupo de trabajo estaba formado por Victoria Martín, Antonio Pallarés, Ricardo García, Pedro J. Miana y el propio Bernardo.

A la vista del éxito de la primera Escuela-Taller y del interés demostrado por alumnos, profesores e investigadores por su continuidad, en el año 2012 se organizó la II *Escuela-Taller de Análisis Funcional*. Se decidió ampliar a treinta alumnos el número máximo de participantes, celebrándose en La Manga del Mar Menor (Murcia), del 16 al 18 de abril. Colaboraron en la organización miembros del grupo de la Universidad de Murcia, entre ellos Antonio Pallarés. Finalmente asistieron veintiocho alumnos de diez universidades españolas, y precedió al VIII *Encuentro de la Red de Análisis Funcional y Aplicaciones*, acontecido en el mismo lugar del 19 al 21 de abril.

En esta ocasión se eligieron los siguientes dos talleres, con un total de cinco apartados que fueron desarrolladas por grupos de hasta seis estudiantes:

Taller 1. «Teoremas de punto fijo y aplicaciones», con las secciones

- 1.1. «Teorema de Brouwer y aplicación a equilibrios de Nash»,
- 1.2. «Teorema minimax y consecuencias» y
- 1.3. «Problema del subespacio invariante»;

Taller 2. «Análisis convexo», con las secciones

- 2.1. «Desigualdad de Brunn-Minkowski e isoperimétrica» y
- 2.2. «Desigualdad isoperimétrica vs. desigualdades de Sobolev».

Los profesores responsables de los cinco grupos de trabajo fueron

- Jesús Bastero (Universidad de Zaragoza),
- Joan Cerdá (Universitat de Barcelona),
- Manuel Maestre (Universitat de València),
- Antonio Martinón (Universidad de La Laguna) y
- Bernardo Cascales (Universidad de Murcia).

Participantes de la II *Escuela-Taller de Análisis Funcional*, La Manga del Mar Menor, 16-18 de abril de 2012.

Al igual que en la anterior Escuela-Taller, los alumnos desarrollaron los temas asignados y, a continuación, los expusieron a sus compañeros y a los miembros de la Red asistentes. Posteriormente asistieron al Encuentro de la Red y en la cena del Encuentro disfrutaron del entretenimiento «Parecidos Razonables», ideado por miembros del comité organizador. El «pájaro loco» con su cresta en punta fue elegido para representar al coordinador de la Red en ese momento, nuestro amigo Bernardo.

Durante el período 2010-2012, las actividades de la Red fueron financiadas por las acciones complementarias MTM2009-08489-E y MTM2010-11906-E.

5. Las escuelas de Zafra (2013), Sevilla (2014) y Madrid (2015)

En la reunión de coordinación de la Red en abril de 2012 se acordó continuar con la Escuela-Taller e incorporar a Tomás Domínguez, de la Universidad de Sevilla, como coordinador de la Red, quedando este como coordinador único desde abril de 2013 hasta marzo de 2014. Además, por cuestiones organizativas y limitaciones presupuestarias, se decidió limitar el número de alumnos de la escuela a veinticinco, creando una lista de reserva en caso de que hubiera más solicitudes. También se priorizaría a los estudiantes que no hubieran asistido a ninguna edición frente a los ya participantes en ocasiones anteriores.

Inicialmente, la III *Escuela-Taller de Análisis Funcional* se iba a celebrar en Aracena (Huelva), pero finalmente se realizó en Zafra (Badajoz) del 9 al 13 de abril de 2013. Se organizó integrada en el IX *Encuentro de la Red de Análisis Funcional y Aplicaciones*, celebrado en la misma localidad de Zafra del 11 al 13 de abril. Así, las exposiciones finales de los equipos de la Escuela fueron incluidas en el programa oficial del Encuentro. Para la financiación de esta escuela se contó con la ayuda de la acción complementaria del Ministerio MTM2011-15726-E.

A esta edición de la Escuela-Taller fueron invitados veintiún alumnos de nueve universidades, seleccionados por los investigadores principales de los distintos grupos de investigación que forman parte de la Red. Los dos temas elegidos, divididos en cuatro secciones para ser desarrollados, fueron los siguientes:

Taller 1. «Algunas desigualdades notables del Análisis», subdividido en

- 1.1. «Desigualdad de Hardy» y
- 1.2. «Desigualdad de Grothendieck»;

Participantes de la III *Escuela-Taller de Análisis Funcional* y el IX *Encuentro de la Red de Análisis Funcional y Aplicaciones*, Zafra, 9-13 de abril de 2013.

Taller 2. «Métodos topológicos en Análisis Funcional», con las secciones

2.1. «Aplicaciones del conjunto de Cantor» y
2.2. «Límites en espacios de Banach».

En esta ocasión, los profesores encargados de la Escuela-Taller fueron

- José P. Moreno (Universidad Autónoma de Madrid),
- David Pérez-García (Universidad Complutense de Madrid),
- Antonio Fernández Carrión (Universidad de Sevilla) y
- Jesús M. F. Castillo (Universidad de Extremadura).

Además de la excursión a la ruta de los Castillos y la inolvidable visita a un secadero de jamones en Jerez de los Caballeros con degustación incluida, en la cena social del IX Encuentro se realizó una edición exclusivamente matemática del juego «Pasapalabra». Victoria Martín colaboró notablemente para que la Escuela-Taller y el Encuentro fueran un éxito.

La IV *Escuela-Taller de Análisis Funcional* se celebró en Sevilla del 4 al 8 de marzo de 2014 y se organizó en colaboración con el Instituto de Matemáticas de la Universidad de Sevilla «Antonio de Castro Brzezicki» (IMUS). La Escuela contó con la participación de veinticuatro alumnos de once universidades, seleccionados una vez más por los investigadores principales de los grupos de investigación que conforman la Red. Al igual que en la anterior edición, los alumnos expusieron su trabajo final integrado en el programa del X Encuentro de la Red, celebrado del 6 al 8 de marzo.

Los dos temas seleccionados fueron los siguientes, divididos en dos secciones:

Taller 1. «Operadores Lineales», subdividido en

1.1. «Operadores no acotados en Espacios de Hilbert» y
1.2. «Aplicaciones Lineales que preservan Funciones Ortogonales»;

Taller 2. «Aplicaciones del Análisis Funcional», subdividido en

2.1. «Movimiento Browniano y la fórmula de Itô» y
2.2. «El problema de Monge-Kantorovich».

En esta ocasión, la Escuela-Taller estuvo a cargo de los siguientes profesores:

- José L. González Llavona (Universidad Complutense de Madrid),
- Alberto Ibort (Universidad Carlos III de Madrid e ICMAT),
- Antonio Pallarés (Universidad de Murcia),
- Antonio Peralta (Universidad de Granada) y
- Rafael Espínola (Universidad de Sevilla).

En la reunión de la Red, se incorporaron como coordinadores Jesús M. Fernández Castillo y Ricardo García, de la Universidad de Extremadura, manteniéndose Tomás Domínguez también como cocoordinador hasta el año siguiente. Al no poseer fuente de financiación por parte del Ministerio para la realización de la siguiente Escuela-Taller, se decidió contactar con el Instituto de Ciencias Matemáticas (Madrid) para la realización de la actividad en sus instalaciones.

La V *Escuela-Taller de Análisis Funcional* (y el XI *Encuentro de la Red de Análisis Funcional y Aplicaciones*) tuvo lugar del 2 al 7 de marzo 2015 en las instalaciones del ICMAT (Madrid), que, además, la subvencionó. Esta edición contó con veinticuatro alumnos de trece universidades. Los temas tratados en la Escuela-Taller fueron, en esta ocasión, cuatro talleres independientes, cuyos profesores asignados fueron los siguientes:

Taller 1. «Inversión Global de Funciones». Profesor encargado: Jesús Ángel Jaramillo Aguado (Universidad Complutense de Madrid).

Taller 2. «Desigualdades Variacionales en Espacios de Hilbert y Aplicaciones». Profesor encargado: Jesús García-Falset (Universitat de València).

Taller 3. «Teorema de Mazur-Ulam». Profesor encargado: Miguel Martín Suárez (Universidad de Granada).

Taller 4. «Principios Variacionales y Aplicaciones». Profesor encargado: Daniel Azagra Rueda (Universidad Complutense de Madrid).

6. Las Escuelas-Taller de Cáceres (2016 y 2017)

La VI *Escuela-Taller de Análisis Funcional* (y el XII *Encuentro de la Red de Análisis Funcional y Aplicaciones*) tuvo lugar en Cáceres del 29 de febrero al 4 de marzo de 2016. El lugar elegido fue el complejo cultural de San Francisco. A lo largo de su historia, este lugar ha tenido diferentes usos: sede del Colegio de Teología Escolástica, cuartel y caballerizas en épocas de guerras, hospital provincial o casa de misericordia.

Para esta Escuela-Taller, los temas y profesores seleccionados fueron los siguientes:

Taller 1. «Midiendo conjuntos no medibles y espacios de Banach universales». Profesor encargado: Antonio Avilés (Universidad de Murcia).

Taller 2. «Análisis armónico de la A a la A». Profesor encargado: Félix Cabello Sánchez (Universidad de Extremadura).

Taller 3. «Teoremas de puntos fijos y aplicaciones». Profesor encargado: Juan José Nieto (Universidade de Santiago de Compostela).

Taller 4. «Mecánica Cuántica - Principio de Incertidumbre - Ecuación de Schrödinger - Ecuación de Dirac». Profesor encargado: Luis Vega (Universidad del País Vasco / Euskal Herriko Unibertsitatea).

Participaron veintitrés alumnos provenientes de once universidades diferentes. Esta edición (y la anterior) fue financiada parcialmente con el proyecto de redes de excelencia MTM2014-53152-REDT. Es destacable el bonito diseño del póster conjunto de la Escuela-Taller y del Encuentro que puede verse en el Anexo B.

La VII *Escuela-Taller de Análisis Funcional* (y el XIII *Encuentro de la Red de Análisis Funcional y Aplicaciones*) tuvo lugar del 6 al 11 de marzo de 2017. Se volvió a celebrar en el complejo cultural San Francisco, y el póster conjunto de la Escuela-Taller y del Encuentro no defraudó.

Los cuatros temas tratados en la Escuela-Taller y los cuatro profesores que los coordinaron fueron los siguientes:

Taller 1. «Teoremas de Ramsey y espacios de Banach». Profesor encargado: Jordi López Abad (ICMAT y Université Pierre et Marie Curie).

Taller 2. «Análisis de las desigualdades de Poincaré-Sobolev». Profesor encargado: Carlos Pérez Moreno (BCAM y Universidad del País Vasco / Euskal Herriko Unibertsitatea).

Taller 3. «Factorizando, que es gerundio». Profesor encargado: Pedro Tradacete (Universidad Complutense de Madrid).

Taller 4. «Desigualdades funcionales y convergencia de procesos de difusión». Profesor encargado: Juan Luis Vázquez (Universidad Autónoma de Madrid).

Participaron veintiún alumnos provenientes de doce universidades nacionales. En este encuentro, en la reunión de coordinación de la Red, fueron nombrados coordinadores de la Red de Análisis Funcional y Aplicaciones Domingo García Rodríguez y Manuel Maestre Vera (Universitat de València), encargándose de las siguientes ediciones.

7. Las escuelas de Bilbao (2018 y 2019) y seguimos en La Laguna (2020)

En la Escuela-Taller de Cáceres (2017), Carlos Pérez ofreció su colaboración para celebrar las próximas ediciones en el Basque Center for Applied Mathematics (BCAM) en Bilbao. Así, se lograría una vieja aspiración de la Red, crecer en el norte de la península. A excepción de la primera escuela en Jaca, las demás ediciones se habían celebrado en Madrid y al sur de Madrid.

La VIII *Escuela-Taller de Análisis Funcional* (y el XIV *Encuentro de la Red de Análisis Funcional y Aplicaciones*) se celebró en Bilbao del 5 al 9 marzo de 2018. Ha sido la edición más numerosa celebrada, con treinta y un alumnos participantes de doce universidades. Las estancias de los alumnos fueron cofinanciadas por el proyecto de redes de excelencia MTM2016-81726-REDT y el BCAM. Javier Falcó, de la Universitat de València, también ha colaborado intensamente en la organización de las siguientes escuelas.

Se seleccionaron los siguientes cinco temas y cinco profesores responsables:

Taller 1. «Semigroup Theory in Quantum Mechanics». Profesor encargado: Jean-Bernard Bru (BCAM y Universidad del País Vasco / Euskal Herriko Unibertsitatea).

Taller 2. «El teorema de Müntz-Szász y algunas de sus extensiones». Profesor encargado: Pedro J. Miana (IUMA y Universidad de Zaragoza).

Taller 3. «El laplaciano fraccionario desde distintos puntos de vista». Profesora encargada: Luz Roncal (BCAM y Universidad del País Vasco / Euskal Herriko Unibertsitatea).

Taller 4. «En busca de la linealidad en matemáticas». Profesor encargado: Juan B. Seoane-Sepúlveda (IMI y Universidad Complutense de Madrid).

Taller 5. «Linear dynamics: Somewhere dense orbits are everywhere dense». Profesor encargado: Alfred Peris Manguillot (IUMPA y Universitat Politècnica de València).

El 5 de abril del 2018 fallecía, de forma inesperada, Bernardo Cascales, inspirador y *alma mater* de la *Escuela-Taller de Análisis Funcional*. Desde el primer momento, numerosos compañeros recordamos a nuestro amigo de la sonrisa permanente y nos unimos a la familia en tan terrible pérdida. Como a Bernardo le hubiera gustado, del 4 al 8 de marzo de 2019 se celebró la IX *Escuela-Taller de Análisis Funcional*, dedicada a la memoria del profesor y amigo Bernardo Cascales. La Escuela-Taller estuvo integrada, siguiendo la tradición de los últimos años, en el XV *Encuentro de la Red de Análisis Funcional y Aplicaciones*. De nuevo se celebró en el BCAM, y participaron veintitrés alumnos de trece centros universitarios.

Los cuatro talleres que se desarrollaron, así como los profesores responsables, fueron

Participantes de la IX *Escuela-Taller de Análisis Funcional* y el XV *Encuentro de la Red de Análisis Funcional y Aplicaciones*, Bilbao, 4-8 de marzo de 2019.

Taller 1. «Funciones cuadrado y cálculo funcional H^∞ para operadores vectoriales». Profesor encargado: Jorge Betancor Pérez (Universidad de La Laguna).

Taller 2. «Geometría de espacios de polinomios y desigualdades polinomiales». Profesor encargado: Gustavo Muñoz Fernández (IMI y Universidad Complutense de Madrid).

Taller 3. «Maximal averaging operators: from geometry to boundedness through duality». Profesor encargado: Ioannis Parissis (BCAM y Universidad del País Vasco / Euskal Herriko Unibertsitatea).

Taller 4. «Espacios de Hardy y funciones holomorfas en infinitas variables». Profesor encargado: Pablo Sevilla-Peris (IUMPA y Universitat Politècnica de València).

En la reunión de coordinación de la Red se tomaron las siguientes decisiones. Se acordó que el siguiente coordinador sería Jorge Betancor Pérez, de la Universidad de La Laguna, para el bienio 2020-2021. A continuación, sería Alfred Peris Manguillot, de la Universitat Politècnica de València, quien asumiría la dirección de la Red para los años 2022 y 2023. Por unanimidad, se decidió que, a partir de este momento, la Escuela-Taller de la Red de Análisis Funcional y Aplicaciones se denominará *Escuela-Taller de Análisis Funcional «Bernardo Cascales»*.

La X *Escuela-Taller de Análisis Funcional «Bernardo Cascales»* tendrá lugar del 2 al 6 de marzo de 2020 en la Universidad de La Laguna (Tenerife). Para conocer más datos de esta próxima escuela o de las anteriores puede consultarse la web `https://www.uv.es/functanalys/conferencias-escuela-taller.html`.

Con fecha 25 de noviembre de 2019, se concedió a la Red de Análisis Funcional y Aplicaciones el proyecto MTM2018-102535-REDT para los años 2020 y 2021, y que asegura la existencia de la Red y la realización de la *Escuela-Taller de Análisis Funcional «Bernardo Cascales»*.

A. Alumnos de las Escuelas-Taller (2011-2019)

Acrónimo	Centro
UA	Universitat d'Alacant / Universidad de Alicante
UAB	Universitat Autònoma de Barcelona
UAM	Universidad Autónoma de Madrid
UAL	Universidad de Almería
UB	Universitat de Barcelona
U. Bonn	Universität Bonn
UCA	Universidad de Cádiz
UCM	Universidad Complutense de Madrid
UEX	Universidad de Extremadura
UGR	Universidad de Granada
ULL	Universidad de La Laguna

Acrónimo	Centro
UM	Universidad de Murcia
UMA	Universidad de Málaga
UOV	Universidad de Oviedo / Universidá d'Uviéu
UPC	Universitat Politècnica de Catalunya
UPV	Universitat Politècnica de València
UPV/EHU	Universidad del País Vasco / Euskal Herriko Unibertsitatea
US	Universidad de Sevilla
USC	Universidade de Santiago de Compostela
UV	Universitat de València
UZ	Universidad de Zaragoza

I Escuela-Taller, Jaca, 2011

Estudiante	Centro
A. Castro Castilla	ULL
V. Pérez Calabuig	UPV
J. M. Ribera Puchades	UPV
C. Pastor Alcoceba	UPV
A. C. Márquez García	UAL
N. Quereda Castañeda	UAL
Á. D. Martínez Martínez	UV
D. Zorio Ventura	UV
J. Ginés Espín	UM

Estudiante	Centro
A. Pérez	UM
X. Ros Oton	UPC
F. Mazaira Font	UPC
C. Zaragoza	UV
M. Castellano Rodríguez	UB
M. Farré	UB
L. Abadías Ullod	UZ
M. Chica Rivas	UGR

Estudiante	Centro
L. A. Urrutia Matarín	UGR
F. Cortez	UCM
B. Medina Pérez	ULL
C. Hernández Trujillo	ULL
J. Alberto Alonso	UA
C. Casorran Amilburu	UA
A. Molino	UGR
I. Zaplana	UM

II Escuela-Taller, La Manga, 2012

Estudiante	Centro
C. Domingo	UB
C. Hernández	ULL
L. C. García	UMA
A. Jiménez	US
M. Latorre	UV
S. Montaner	UZ
L. Abadías Ullod	UZ
I. Álvarez Romero	UCM
J. R. Balaguer Tornel	UM
D. Lear Claveras	UZ

Estudiante	Centro
E. Primo Tárraga	UV
J. L. Ródenas Pedregosa	UCM
A. M. Cabrera	UGR
R. Jiménez	UPV
J. Martín	UCM
G. Martínez	UMA
B. Medina	ULL
D. Rodríguez	UCM
A. Gómez	US

Estudiante	Centro
V. M. Jiménez	ULL
P. Jiménez	UCM
A. Leonhardt	UB
A. Molino	UGR
E. Camacho	US
I. Fernández	UCM
A. González	ULL
J. Granados	US
L. Urrutia	UGR

III Escuela-Taller, Zafra, 2013

Estudiante	Centro
D. Lear Claveras	UB
E. Primo Tárraga	UV
F. Fenoll Ballesteros	UV
I. Fernández Varas	UCM
I. Álvarez Romero	UCM
D. Beltran Portalés	UB
E. Soto Ballesteros	UB

Estudiante	Centro
J. Castillo Medina	UV
J. Sánchez Fernández	UPV
S. Barahona Albiol	UV
Á. Capel Cuevas	UGR
A. Bethencourt de León	ULL
E. Camacho Aguilar	US
J. J. Marín García	UM

Estudiante	Centro
P. Jiménez Rodríguez	UCM
A. Rueda Zoca	UGR
A. Gómez Pachón	US
J. Ocáriz Gallego	UM
M. Prado Rodríguez	US
M. Cueca Ten	UV
Y. Puig de Dios	UPV

IV Escuela-Taller, Sevilla, 2014

Estudiante	Centro
S. Barahona Albiol	UV
G. L. Bello Burguet	UZ
R. Blasco García	UZ
L. M. Bonilla Bermúdez	ULL
M. de G. Cabrera Padilla	UAL
A. Capel Cuevas	UGR
M. de León Contreras	ULL
D. Díaz Mellado	US

Estudiante	Centro
J. L. Durán Batalla	UEX
J. J. González López	UM
R. Grande Izquierdo	UPV/EHU
L. Martín Valverde	UAL
A. J. Molero del Río	US
L. Pardo Simón	UM
P. Pérez López	UAL
A. Piedrafita Postigo	UAB

Estudiante	Centro
A. Poveda Ruzafa	UV
M. Prado Rodríguez	US
J. Prieto Garralda	UGR
E. Roure Perdices	UB
A. Rueda Zoco	UGR
J. Sánchez Fernández	UV
M. M. Sillero Denamiel	US
M. Solera Diana	UV

V Escuela-Taller, Madrid, 2015

Estudiante	Centro
T. Aguilar Hernández	ULL
P. M. Berná Larrosa	UV
M. G. Cabrera Padilla	UAL
M. Caelles Vidal	UB
C. A. Cruz Rodríguez	ULL
D. Eceizabarrena Pérez	UPV/EHU
M. García Barjollo	UEX
P. J. Gerlach Mena	US

Estudiante	Centro
R. González Fariña	ULL
L. Martín Valverde	UAL
A. Mas Mas	UA
F. Mengual Bretón	UZ
G. Mestre Marcos	UA
D. Morales González	UEX
C. J. Moreno Ávila	UEX
L. Orellana	UZ

Estudiante	Centro
C. Parreño Torres	UV
M. Pelegrín García	UM
J. Rebollo García	US
J. A. Salmerón Garrido	UM
M. Solera Diana	UAM
M. Soria Carro	UAB
M. Terrón Villalba	UEX
J. Toboso Flores	UAB

VI Escuela-Taller, Cáceres, 2016

Estudiante	Centro
A. Llinares Romero	UA
A. Álvarez Carmona	UEX
A. Rodríguez Arenas	UPV
A. J. Román Salvatierra	US
A. Zarauz Moreno	UAL
D. Mompeán Rueda	UM
D. Morales González	UEX
D. Lloria Albiñana	UPV

Estudiante	Centro
I. Pérez Arribas	UPV/EHU
I. López Bailón	UGR
J. Canto Llorente	UPV/EHU
J. C. Bastons García	UB
J. Santos Barragan	UEX
J. A. Gutiérrez Sagredo	UM
J. A. Salmerón Garrido	UM
J. C. García Merino	UEX

Estudiante	Centro
J. A. Fernández Torvisco	UEX
J. C. Cantero Guardeño	UAB
M. Santos Gutiérrez	UPV/EHU
M. A. Andrés Mañas	UAL
P. J. Gerlach Mena	US
P. Piniella Cerón	UCA
V. González López	UM

VII Escuela-Taller, Cáceres, 2017

Estudiante	Centro
S. Arias García	UB
J. Becerra Garrido	UEX
B. Fernández Besoy	UCM
E. Gómez Orts	UPV-UV
C. Lois Prados	USC
J. Martínez Perales	UPV/EHU
M. C. Molero del Ría	US

Estudiante	Centro
P. Palacios Herrero	UZ
O. Roldan Blay	UV
G. Sala Fernández	UB
A. Salguero Alarcón	UEX
D. Santacreu Ferra	UV
C. Valverde Martín	US
V. Asensio López	UPV

Estudiante	Centro
F. J. Carmona Molero	US
J. D. Rodríguez Abellán	UM
A. Rosales Tristancho	US
J. Suárez Quero	UA
A. Torregrosa	UCM
A. Torres Ruiz	UAL
J. M. Uzal Couselo	USC

VIII Escuela-Taller, Bilbao, 2018

Estudiante	Centro
N. Accomazzo Scotti	UPV/EHU
J. Aguado López	US
B. Amador Medina	ULL
S. Baena Miret	UB
A. Becerra Tomé	US
D. Bolón Rodríguez	USC
A. I. Cano Mármol	UM
A. Carballido Costas	USC
M. L. Castillo Godoy	UGR
C. Constantino Otaivén	US
C. Corbalán Mirete	UM

Estudiante	Centro
M.Cueto Avellaneda	UAL
R. Chiclana Vega	UGR
M. García Fernández	UB
F. Gómez Marín	UV
E. Gómez Orts	UV
F. J. González Doña	US
H. Jardón Sánchez	UOV
J. Llorente Jorge	UCM
M. E. Martínez Gómez	UCM
J. Martínez Perales	UPV/EHU

Estudiante	Centro
D. Nieves Roldán	UM
A. Quero de la Rosa	UGR
A. Ratsimanetrimanana	UPV/EHU
A. Rodríguez Abella	UCM
D. L. Rodríguez Vidanes	UCM
E. Sáez Maestro	UCM
M. Sanchiz Alonso	UCM
D. Santacreu Ferrà	UPV
I. Soler Albadalejo	UM
W. Vaconcelos Cavalcante	UV

IX Escuela-Taller, Bilbao, 2019

Estudiante	Centro
M. Bernardino del Pino	US
R. Chiclana Vega	UGR
L. Gómez Espinosa	UEX
S. Ndiaye	UM
H. Ariza Remacha	UV
G. Cao Labora	UPC
A. López Martínez	UV
H. Méndez Gómez	UPV

Estudiante	Centro
A. Arraz Almirall	UB
P. Ribera Baraut	UB
C. Cobollo Gómez	UV
A. Quilis Sandemetrio	UPV
J. Chirinos Rodríguez	ULL
H. Jardón Sánchez	UOV
E. Lavado Santiago	UEX
C. I. Martínez Artero	UV

Estudiante	Centro
L. Cabezas Rosa	UAL
M. M. Utrera Torres	UAL
N. González Barral	USC
A. Quero de la Rosa	UGR
M. L. Castillo Godoy	UGR
N. Storch de Gracia	U. Bonn
C. Martín Murillo	UEX

B. Pósteres de las Escuelas-Taller (2011-2019)

IX encuentro + III escuela taller
alumnos de grado, licenciatura y máster
http://www.um.es/functanalysis/
functional.analysis.spain@gmail.com
Red de Análisis Funcional
y aplicaciones
Grupos de la Red
Felix Cabello (UEx)
Joan Cerdá (UB)
Domingo García (UV)
Enrique Jordá (UPVA)
M. Aránzazu Juan (UPVA)
Chong Li (U. of Hangzhou/US)
Antonio M. Peralta (UGR)
Pedro Tradacete (UC3M)
Escuela Taller:
Jesús Castillo (UEx)
Antonio Fernández (US)
José P. Moreno (UAM)
David Pérez-García (UCM)
Tesis Doctorales
Luciano Abadias (UZA)
David Ariza (US)
Antonio Pérez (UMU)
Rafael Payá (UGR) (apertura)
Antonio Córdoba (UAM) (clausura)
9 al 13 de abril
2013
Zafra
Badajoz

IV escuela taller
alumnos de grado, licenciatura y máster
Red de Análisis Funcional
y aplicaciones
Profesores:
Rafael Espínola (US)
José L. González Llavona(UCM)
José Orihuela (UMU)
Antonio Peralta (UGR)
4 al 8 de marzo
http://www.um.es/functanalysis/school
functional.analysis.spain@gmail.com
2014
Sevilla

XI encuentro + V escuela taller
alumnos de grado, licenciatura y máster
http://www.um.es/functanalysis/school
functional.analysis.spain@gmail.com
Red de Análisis Funcional
y aplicaciones
Manuel de León (ICMAT) (apertura)
Juan José Nieto (USC) (clausura)
Profesores:
Daniel Azagra (UCM)
Jesús García-Falset (UV)
Jesús Ángel Jaramillo (UCM)
Miguel Martín (UGR)
Grupos de la Red
María Acosta (UGR)
Fernando Cobos (UCM)
Xabier Domínguez (UDC)
Fernando García-Castaño (UA)
María Ángeles Japón (US)
El Amin Kaidi (UAL)
Ginés López (UGR)
Pedro Miana (UNIZAR)
Kishin Sadarangani (ULPGC)
Tesis Doctorales
Javier Cabello (UEX)
Nadia Clavero (UB)
Ismael García Bayona (UV)
Oscar Madiedo (UCM)
3 al 7 de marzo
2015
Madrid

XII ENCUENTRO + VI ESCUELA TALLER
2 a 4 de Marzo
29 de febrero a 2 de Marzo
RED DE
ANÁLISIS
FUNCIONAL
Y APLICACIONES
Profesores:
Antonio Avilés (UM)
Félix Cabello (UEX)
Juan José Nieto (USC)
Luis Vega (UPV/EHU)
Conferencias:
José Bonet (UPV)
Bernardo Cascales (UM)
María Jesús Carro (UB)
Eva Gallardo (UCM)
Valentín Ferenczi (Sao Paulo)
Jesús Jaramillo (UCM)
Pier Luigi Papini (Bolonia)
Rafael Payá (UGR)
Luis Vega (UPV/EHU)
http://www.um.es/functanalysis/school
functional.analysis.spain@gmail.com
CÁCERES 2016
Complejo Cultural San Francisco
XIII ENCUENTRO + VII ESCUELA TALLER
9 a 11 de Marzo
6 a 11 de Marzo
2017
Cáceres
Complejo Cultural San Francisco
Red de Análisis
Funcional y aplicaciones
Profesores:
Jordi López-Abad (París VII)
Carlos Pérez (BCAM/UPV)
Pedro Tradacete (UC3)
Juan Luis Vázquez (UAM)
Conferencias:
Fernando Cobos (UCM)
Vicente Montesinos (UPV)
Javier Parcet (CSIC)
David Pérez (UCM)
Alfred Peris (UPV)
Juan Luis Vázquez (UAM)
http://www.um.es/functanalysis/school
functional.analysis.spain@gmail.com

C. Proyectos Nacionales financiadores de las Escuelas-Taller (2011-2019)

Referencia	Fecha de inicio	Fecha de fin	Cantidad (euros)
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Semigroup theory in quantum mechanics

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Abstract: In mathematics, the concept of strongly continuous one-parameter semigroup (C^0-semigroup) appears intuitively to be the generalization of the (usual) exponential function. Roughly speaking, this class of semigroups provides solutions of linear ordinary differential equations with constant coefficients in Banach spaces, see *Schrödinger equation* (1)–(2). Semigroup theory turns out to be fundamental in order to understand the time evolution in quantum mechanics, and is necessary in order to generate the dynamics of both well-known formulations (*Shrödinger picture* and *Heisenberg picture*). Within this paper, the main result that we present is the *Hille–Yosida theorem*, see section 5, which characterizes the generators of C^0-semigroups of linear operators on Banach spaces. It is named after the mathematicians Einar Hille and Kōsaku Yosida who independently stated it around 1948. This manuscript is highly inspired by Engel and Nagel's notes [2].

Resumen: En matemáticas, el concepto de semigrupo uniparamétrico fuertemente continuo (C^0-semigrupo) puede entenderse intuitivamente como generalización de la función exponencial. A grandes rasgos, esta clase de semigrupos ofrece soluciones a ecuaciones diferenciales ordinarias con coeficientes constantes en espacios de Banach, véase la *ecuación de Schrödinger* (1)–(2). La teoría de semigrupos resulta fundamental a la hora de comprender la evolución temporal en mecánica cuántica, y es necesaria para generar la dinámica de ambas formulaciones conocidas (*imagen de Schrödinger* e *imagen de Heisenberg*). En este artículo, el principal resultado presentado es el *teorema de Hille-Yosida*, ver sección 5, que caracteriza los generadores de los C^0-semigrupos de operadores lineales sobre espacios de Banach. Este teorema debe su nombre a los matemáticos Einar Hille y Kōsaku Yosida, quienes lo enunciaron independientemente en torno a 1948. El presente texto se inspira altamente en el libro de Engel y Nagel [2].

Keywords: Hille-Yosida theorem, semigroup theory, quantum mechanics.

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Notations

- For a normed generic vector space $\mathcal{X}$, its norm is denoted by $\|\cdot\|_{\mathcal{X}}$.
- The identity element of a generic vector space $\mathcal{X}$ is denoted by $\mathbf{1}_{\mathcal{X}}$.
- The set of linear operators from $\mathcal{X}$ into $\mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X})$.
- The set of all bounded linear operators on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. For an operator $A \in \mathcal{B}(\mathcal{X})$, its norm is defined by
$$\|A\|_{\mathcal{B}(\mathcal{X})} := \sup_{u\in\mathcal{X}} \frac{\|Au\|_{\mathcal{X}}}{\|u\|_{\mathcal{X}}}.$$
- If $\mathcal{X}$ is a Hilbert space, then its norm is associated to a scalar product denoted by $\langle\cdot,\cdot\rangle_{\mathcal{X}}$.
- For all $A, B \in \mathcal{B}(\mathcal{X})$, we define
$$[A, B] := AB - BA \qquad \text{and} \qquad \{A, B\} := AB + BA.$$
- For any complex number z, its conjugate is denoted by $\overline{z}$.

1. Introduction

The foundations of quantum mechanics were established during the first half of the 20$^{\text{th}}$ century. In the mid-twenties, two main formulations of quantum physics appeared, both meant to establish the principles of quantum theory. These two directions were taken by W. K. Heisenberg and by E. Schrödinger, respectively. After being in opposition, they turned out to be equivalent after several contributions of J. von Neumann on the foundation of quantum mechanics in the following years. Both formulations are currently used in any standard textbook on quantum physics. For the sake of clarity, we will first set the so-called *Schrödinger picture* of quantum mechanics. Indeed, it is widely known, used and commented in fields such as that of partial differential equations (PDEs), for instance, through the celebrated *Schrödinger equation*.

2. Schrödinger picture of quantum mechanics

In 1925, following de Broglie's hypothesis on wave property of matter, E. Schrödinger derived his celebrated equation, describing a time-dependent wave behavior of quantum objects. In fact, the state of the quantum system is completely described by a family of time dependent wave functions $\{\psi(t)\}_{t\in\mathbb{R}}$ within a Hilbert space $\mathcal{H}$. For instance, for the one-particle case, one generally considers the case $\mathcal{H} := L^2(\mathbb{R}^3)$ or $\mathcal{H} := \ell^2(\mathbb{Z}^3)$, respectively, for the continuum quantum system or the discrete one. This time evolution is fixed by a self-adjoint operator H acting on $\mathcal{H}$. Indeed, for any time $t \in \mathbb{R}$, the wave function is determined by the well-known *Schrödinger equation*:

$$(SE) \quad \begin{cases} \mathrm{i}\partial_t\psi(t) = H\psi(t), \\ \psi(0) = \psi_0 \in \mathcal{H}. \end{cases} \tag{1}$$

This implies that

$$\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0, \qquad t \in \mathbb{R}. \tag{2}$$

Note that the fact that H is self-adjoint is important to give a sense to equations (1) and (2). It is described through Stone's theorem, see theorem 19, which sets that having a self-adjoint operator, acting on some Hilbert space, is a sufficient condition in order to define a strongly continuous one-parameter group (also denoted C_0-group). We will say some words on them later but, at this point, the aim is to give an intuition to the reader about the different ways to formulate quantum mechanics. A standard example taught to every student in quantum mechanics is brought by the case where $\mathcal{H} := L^2(\mathbb{R}^3)$ and $\|\psi(t)\|_{\mathcal{H}} = \|\psi_0\|_{\mathcal{H}} = 1$. Then, $|\psi(t,x)|^2$ is interpreted as the probability for the particle to be at a position $x \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. As mentioned in the introduction above, a widely studied standard example in the field of PDEs is given by the case where the operator $H := -\Delta$ (the usual Laplacian operator). The same interpretation can be done on the lattice $\mathbb{Z}^3$, instead of taking $\mathbb{R}^3$.

3. Heisenberg picture of quantum mechanics

Physical quantities such as position, speed, energy, etc., are self-adjoint operators acting on the Hilbert space $\mathcal{H}$. They are called *observables*, being all quantities of the physical system that can be measured. For instance, one of the most important observables is the celebrated self-adjoint *Hamiltonian* H that describes the time evolution of the wave function in the Schrödinger equation (1)–(2). This Hamiltonian is associated with the energy observable.

The measurement of a physical quantity (observable) has, from this point of view, a random character. The statistical distribution of its value is described by the family of wave functions $\{\psi(t)\}_{t\in\mathbb{R}}$ (see equation (1)). The expectation value of any observable B acting on $\mathcal{H}$ is given by

$$\langle\psi(t), B\psi(t)\rangle_{\mathcal{H}}.$$

By equation (2), it equals

$$\langle\psi(t), B\psi(t)\rangle_{\mathcal{H}} = \langle\psi_0, \mathrm{e}^{\mathrm{i}tH}B\mathrm{e}^{-\mathrm{i}tH}\psi_0\rangle_{\mathcal{H}}. \tag{3}$$

At this point, it turns out that, instead of considering the wave functions as being time-dependent, like in the Schrödinger picture of quantum mechanics, one can take them as fixed in time and assume a time evolution of the so-called observables. Both methods lead to the same statistical distribution as one can see in equation (3). Indeed, for the time evolution of any observable B, we apply on it the map $\tau_t(B) \coloneqq \mathrm{e}^{\mathrm{i}tH}B\mathrm{e}^{-\mathrm{i}tH}$ for $t\in\mathbb{R}$. For an operator H acting on the Hilbert space $\mathcal{H}$, the family $\{\tau_t\}_{t\in\mathbb{R}}$ defines a strongly continuous group acting on $\mathcal{B}(\mathcal{H})$ and satisfies the following evolution equation for all $t\in\mathbb{R}$:

$$\partial_t\tau_t = \tau_t\circ\delta = \delta\circ\tau_t, \qquad \tau_0 = \mathbf{1}_{\mathcal{B}(\mathcal{H})}, \tag{4}$$

where $\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ is the identity operator on $\mathcal{B}(\mathcal{H})$ and the generator δ is defined on some dense subset $\mathcal{D}$ of $\mathcal{B}(\mathcal{H})$. Note that, if H is a bounded operator on $\mathcal{H}$, then $\mathcal{D} = \mathcal{B}(\mathcal{H})$ and

$$\delta(B) \coloneqq i\,[H,B], \qquad B\in\mathcal{B}(\mathcal{H}).$$

$\{\tau_t\}_{t\in\mathbb{R}}$ is a family of isomorphisms of $\mathcal{B}(\mathcal{H})$ and, for all $A,B\in\mathcal{D}$, one has

$$\delta(A^*) = \delta(A)^* \qquad \text{and} \qquad \delta(AB) = \delta(A)B + A\delta(B). \tag{5}$$

An operator satisfying (5) is called a *symmetric derivation* or $*$*-derivation*. A^* is the usual adjoint operator of A. Once again, more precise definitions of the mathematical tools that are involved to formulate quantum time evolution will be given later, since it is not necessary for the moment. Indeed, the aim of this section is to give the readers intuition about the different approaches that can be taken. At this point, the knowledge of semigroup properties turns out to be fundamental in order to understand the dynamics in quantum mechanics. Observe that, in the Schrödinger picture, one has a semigroup acting on a Hilbert space, while in the case of the Heisenberg picture, the semigroup acts on a Banach space. Within the next sections, we introduce the main results in relation to semigroup theory.

4. Semigroups and generators

First of all, let us give the definitions and basic results of semigroup theory as they are given in Engel and Nagel's notes [2]. These will provide the basis required to prove the main theorems studied in this article. Let X be a Banach space.

Definition 1. A **strongly continuous one-parameter semigroup**, also called C_0-semigroup, is a family $(T(t))_{t\geq 0}$ of bounded operators $T(t)\colon X\to X$ satisfying the functional equation

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t,s\geq 0, \\ T(0) = \mathbf{1}_X \end{cases}$$

and the strong continuity property, which is nothing else but the continuity of the orbit maps

$$\begin{aligned} \xi_x : \mathbb{R}^+ &\longrightarrow X \\ t &\longmapsto \xi_x(t) := T(t)x \end{aligned}$$

for each $x \in X$. If these properties hold not only in $\mathbb{R}^+$ but also in $\mathbb{R}$, we call $(T(t))_t$ a **strongly continuous group**, or C_0-group. ◂

Lemma 2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. Then, there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that, for all $t \geq 0$,*

$$\|T(t)\|_{\mathcal{B}(X)} \leq M\,\mathrm{e}^{\omega t}.$$

Proof. From the uniform boundedness, there exists $M \geq 1$ such that $\|T(s)\| \leq M$ for all $0 \leq s \leq 1$. Writing any $t \geq 0$ as $t = s + n$ with $n \in \mathbb{N}$ and $s \in [0,1]$,

$$\|T(t)\|_{\mathcal{B}(X)} \leq \|T(s)\|_{\mathcal{B}(X)}\,\|T(1)\|^n_{\mathcal{B}(X)} \leq M^{n+1} = M\mathrm{e}^{n\log M} \leq M\mathrm{e}^{wt}$$

holds for $\omega := \log M$ and $t \geq 0$. ∎

Definition 3. If lemma 2 holds for $\omega = 0$ and $M = 1$, the semigroup is called **contractive**. It means that $\|T(t)\|_{\mathcal{B}(X)} \leq 1$ for all $t \geq 0$. ◂

Example 4. Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{B}(\mathcal{H}) := X$. It can be easily shown that the series

$$\mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n$$

converges and that $T(t) := \mathrm{e}^{tA}$ defines a C_0-group. From the triangle inequality, we deduce that

$$\|T(t)\|_{\mathcal{B}(X)} \leq \mathrm{e}^{t\|A\|_X},$$

and therefore lemma 2 holds for $M = 1$ and $\omega = \|A\|_X \in \mathbb{R}$. ◂

Remark 5 (abstract Cauchy problem). In example 4, we have been able to define a C_0-group from a bounded operator. This group satisfies

$$\begin{cases} \dot{T}(t) = AT(t) & \text{for all } t \geq 0, \\ T(0) = \mathbf{1}_X. \end{cases} \tag{6}$$

The main topic studied in this article is the existence and properties of such an A for a general C_0-semigroup $(T(t))_{t\geq 0}$ by using the abstract Cauchy problem (6). ◂

Definition 6. A C_0-semigroup $(T(t))_{t\geq 0}$ is called **uniformly continuous** if the map

$$\begin{aligned} \mathbb{R}^+ &\longrightarrow \mathcal{B}(X) \\ t &\longmapsto \|T(t)\|_{\mathcal{B}(X)} \end{aligned}$$

is continuous. ◂

Proposition 7. *Let $(T(t))_{t\geq 0}$ be a uniformly continuous semigroup. Then, there exists a bounded operator A on X such that $T(t) = \mathrm{e}^{tA}$ for all $t \geq 0$.*

For more details, see theorem 2.12 in Engel and Nagel's notes [2]. Within this article, we focus our study on the general case of strong continuity. In this case, the existence of such a bounded operator A requires a deeper study of operator semigroups, see remark 5. We start by defining the generator of a C_0-semigroup.

Definition 8 (generator). The **generator** $A : D(A) \subseteq X \to X$ of a C_0-semigroup $(T(t))_{t\geq 0}$ is the operator

$$Ax := \dot{\xi}_x(0) = \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x)$$

with domain

$$D(A) = \left\{ x \in X \,:\, \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) \text{ exists} \right\}.$$ ◂

Note that the orbit map ξ_x is differentiable on $\mathbb{R}^+$ if and only if it is right-differentiable at $t = 0$. Indeed, the derivative of $\xi_x(t)$ at any t depends only on the derivative at $t = 0$ in the following way:

$$\dot{\xi}_x(t) = T(t)\dot{\xi}_x(0).$$

The following lemma summarizes some (basic) properties of the generator. They will be used throughout the proofs of the upcoming results.

Lemma 9. *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then,*

(i) *$A : D(A) \to X$ is a linear operator.*

(ii) *If $x \in D(A)$, then $T(t)x \in D(A)$ and, for all $t \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)x = T(t)Ax = AT(t)x.$$

(iii) *For all $t \geq 0$ and $x \in X$,*

$$\int_0^t T(s)x\,\mathrm{d}s \in D(A).$$

(iv) *For all $t \geq 0$,*

$$T(t)x - x = \begin{cases} A\displaystyle\int_0^t T(s)x\,\mathrm{d}s & \text{if } x \in X, \\ \displaystyle\int_0^t T(s)Ax\,\mathrm{d}s & \text{if } x \in D(A). \end{cases}$$

Proof. (i) From definition 8, it is clear that A is a linear operator and that $D(A)$ is a linear subspace of X.

(ii) Let $x \in D(A)$. Since $T(t)$ is bounded for all $t \geq 0$,

$$T(t)Ax = T(t)\lim_{h\downarrow 0}\frac{1}{h}(T(h)x - x) = \lim_{h\downarrow 0}\frac{1}{h}\left(T(h)T(t)x - T(t)x\right) = AT(t)x$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(t)x) \coloneqq T(t)\lim_{h\downarrow 0}\frac{1}{h}(T(h)x - x),$$

and hence

$$T(t)Ax = \frac{\mathrm{d}}{\mathrm{d}t}(T(t)x) = AT(t)x.$$

(iii) For all $t \geq 0$ and $x \in X$,

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}\left(T(h)\int_0^t T(s)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right) &= \lim_{h\downarrow 0}\left(\frac{1}{h}\int_0^t T(s+h)x\,\mathrm{d}s - \frac{1}{h}\int_0^t T(s)x\,\mathrm{d}s\right)\\ &= \lim_{h\downarrow 0}\left(\frac{1}{h}\int_t^{t+h} T(s)x\,\mathrm{d}s - \frac{1}{h}\int_0^h T(s)x\,\mathrm{d}s\right)\\ &= T(t)x - x\end{aligned}$$

(note that the last limit holds from the fundamental theorem of calculus).

(iv) Note that from what we have just seen, for any $x \in X$,

$$Tx - x = \lim_{h\downarrow 0}\frac{1}{h}\left(T(h)\int_0^t T(s)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right) = A\int_0^t T(s)x\,\mathrm{d}s.$$

Moreover, if $x \in D(A)$, note that

$$\left\|T(s)\frac{T(h)x - x}{h} - T(s)Ax\right\|_X \leq \|T(s)\|_{\mathcal{B}(X)}\left\|\frac{T(h)x - x}{h} - Ax\right\|_X.$$

Hence, on $s \in [0,t]$, for any $t \geq 0$ we have the following convergence, which is uniform with respect to s:

$$T(s)\frac{T(h)x - x}{h} \xrightarrow[h\downarrow 0]{u} T(s)Ax.$$

Therefore, for any $x \in D(A)$,

$$\lim_{h\downarrow 0}\frac{1}{h}(T(h) - \mathbf{1}_X)\int_0^t T(s)x\,\mathrm{d}s = \int_0^t T(s)\lim_{h\downarrow 0}\left(\frac{1}{h}T(h) - \mathbf{1}_X\right)x\,\mathrm{d}s = \int_0^t T(s)Ax\,\mathrm{d}s.$$

This concludes the proof. ∎

The following theorems give us further properties of the generator.

Theorem 10. *The generator A of a C^0-semigroup $(T(t))_{t\geq 0}$ is closed, densely defined and it determines the semigroup uniquely.*

Proof. Let us prove A is closed. Suppose there is a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \to x \in X$, for $n \to \infty$. Suppose that $Ax_n \to y \in X$, for $n \to \infty$. It suffices, by the characterisation of closed operators, to show that $x \in D(A)$ and $Ax = y$. Since $x_n \in D(A)$, for $t > 0$, one has (see lemma 9)

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n\,\mathrm{d}s. \tag{7}$$

We check now that $\int_0^t T(s)Ax_n\,\mathrm{d}s$ converges to $\int_0^t T(s)y\,\mathrm{d}s$ as $n \to \infty$. By strong continuity, the map $s \in [0,t] \mapsto T(s)y$ is integrable over $[0,t]$. By the triangular inequality of integrals,

$$\begin{aligned}\left\|\int_0^t T(s)Ax_n\,\mathrm{d}s - \int_0^t T(s)y\,\mathrm{d}s\right\|_X &= \left\|\int_0^t T(s)(Ax_n - y)\,\mathrm{d}s\right\|_X \leq \int_0^t \|T(s)(Ax_n - y)\|_X\,\mathrm{d}s\\ &\leq \int_0^t \|T(s)\|_{\mathcal{B}(X)}\|Ax_n - y\|_X\mathrm{d}s \leq \left(\int_0^t M\mathrm{e}^{\omega s}\,\mathrm{d}s\right)\|Ax_n - y\|_X.\end{aligned}$$

Here we have used lemmas 9 and 2. The sequence of inequalities follows from the triangular inequality for integrals, the boundedness of $T(s)$ (by strong continuity) and the exponential growth bound. Since Ax_n converges to y, we deduce that $\lim_{n\to\infty}\int_0^t T(s)Ax_n\,\mathrm{d}s = \int_0^t T(s)y\,\mathrm{d}s$. But the strong continuity of the semigroup yields that $\lim_{n\to\infty} T(t)x_n - x_n = T(t)x - x$. By uniqueness of limits, we conclude from equation (7) that, for all $t \geq 0$,

$$T(t)x - x = \int_0^t T(s)y\,\mathrm{d}s.$$

When t is taken to be positive, we have

$$\frac{1}{t}\left(T(t)x - x\right) = \frac{1}{t}\int_0^t T(s)y\,\mathrm{d}s.$$

When t approaches zero we are simply taking the derivative of $T(t)x$ at $t = 0$. That limit exists by the fundamental theorem of vector calculus (the integrand $T(s)y$ is continuous). This implies that

$$Ax = \frac{\mathrm{d}}{\mathrm{d}t}\left(T(t)x\right)|_{t=0} = T(0)y = y.$$

This means that $Ax = y$ is well-defined, thus $x \in D(A)$. Hence A is closed.

To see that A is densely defined, let us consider $x \in X$. By lemma 9,

$$\int_0^t T(s)x\,\mathrm{d}s \in D(A)$$

for all $t > 0$. Moreover, since T is strongly continuous, taking the limit $t \to 0$ is possible again by the fundamental theorem of vector calculus. This yields

$$\lim_{t\to 0} \frac{1}{t} \int_0^t T(s)x \,\mathrm{d}s = T(0)x = x.$$

Thus, $D(A)$ is a dense subspace of X.

Finally, to prove that A determines the semigroup uniquely, we suppose there is another strongly continuous semigroup $(S(t))_{t\geq 0}$ such that its generator is $A : D(A) \to X$. Let $x \in D(A)$ and $t \geq 0$ be fixed. To see that they are the same semigroup, we define the auxiliary function

$$s \in [0, t] \mapsto \psi_{t,x}(s) = T(t - s)S(s)x.$$

We differentiate this at $s \in [0, t]$. Consider the quotient

$$\begin{aligned}\frac{1}{h}(\psi_{t,x}(s + h) - \psi_{t,x}(s)) &= \frac{1}{h}(T(t - s - h)S(s + h)x - T(t - s)S(s)x)\\ &= \left[T(t - s - h)\frac{1}{h}(S(s + h)x - S(s)x)\right] + \left[\frac{1}{h}(T(t - s - h) - T(t - s))S(s)x\right].\end{aligned}$$

The second term converges to $-AT(t - s)S(s)x$ as $h \to 0$, since $S(s)x \in D(A)$ by lemma 9. The minus sign comes from the chain rule: $-A$ is the generator of $s \mapsto T(t - s)$.

The first term converges to $T(t - s)AS(s)x$ due to the fact that $||T(t - s - h)||_{\mathcal{B}(X)}$ is exponentially bounded by $M\mathrm{e}^{\omega(t-s-h)}$ and the strong continuity of the semigroups. See lemma A.19 in Engel and Nagel's notes [2], where this version of the product rule for semigroups composition is proved in detail. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi_{t,x}(s) = T(t - s)AS(s)x + -AT(t - s)S(s)x.$$

Since semigroups and generators commute (here $-A$ is the generator of $T(t - s)$), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi_{t,x}(s) = 0$$

for all $s \in [0, t]$. Therefore, $\psi_{t,x}$ is constant:

$$T(t)x = \psi_{t,x}(0) = \psi_{t,x}(t) = S(t)x.$$

Hence, $T(t)$ and $S(t)$ agree on $D(A)$, which is dense on X. Thus, they agree on all X. ■

Now, we are going to see some definitions and properties to prove the Hille-Yosida theorem.

Definition 11. Let $\lambda \in \mathbb{C}$ and let A be a closed linear operator. The **resolvent set** of A is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda \mathbf{1}_X - A) \text{ is bijective}\},$$

and $R(\lambda, A) := (\lambda \mathbf{1}_X - A)^{-1}$ is called the *resolvent map* of A. ◂

Remark 12. Let $(T(t))_{t\geq 0}$ be a semigroup. For $\mu \in \mathbb{C}$ and $\alpha > 0$, we define the rescaled semigroup $(S(t))_{t\geq 0}$ by

$$S(t) = \mathrm{e}^{\mu t}T(\alpha t), \qquad t \geq 0.$$

Note that, if $(A, D(A))$ is the generator of $(T(t))_{t\geq 0}$, then $(\alpha \mathbf{1}_X, D(\mu \mathbf{1}_X + \alpha A))$ is the generator of $(S(t))_{t\geq 0}$ and the resolvent map is $R(\lambda, \mu \mathbf{1}_X + \alpha A) = \frac{1}{\alpha}R(\frac{\lambda}{\alpha} - \frac{\mu}{\alpha}, A)$ for $\lambda \in \rho(\mu \mathbf{1}_X + \alpha A)$ ◂

Theorem 13. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space X, and let $(A, D(A))$ be its generator. If $\lambda \in \mathbb{C}$ is such that*

$$R(\lambda) := \int_0^{+\infty} \mathrm{e}^{-\lambda s}T(s)x \,\mathrm{d}s$$

is well-defined for all $x \in X$, then $\lambda \in \rho(A)$ and $R(\lambda) = R(\lambda, A)$.

Proof. Without loss of generality, we can assume that $\lambda = 0$. Therefore, one needs to show that $0 \in \rho(A)$. In particular, we will show that $R(0) = R(0, A) = (-A)^{-1}$. For all $x \in X$ and $h > 0$,

$$\frac{T(h) - \mathbf{1}_X}{h} R(0)x = \frac{T(h) - \mathbf{1}_X}{h} \int_0^{+\infty} T(s)x \, \mathrm{d}s = \frac{1}{h} \int_0^{+\infty} T(s+h)x \, \mathrm{d}s - \frac{1}{h} \int_0^{+\infty} T(s)x \, \mathrm{d}s$$
$$= \frac{1}{h} \int_h^{+\infty} T(s)x \, \mathrm{d}s - \frac{1}{h} \int_0^{+\infty} T(s)x \, \mathrm{d}s = -\frac{1}{h} \int_0^{h} T(s)x \, \mathrm{d}s.$$

Moreover,

$$\lim_{h \to 0} \left(\frac{1}{h} \int_0^h T(s)x \, \mathrm{d}s \right) = x.$$

Thus, $R(0)x \in D(A)$ and $A\,R(0) = -\mathbf{1}_X$. Furthermore, if $x \in D(A)$, we have that

$$\lim_{t \to +\infty} \int_0^t T(s)x \, \mathrm{d}s = R(0)x$$

and, by lemma 9,

$$\lim_{t \to +\infty} A \int_0^t T(s)x \, \mathrm{d}s = \lim_{t \to +\infty} \int_0^t T(s)Ax \, \mathrm{d}s = R(0)Ax.$$

By theorem 10, we deduce that

$$R(0)Ax = AR(0)x = -x, \qquad \text{for } x \in D(A).$$

This concludes the proof. ∎

Corollary 14. *Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup such that*

$$\|T(t)\|_{\mathcal{B}(X)} \leq M\mathrm{e}^{\omega t} \qquad \omega \in \mathbb{R},\ M \geq 1.$$

If $\lambda \in \mathbb{C}$ and $\omega < \operatorname{Re}\lambda$, then

$$\|R(\lambda, A)\|_{\mathcal{B}(X)} \leq \frac{M}{\operatorname{Re}\lambda - \omega}.$$

Proof. For $t, t' \geq 0$,

$$\left\| \int_{t'}^t \mathrm{e}^{-\lambda s} T(s) \, \mathrm{d}s \right\|_{\mathcal{B}(X)} \leq M \int_{t'}^t \mathrm{e}^{(\omega - \operatorname{Re}\lambda)s} \, \mathrm{d}s.$$

By the Cauchy criterium, for $\omega < \operatorname{Re}\lambda$,

$$\int_0^\infty \mathrm{e}^{(\omega - \operatorname{Re}\lambda)s} \, \mathrm{d}s$$

exists. Therefore, by theorem 13, $\lambda \in \rho(A)$ and

$$R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda s} T(s) \, \mathrm{d}s.$$

Obviously,

$$\|R(\lambda, A)\|_{\mathcal{B}(X)} \leq M \int_0^\infty e^{(\omega - \operatorname{Re}\lambda)s} \, \mathrm{d}s = \frac{M}{\operatorname{Re}\lambda - \omega}.$$

This concludes the proof. ∎

5. Hille-Yosida generation theorem

So far, we have given necessary properties for an operator to be a generator of a strongly continuous semigroup on X. In particular, for a strongly continuous contraction semigroup $(T(t))_{t\geq 0}$, we know by theorem 10 that its generator $(A, D(A))$ is closed and densely defined. Moreover, because of corollary 14 and definition 3, for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$, $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)\|_{\mathcal{B}(X)} \leq \frac{1}{\operatorname{Re}\lambda}.$$

Now we are going to show that these conditions are sufficient for contraction semigroup. First we recall a result that will be useful in the sequel.

Lemma 15 ([2, ch. I, §1, proposition 1.3]). *Let $(T(t))_{t\geq 0}$ be a semigroup. If there exist a dense subset $D \subset X$, $\delta > 0$ and $M \geq 1$ such that*

(i) $\|T(t)\|_{\mathcal{B}(X)} \leq M$ *for all* $t \in [0, \delta]$ *and*

(ii) $\lim\limits_{t\downarrow 0} T(t)x = x$ *for all* $x \in D$,

then $(T(t))_{t\geq 0}$ is a strongly continuous semigroup.

Theorem 16 (Hille-Yosida, 1948). *Let $(A, D(A))$ be a linear operator on a Banach space X. The following statements are equivalent:*

(i) $(A, D(A))$ *generates a strongly continuous contraction semigroup.*

(ii) $(A, D(A))$ *is closed, densely defined, and for all* $\lambda > 0$, $\lambda \in \rho(A)$ *and* $\|\lambda R(\lambda, A)\|_{\mathcal{B}(X)} \leq 1$.

(iii) $(A, D(A))$ *is closed, densely defined, and for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > 0$, $\lambda \in \rho(A)$ *and* $\|R(\lambda, A)\|_{\mathcal{B}(X)} \leq \frac{1}{\operatorname{Re}\lambda}$.

Proof. Note that (i) yields (iii) by an application of corollary 14. Moreover, (ii) is a straightforward conclusion of (iii). Thus, it remains to prove that (ii) implies (i).

To that purpose, we define the *Yosida approximants*

$$A_n := nAR(n, A) = n^2 R(n, A) - nI, \qquad n \in \mathbb{N}.$$

Note that, for each $n \in \mathbb{N}$,

$$\|A_n\|_{\mathcal{B}(X)} \leq n\|nR(n, A)\|_{\mathcal{B}(X)} + n \leq 2n.$$

Moreover, since

$$(n - A)(m - A) = (m - A)(n - A),$$

one has

$$R(n, A)R(m, A) = R(m, A)R(n, A) \qquad \text{and} \qquad [A_n, A_m] = 0.$$

These properties imply that the semigroups $(T_n(t))_{t\geq 0}$ given by $T_n(t) := \mathrm{e}^{tA_n}$, $t \geq 0$, are uniformly continuous, and mutually commute.

Because of the fact that $A_n x = nAR(n, A)x = n^2 R(n, A)x - nI$ converges to Ax for $x \in D(A)$ (see Engel and Nagel's notes [2, ch. II, §3, proposition 3.4]), we can anticipate the following properties:

(a) $T(t)x := \lim_{n\to\infty} T_n(t)x$ exists for each $x \in X$.

(b) $(T(t))_{t\geq 0}$ is a C_0-contraction semigroup on X.

(c) This semigroup has generator $(A, D(A))$.

In order to prove (a), we observe that $(T_n(t))_{t\geq 0}$ is a contraction semigroup for each $n \in \mathbb{N}$:

$$\|T_n(t)\|_{\mathcal{B}(X)} \leq \mathrm{e}^{-nt}\mathrm{e}^{\|n^2 R(n,A))\|_{\mathcal{B}(X)} t} \leq \mathrm{e}^{-nt}\mathrm{e}^{nt} = 1 \tag{8}$$

for $t \geq 0$, by assumption (ii).

Now, by using the mutual commutativity of the semigroups $(T_n(t))_{t\geq 0}$ for all $n \in \mathbb{N}$ and the vector-valued version of the fundamental theorem of calculus, for $x \in D(A)$, $t \geq 0$, $m, n \in \mathbb{N}$,

$$T_n(t)x - T_m(t)x = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(T_m(t-s)T_n(s)x)\,\mathrm{d}s = \int_0^t T_m(t-s)T_n(s)(A_n x - A_m x)\,\mathrm{d}s.$$

By using the triangle inequality and (8), we obtain that

$$\|T_n(t)x - T_m(t)x\|_{\mathcal{B}(X)} \leq t\|A_n x - A_m x\|_{\mathcal{B}(X)}. \tag{9}$$

For $x \in D(A)$, since $(A_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence, $(T_n(t)x)_{n\in\mathbb{N}}$ is a Cauchy sequence, i. e., $(T_n(t)x)_{n\in\mathbb{N}}$ converges to some $T(t)x$. Now let $x \in X$. Since $D(A)$ is dense in X, one has

$$\forall \varepsilon > 0, \exists y \in D(A) : \|x - y\|_X < \varepsilon.$$

Therefore,

$$\|T_n(t)x - T_m(t)x\|_{\mathcal{B}(X)} \leq \|T_n(t)(x-y)\|_{\mathcal{B}(X)} + \|T_n(t)y - T_m(t)y\|_{\mathcal{B}(X)} + \|T_m(t)(y-x)\|_{\mathcal{B}(X)}.$$

Observe that the right side of the inequality is arbitrarily small as n, m go to ∞ because $(T_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence, $(T_n(t)(x))_{n\in N}$ is a Cauchy sequence, for all $t \geq 0$ and $x \in X$. Therefore, it converges to some $T(t)x$, for all $x \in X$.

In (b), one needs to prove that the family of operators defined above is a strongly continuous contraction semigroup. First, observe that

$$x = \lim_{n\to\infty} T_n(0)x = T(0)x.$$

Hence,

$$T(0) = \mathbf{1}_X.$$

Moreover, for $t, s \geq 0$,

$$T(t+s)x = \lim_{n\to\infty} T_n(t+s)x = \lim_{n\to\infty} T_n(t)T_n(s)x. \tag{10}$$

Furthermore, for $t, s \geq 0$,

$$T_n(t)T_n(s)x = T_n(t)T(s)x + T_n(t)(T_n(s) - T(s))x.$$

Finally, observe that

$$\lim_{n\to\infty} \|T_n(t)(T_n(s) - T(s))x\|_X = 0$$

and

$$\lim_{n\to\infty} T_n(t)T(s)x = T(t)T(s)x.$$

By (10) we thus deduce the semigroup property.

To prove that the family $(T(t))_{t\geq 0}$ is strongly continuous, note that, by (9), for all $x \in D(A)$, $T(s)x$ is actually the uniform limit of $T_n(s)x$ on the interval $[0,t]$. The maps $s \in [0,t] \mapsto T_n(s)x$ are continuous. Hence, the uniform limit $s \in [0,t] \mapsto T(s)x$ is also continuous. From (8), $\|T(t)\| \leq 1$ for all $t \geq 0$. Thus, by using lemma 15 with $D = D(A)$ we conclude the family is strongly continuous.

To finish the proof, it remains to show that the generator of $(T(t))_{t\geq 0}$, namely $(B, D(B))$, is $(A, D(A))$. Fix any $x \in D(A)$. The orbit map

$$\xi_x : t \in [0, t_0] \mapsto \xi_x(t) = T(t)x$$

is the uniform limit of

$$\xi_x^n : t \in [0, t_0] \mapsto \xi_x^n(x) = T_n(t)x.$$

Also, their derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_x^n : t \in [0, t_0] \mapsto T_n(t)A_n x$$

converge uniformly to

$$\eta_x : t \mapsto T(t)Ax.$$

Indeed, for $t \in [0, t_0]$

$$\|T_n(t)A_n x - T(t)Ax\|_{\mathcal{B}(X)} \leq \|T_n(t)(A_n x - Ax)\|_{\mathcal{B}(X)} + \|(T_n(t) - T(t))Ax\|_{\mathcal{B}(X)}$$

and the right hand side vanishes as n goes to ∞ uniformly with respect to $t \in [0, t_0]$. Since

$$\xi_x^n(t) = x + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\xi_x^n(s)\,\mathrm{d}s = x + \int_0^t T_n(s)A_n x\,\mathrm{d}s,$$

by taking $n \to \infty$, we have

$$\xi_x(t) = \lim_{n\to\infty} \xi_x^n(t) = x + \int_0^t T(s)Ax\,\mathrm{d}s = x + \int_0^t \eta_x(s)\,\mathrm{d}s.$$

Thus, ξ_x is differentiable with $\frac{\mathrm{d}}{\mathrm{d}t}\xi_x(t)|_{t=0} = \eta(0) = Ax$, i. e., $D(A) \subseteq D(B)$ and $Ax = Bx$, for $x \in D(A)$.

Now let $\lambda > 0$. By hypothesis, $\lambda \in \rho(A)$. Since $(B, D(B))$ is the generator of the contraction semigroup $(T(t))_{t\geq 0}$, $\lambda \in \rho(B)$. Thus, both $(\lambda - A)$ and $(\lambda - B)$, possibly unbounded, admit a bounded inverse operator mapping the whole space onto the domain of the generator. Then, for every $y \in D(B)$, we get that

$$(\lambda - B)y = \mathbf{1}_X(\lambda - B)y = (\lambda - A)\underbrace{R(\lambda, A)(\lambda - B)y}_{\in D(A)}.$$

Moreover, since A and B agree on $D(A)$,

$$(\lambda - B)y = (\lambda - B)R(\lambda, A)(\lambda - B)y.$$

By applying $R(\lambda, B)$ on both sides we get

$$y = R(\lambda, A)(\lambda - B)y \in D(A).$$

This implies that $D(B) \subset D(A)$, thus $(A, D(A)) = (B, D(B))$. This concludes the proof. ∎

A generalization of the Hille-Yosida theorem was set in 1952 by Feller, Miyadera and Phillips. Its proof relies on the generation theorem proved by Hille and Yosida, which can be applied after a rescaling argument and a renormalization of the space.

Theorem 17 (general generation theorem, Feller-Miyadera-Phillips, 1952). *Let $(A, D(A))$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$, $M \geq 1$ be constants. Then, the following properties are equivalent.*

(i) *$(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying that, for all $t \geq 0$,*

$$\|T(t)\| \leq M\mathrm{e}^{\omega t}.$$

(ii) *$(A, D(A))$ is closed, densely defined, and for all $\lambda > \omega$, $\lambda \in \rho(A)$ and*

$$\forall n \in \mathbb{N} \quad \|\,[(\lambda - \omega)R(\lambda, A)]^n\,\| \leq M.$$

(iii) *$(A, D(A))$ is closed, densely defined, and for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$, $\lambda \in \rho(A)$ and*

$$\forall n \in \mathbb{N} \quad \|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}.$$

Proof. The fact that (i) implies (iii) is proved in corollary 1.11 of Engel and Nagel's notes [2]. We shall omit this proof for a matter of space. Then, (iii) immediately implies (ii). Thus, we will detail the fact that (ii) implies (i).

We have already seen that, if A generates $(T(t))_{t\geq 0}$, then $A-\omega$ generates $\left(e^{-\omega t}T(t)\right)_{t\geq 0}$. Furthermore, the resolvent satisfies

$$R(\lambda, A-\omega) = R(\lambda+\omega, A).$$

Hence, for any $\lambda > 0$, $\lambda \in \rho(A-\omega)$. One can assume without loss of generality that $\omega = 0$. Therefore, by hypothesis

$$\forall n \in \mathbb{N} \quad \|\lambda^n R(\lambda, A)^n\| \leq M. \tag{11}$$

Note that throughout the rest of the proof, as it has already been defined previously, $R(\lambda, A)$ is denoted by $R(\lambda)$.

Now, we define, for any $\mu > 0$, the following norm on X:

$$\|x\|_\mu := \sup_{n\geq 0} \|\mu^n R(\mu)^n x\|_X, \tag{12}$$

which is equivalent to $\|\cdot\|_X$. In fact, the estimate $\|x\|_\mu \leq M\|x\|_X$ follows from equation (11). By taking $n=0$ in (12) we get the equivalence of norms:

$$\forall x \in X, \quad \|x\|_X \leq \|x\|_\mu \leq M\|x\|_X. \tag{13}$$

Moreover,

$$\|\mu R(\mu)x\|_\mu = \sup_{n\geq 1} \|\mu^n R(\mu)^n x\|_X \leq \sup_{n\geq 0} \|\mu^n R(\mu)^n x\|_X = \|x\|_\mu. \tag{14}$$

Let $0 < \lambda \leq \mu$ and fix some $x \in X$. Observe that, for $R(\lambda)x \in D(A)$ and $R(\mu)(\mu - A)$ acting as the identity on $D(A)$,

$$R(\lambda)x = R(\mu)(\mu-A)R(\lambda)x = R(\mu)(\mu-\lambda)R(\lambda)x + R(\mu)(\lambda - A)R(\lambda)x = R(\mu)(x + (\mu-\lambda)R(\lambda)x).$$

By the triangle inequality on μ-norms,

$$\|R(\lambda)x\|_\mu \leq \|R(\mu)x\|_\mu + \|(\mu-\lambda)R(\mu)R(\lambda)x\|_\mu,$$

and, by using equation (14), we obtain that

$$\|\lambda R(\lambda)x\|_\mu \leq \|x\|_\mu.$$

Together with the norm equivalence in (13), this inequality implies

$$\|\lambda^n R(\lambda)^n x\|_X \leq \|\lambda^n R(\lambda)^n x\|_\mu \leq \|x\|_\mu.$$

By considering the supremum over n of the left hand side, we obtain the following property of the μ-norms:

$$\forall x \in X, \quad \|x\|_\lambda \leq \|x\|_\mu \text{ for } 0 < \lambda \leq \mu.$$

Because of equation (13),

$$|||x||| := \sup_{\mu > 0} \|x\|_\mu$$

is well-defined and actually defines another norm on X. Because of the equivalence relation of the μ-norms, the norm $|||\cdot|||$ satisfies

$$\forall x \in X, \quad \|x\|_X \leq |||x||| \leq M\|x\|_X. \tag{15}$$

One concludes that $|||\lambda R(\lambda)||| \leq 1$. Thus, $(A, D(A))$ satisfies the hypothesis of theorem 16 and generates a $|||\cdot|||$-contraction semigroup $(T(t))_{t\geq 0}$ in the Banach space $(X, |||\cdot|||)$. It follows from the equivalence of the $|||\cdot|||$-norm and the previous norm established in equation (15) that, for every $t \geq 0$,

$$\|T(t)\|_{\mathcal{B}(X)} \leq M.$$

This concludes the proof. ∎

6. Hilbert space generation theorems

In this section, let $\mathcal{H}$ be a Hilbert space. First of all, given a strongly continuous semigroup $(T(t))_{t\geq 0} \subset \mathcal{B}(\mathcal{H})$, one shall define its *adjoint semigroup* as $(T(t)^*)_{t\geq 0}$. Note that, since $T(t)^*T(s)^* = (T(s)T(t))^* = T(t+s)^*$, for $t, s \geq 0$, the adjoint semigroup is well-defined.

Proposition 18. *Let $(A, D(A))$ be the generator of the C_0-semigroup $(T(t))_{t\geq 0}$ acting on a Hilbert space $\mathcal{H}$. Then, its adjoint semigroup is strongly continuous with generator $(A^*, D(A^*))$.*

Proof. For $(T(t))_{t\geq 0}$ being strongly continuous there exist $M \geq 0$, $\omega \in \mathbb{R}$ such that the growth bound $\|T(t)\|_{\mathcal{B}(\mathcal{H})} \leq M\mathrm{e}^{\omega t}$ holds. Since $\|T(t)^*\|_{\mathcal{B}(\mathcal{H})} = \|T(t)\|_{\mathcal{B}(\mathcal{H})}$, the adjoint semigroup satisfies the same inequality. Let $x \in D(A)$ be a normalised vector and $z \in D(A^*)$. Then, by the properties stated in lemma 9,

$$\langle x, T(t)^*z - z\rangle = \langle T(t)x - x, z\rangle = \int_0^t \langle AT(\tau)x, z\rangle\,\mathrm{d}\tau = \int_0^t \langle x, T(\tau)^*A^*z\rangle\,\mathrm{d}\tau. \tag{16}$$

Thus, by the Cauchy-Schwarz and triangle inequalities[1]

$$|\langle x, T(t)^*z - z\rangle| \leq \int_0^t \|T(\tau)^*\|_{\mathcal{B}(\mathcal{H})}\|A^*z\|_{\mathcal{H}}\,\mathrm{d}\tau \leq Mt\mathrm{e}^{\omega t}\|A^*z\|_{\mathcal{H}}. \tag{17}$$

Since $D(A)$ is dense in $\mathcal{H}$, it follows from above that $\|T(t)^*z - z\|_{\mathcal{H}} \leq Mt\mathrm{e}^{\omega t}\|A^*z\|_{\mathcal{H}}$. Therefore, $\lim_{t\downarrow 0} T(t)^*z = z$ for every $z \in D(A)$. Moreover, for $t_0 > 0$ and $t \in [0, t_0]$, $\|T(t)^*\| \leq M\mathrm{e}^{\omega t_0}$ ($M = 1$, $\omega = 0$ in the contraction case). By lemma 15, the adjoint semigroup is strongly continuous.

Suppose that $(B, D(B))$ is the generator of the adjoint semigroup. Let $x \in D(A)$ and $y \in D(B)$. Observe that

$$\langle Ax, y\rangle = \lim_{t\downarrow 0}\frac{1}{t}\langle T(t)x - x, y\rangle = \lim_{t\downarrow 0}\frac{1}{t}\langle x, T(t)^*y - y\rangle = \langle x, By\rangle.$$

Therefore, $D(B) \subset D(A^*)$, by definition of $D(A^*)$. Moreover, since $D(A)$ is dense, (16) implies that, for $z \in D(A^*)$,

$$T(t)^*z - z = \int_0^t T(\tau)^*A^*z\,\mathrm{d}\tau.$$

Hence, for $z \in D(A^*)$,

$$Bz = \lim_{h\downarrow 0}\frac{1}{h}(T(h)^*z - z) = \lim_{h\downarrow 0}\frac{1}{h}\int_0^t T(\tau)^*A^*z\,\mathrm{d}\tau = A^*z$$

holds and $D(A^*) \subset D(B)$ and $A^* = B$. ∎

A (possibly unbounded) operator A acting on a Hilbert space is said to be *skew-adjoint* whenever $A^* = -A$. The next theorem, due to Stone, deals with generators satisfying this property. The proof we provide relies on the Hille-Yosida contraction generation theorem. As we will see in the next section, the generators of evolution groups in quantum mechanics are skew-adjoint.

Theorem 19 (Stone, 1932). *Let $(A, D(A))$ be an operator acting on a Hilbert space. Then, $(A, D(A))$ generates a unitary C_0-group $(U(t))_{t\in\mathbb{R}}$ if and only if A is skew-adjoint.*

Proof. If $(U(t))_{t\in\mathbb{R}}$ is a unitary C_0-group, then A^* is the generator for $U(t)^* = U(t)^{-1} = U(-t)$, as was shown in the previous theorem. Given any $x \in D(A)$,

$$\lim_{h\downarrow 0}\frac{1}{h}(U(h)^*x - x) = \lim_{h\downarrow 0}\frac{1}{h}(U(-h)x - x) = -Ax,$$

[1] Note that the exponential term in the right-hand-side of equation (17) should be omitted in the contraction case.

so $x \in D(A^*)$. Since the left hand side equals A^*x, $D(A) \subset D(A^*)$ and $-A$ agrees with A^* along its domain. One could repeat the same argument for arbitrary $x \in D(A^*)$, obtaining that $D(A^*) \subset D(A)$. Therefore, $D(A) = D(A^*)$ and A is skew-adjoint.

On the other hand, note that, if $(A, D(A))$ is skew-adjoint, then $(iA, D(A))$ is self-adjoint. Thus, both $(A, D(A))$ and $(A^*, D(A^*)) = (-A, D(A))$ have a purely imaginary spectrum (lying on $\mathrm{i}\mathbb{R}$). It follows that

$$\|\lambda R(\lambda, A)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\mu \in \mathrm{i}\mathbb{R}} \frac{\lambda}{|\lambda + \mu|} \leq 1.$$

The same calculation is satisfied by $-A$ trivially. Therefore, because of the Hille-Yosida generation theorem in the contraction case, $(A, D(A))$, respectively $(-A, D(A))$, is the generator of the semigroup $(U(t)^+)_{t\geq 0}$, respectively $(U(t)^-)_{t\geq 0}$.

Now, we proceed to show that $(U(t))_{t\in\mathbb{R}}$ defined by

$$U(t) = \begin{cases} U(t)^+ & t \geq 0, \\ U(-t)^- & t < 0 \end{cases}$$

is a unitary C_0-group. Indeed, the strong continuity follows after its definition. All that is left to prove is that $(U(t))_{t\in\mathbb{R}}$, with composition as a product, is a unitary group.

We proceed to show that $U(t)$, $U(-t)$ are inverse elements with $U(0) = \mathbf{1}_{\mathcal{H}}$ as identity element. To this end, fix any $x \in D(A)$. For $t = 0$, $U(0)^+U(0)^-x = \mathbf{1}_{\mathcal{H}}x = x$. Then, for $t > 0$, because of the derivative properties of C_0-semigroups and the skew-adjointness of $(A, D(A))$,

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t)^+U(t)^-x = [U(t)^+AU(t)^- + U(t)^+A^*U(t)^-]\, x = 0.$$

Thus, for $t > 0$, $U(t)^+U(t)^-x = x$, and $D(A)$ is dense in $\mathcal{H}$, so $U(t)U(-t) = \mathbf{1}_{\mathcal{H}}$.

In order to prove that $(U(t))_{t\in\mathbb{R}}$ is closed under composition, fix any $t, s > 0$. We have $U(t)U(s) = U(t+s)$ and $U(-t)U(-s) = U(-t-s)$, since $(U(t)^+)_{t\geq 0}$ and $(U(t)^-)_{t\geq 0}$ are semigroups. If $t < s$, then $U(t)U(-s) = U(t)U(-t)U(t-s) = U(t-s)$, and the $t > s$ case follows similarly. Since composition is associative, $(U(t))_{t\in\mathbb{R}}$ is a group.

In order for A to be skew-adjoint, $(U(t)^*)_{t\geq 0}$ must be generated by $A^* = -A$, as follows from proposition 18. However, $-A$ generates $(U(-t))_{t\geq 0}$ too, as follows from the construction above. Theorem 10 ensures the uniqueness of the semigroup generated by $-A$ so, for every $t \geq 0$, $U(t)^* = U(-t)$, i. e., the C_0-group $(U(t))_{t\in\mathbb{R}}$ is unitary. ∎

7. Back to quantum mechanics

In the setting of quantum mechanics, as explained in the introduction, the space of all possible states of the system is modelled by a Hilbert space $\mathcal{H}$. The energy of the system, described by the Hamiltonian H (self-adjoint operator), determines the evolution of the system via the Schrödinger equation, previously defined in (1).

As one can see in (2), the solution of the above system has the form $\psi(t, x) = U(t)\psi_0(x)$, where $U(t) \in \mathcal{B}(\mathcal{H})$ for $t \in \mathbb{R}$. Again, by (2), $U(t)$ satisfies

$$\begin{cases} \partial_t U(t) = -\mathrm{i}HU(t), \\ U(0) = \mathbf{1}_{\mathcal{H}}. \end{cases} \tag{18}$$

Since H is a self-adjoint operator, H is densely defined, and so is $-\mathrm{i}H$. Thus, Stone's theorem ensures that $-\mathrm{i}H$ will generate a strongly continuous unitary group $(U(t))_{t\in\mathbb{R}}$ satisfying the functional equation in (18).

In terms of the wave function interpretation, we need that the evolution semigroup preserves the norm of the original state ψ_0. Otherwise, there would be an undesirable loss (or gain) of probability if, for example,

$$\|\psi_t\|_{\mathcal{H}} < \|\psi_0\|_{\mathcal{H}} = 1.$$

Stone's theorem ensures this will not occur. Since the evolution operator is unitary, we are guaranteed that, in $\mathcal{H}$,

$$\|\psi_t\|_{\mathcal{H}} = \|U(t)\psi_0\|_{\mathcal{H}} = \|\psi_0\|_{\mathcal{H}} = 1.$$

In terms of the Heisenberg picture introduced in section 3, the time evolution of an observable B in a system determined by the Hamiltonian H is given by the action of a strongly continuous group $\{\tau_t\}_{t\in\mathbb{R}}$ on $\mathcal{B}(\mathcal{H})$. This time evolution is defined by $\tau_t(B) \coloneqq \mathrm{e}^{\mathrm{i}tH} B \mathrm{e}^{-\mathrm{i}tH}$, for $t \in \mathbb{R}$ and $B \in \mathcal{B}(\mathcal{H})$. These operators satisfy equation (4), where $\delta : D(\delta) \subset \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a symmetric derivation defined on a dense subset $D(\delta)$ of $\mathcal{B}(\mathcal{H})$. It can be proved (in case of interest, see [1]) that these symmetric derivations satisfy the hypothesis of the Hille-Yosida generation theorem. Therefore, they are the generators of the C_0-group $\{\tau_t\}_{t\in\mathcal{R}}$ and determine the evolution of the physical system uniquely.

In fact, the δ operators described above belong to the class of *dissipative operators*, which are contained in the core of the Lumer-Phillips generation theorem [2, theorem 3.15]. This theorem allows to adapt the Hille-Yosida generation theorem to dissipative operators, in a similar way as the Stone theorem, which adjusts our main theorem to self-adjoint operators.

Bru and de Siqueira Pedra [1] show an example of application of symmetric derivations in quantum mechanics generating a C_0-group. These structures are associated to the behaviour of fermions in lattices.

References

[1] Bru, Jean-Bernard and Siqueira Pedra, Walter de. *Lieb-Robinson Bounds for Multi-Commutators and Applications to Response Theory*. SpringerBriefs in Mathematical Physics. Cham: Springer, 2016. https://doi.org/10.1007/978-3-319-45784-0.

[2] Engel, Klaus-Jochen and Nagel, Rainer. *A Short Course on Operator Semigroups*. Universitext. New York: Springer, 2006. https://doi.org/10.1007/0-387-36619-9.

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Approximation and the full Müntz-Szász theorem

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Abstract: The genesis of approximation theory is the classic result of Weierstrass about the approximation of continuous functions by polynomials. Later, the Müntz-Szász theorem characterizes the sequences of positive real numbers which define dense span subspaces of monomials on the space of continuous functions. In this paper we present this important and beautiful result and some of its extensions, known as the full Müntz-Szász theorem in Lebesgue spaces $L^p([0, 1])$ for $1 \leq p < \infty$ and $C([0, 1])$.

Resumen: Uno de los orígenes de la teoría de la aproximación es el teorema clásico de Weierstrass sobre aproximación de funciones continuas mediante polinomios. Posteriormente, el teorema de Müntz-Szász caracteriza las sucesiones de números positivos que definen subespacios densos de monomios en el espacio de las funciones continuas. En este artículo presentamos este importante y hermoso resultado y algunas de sus extensiones, conocidas como el teorema completo de Müntz-Szász en los espacios de Lebesgue $L^p([0, 1])$ para $1 \leq p < \infty$ y $C([0, 1])$.

Keywords: Weierstrass approximation theorem, Müntz-Szász theorem, Lebesgue spaces, dense subspaces.

MSC2010: 41A10, 43A15, 32E30.

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Reference: Bolón, Diego; Corbalán, Clara María; González-Doña, F. Javier; Nieves, Daniel; Oitavén, Carlos C.; Quero, Alicia, and Miana, Pedro J. “Approximation and the full Müntz-Szász theorem”. In: *TEMat monográficos*, 1 (2020): *Artículos de las VIII y IX Escuela-Taller de Análisis Funcional*, pp. 33-45. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol1-p33.

1. Introduction

The genesis of approximation theory is the classic result of Weierstrass about the approximation of continuous functions by polynomials. As Bertrand Russel said: «All exact science is dominated by the idea of approach». When one calculates, one *approximates*.

The first work on this subject is attributed to Leonhard Euler, who was trying to solve the problem of drawing a map of the Russian Empire whose latitudes were accurate. In 1777, he published the work where he gave the best approximation in relation to the altitudes and latitudes considering all the points of a meridian between the given latitudes, that is, over the entire interval. Given the vast area occupied by the Russian Empire, all projections had a large number of errors on the edge of the map, which is why Euler's approach was a great contribution.

A problem shown by Laplace shared the same character. A paragraph from a famous work, first published in 1799, dealt with the question of determining the best ellipsoidal approximation of the Earth's surface. Here, it was relevant to obtain the least possible error at each point on the Earth's surface. Euler solved his problem in the total domain; on the contrary, Laplace assumed a finite amount of points considerably greater than the number of parameters of the problem. Fourier generalized the results of Laplace in his work *Analyse des équations determinées*. It dealt with the problem of solving, through an approximation method, linear systems of equations with a greater number of equations than of parameters. His method was to minimize the error of each equation.

In 1853, Chebyshev was the first to unify all these considerations in a work under the title of 'Theory of functions that became as little as possible of zero'. A well-known problem of that time was the so-called Watts parallelogram, which studied the determination of the parameters of a steam engine mechanism, so that the conversion of rectilinear movement into a circular movement was as accurate as possible. This led to the general problem of the approximation of a real analytical function by a polynomial of any degree. The first objective that Chebyshev achieved was the determination of the degree n polynomial with the first coefficient given whose zero deviation is the smallest possible over the $[-1, 1]$ interval. Today, this polynomial is known as the first species Chebyshev polynomial.

In 1857, Chebyshev presented a work entitled "Sur les questions de minima qui se rattachent à la représentation approximative des fonctions", in which he pays attention to the following problem: determining the value of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ which solve

$$\min_{\lambda_1,\ldots,\lambda_n} \max_{x\in[a,b]} |f(x, \lambda_1, \ldots, \lambda_n)|,$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a given function. He also proved that, under certain hypotheses about partial derivatives, it was possible to demonstrate a necessary condition for the solution of the previous problem.

The objective of this contribution was to find the polynomial that deviates uniformly as little as possible from zero for any given number of coefficients. This goal mainly determined his early contributions at the St. Petersburg Mathematical School in the area of approximation theory.

The Mathematical School of St. Petersburg was characterized by its tendency to solve specific problems with the intention of obtaining an explicit formula or, failing this, an algorithm that suited its purposes adequately. Consequently, all the contributions of the members of this school were oriented towards classical mathematics, due to the exclusive use of algebraic methods. It is relevant to highlight the articles by Zolotarev and the Markov brothers, who emphasized special problems in the field of uniform approximation theory. One of the most striking results of the Markov brothers is the following inequality, which states, that if $P : [-1, 1] \to \mathbb{R}$ is a polynomial of degree at most n, then

$$\max_{x\in[-1,1]} |P^{(k)}(x)| \leq \frac{n^2\left(n^2-1\right)\left(n^2-2\right)\cdot\left(n^2-(k-1)^2\right)}{1\cdot 2\cdot 5\cdots(2k-1)} \max_{x\in[-1,1]} |P(x)|, \qquad k \leq n.$$

The equality is achieved in the first species Chebyshev polynomials.

Sergei Natanovich Bernstein made use of this result to prove one of his theorems. However, due to the nature of the task, their investigations rested, again, on algebraic methods. The last contribution to the early approximation theory of the St. Petersburg Mathematical School, which came from the hand of Andrey Markov, took place in 1906.

Throughout this time, it is remarkable that the results achieved by western mathematicians were not cited. Not even the renowned Weierstrass theorem from 1885 was cited in any of these publications.

Outside of Russia, approximation theory was born in a different way. It had been preferred to address a theory of the more theoretical approach, due to the great interest in some basic questions generated at the end of the 18th century for the problem of the oscillating rope. The interest in defining the most important concept of modern analysis, the concept of *continuous function*, played a fundamental role in the consequences derived from Weierstrass's approach theorem. He defined the objective, and therefore, it was time to explicitly find sequences of algebraic or trigonometric polynomials that converged to a given continuous function. Finally, they tried to determine the speed of convergence with which these sequences could converge, that is, how quickly the approximation error decreased. Such were the objectives of a large series of alternative tests that quickly emerged after Weierstrass's original work.

Theorem 1 (Weierstrass, 1885). *Every continuous function defined in a compact of the real line is uniform limit of polynomials.*

Some proofs of the Weierstrass theorem have been provided by great mathematicians: Lipót Fejèr used harmonic analysis techniques; the proof of Edmund Landau is based on basic tools of real analysis in one variable, and Sergei Bernstein applied a probabilistic method.

2. The Müntz-Szász theorem on $C([0, 1])$

In 1912, at the Cambridge *International Congress of Mathematicians*, Bernstein posed a problem from Weierstrass's result. He asked about the conditions under which a set of positive numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ verifies that the set of finite linear combinations of $\{t^{\lambda_n}\}_{n\in\mathbb{N}}$ is dense in $C([0, 1])$. Bernstein himself gave some partial results and he guessed rightly that the harmonic sum $\sum_{n\in\mathbb{N}} 1/\lambda_n$ would be crucial. Only two years later, in 1914, Müntz confirmed the conjecture and demonstrated what went down in history as the Müntz-Százs theorem.

Theorem 2 (Müntz-Szász theorem). *Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence of positive real numbers. Then, the subspace of finite linear combinations of* $1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots$, i. e.*, the space* $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots\}$*, is dense in* $C[0, 1]$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty.$$

In 1916, Szász published an article where he completed the proof, further improving and simplifying it. Müntz's demonstration uses real variable techniques and is based on estimating the distance between any continuous function to certain finite subspaces of polynomials, which can be made as small as desired. Szász's proof makes use of complex variable techniques combined with some arguments of functional analysis. The proof we present here can be found in Rudin's book [7] and follows Szász's ideas.

The first step towards proving theorem 2 is to present a more practical and complete version which implies the Müntz-Szász theorem.

Theorem 3. *Let* $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ *and*

$$X = \overline{\operatorname{span}}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots\}.$$

(i) *If* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty$*, then* $X = C[0, 1]$.

(ii) *If* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < +\infty$ *and* $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$, $\lambda \neq 0$*, then* $t^{\lambda} \notin X$.

To prove this theorem, we will use the following lemma.

Lemma 4. *Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ be a sequence such that $\sum_n 1/\lambda_n = \infty$, and let μ be a complex Borel measure on $I = (0,1]$ such that $T \in C(I)^* \cong M(I)$, where $C(I)^*$ is the continuous dual of $C(I)$, is a linear and bounded functional associated to μ with*

$$T(t^{\lambda_n}) = \int_0^1 t^{\lambda_n}\,\mathrm{d}\mu(t) = 0, \qquad n = 1,2,3,\dots \tag{1}$$

Then,

$$T(t^k) = \int_0^1 t^k\,\mathrm{d}\mu(t) = 0, \qquad k = 1,2,3,\dots$$

Proof. Suppose that the condition (1) holds. We may assume that the measure μ is concentrated on $I = (0,1]$. We consider the function

$$f(z) = \int_0^1 t^z\,\mathrm{d}\mu(t) = \int_0^1 \mathrm{e}^{z\log t}\,\mathrm{d}\mu(t),$$

which is well-defined and bounded in the right half plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$:

$$|f(z)| \le \int_0^1 |\mathrm{e}^{z\log t}|\,\mathrm{d}|\mu|(t) = \int_0^1 \mathrm{e}^{\Re(z)\log t}\,\mathrm{d}|\mu|(t) = \int_0^1 t^{\Re(z)}\,\mathrm{d}|\mu|(t) \le \|\mu\| < +\infty.$$

Now, we check that the function f is continuous. Let $\varepsilon > 0$. Since the map $t \mapsto t^z$ is uniformly continuous in the compact $[0,1]$, there exists $\delta(\varepsilon) > 0$ such that

$$|f(z) - f(z_0)| \le \int_0^1 |t^z - t^{z_0}|\,\mathrm{d}|\mu|(t) \le \varepsilon \int_0^1 \mathrm{d}|\mu|(t) = \varepsilon\|\mu\|,$$

for $|z - z_0| < \delta$.

Let γ be a regular closed path on $\mathbb{C}^+$. By Fubini's theorem, we obtain that

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_\gamma \int_0^1 t^z\,\mathrm{d}\mu(t)\,\mathrm{d}z = \int_0^1 \oint_\gamma t^z\,\mathrm{d}z\,\mathrm{d}\mu(t) = 0,$$

and we conclude that f is a bounded analytic function on $\mathbb{C}^+$.

We define the function

$$g(z) := f\left(\frac{1+z}{1-z}\right), \qquad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Note that $g \in H^\infty$, *i. e.*, it is a bounded analytic function on the disc. By the hypothesis (1) we conclude that $g(\alpha_n) = 0$, where

$$\alpha_n := \frac{\lambda_n - 1}{\lambda_n + 1}.$$

We claim that

$$\sum_{n=1}^\infty \frac{1}{\lambda_n} = +\infty \implies \sum_{n=1}^\infty 1 - |\alpha_n| = +\infty.$$

Note that

$$\sum_{n=1}^\infty 1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right| = \sum_{n=1}^\infty \frac{\lambda_n + 1 - |\lambda_n - 1|}{\lambda_n + 1}.$$

We split in two different cases.

- If $0 < \lambda_n < 1$ for all $n \in \mathbb{N}$, then $\lambda_n + 1 - |\lambda_n - 1| = 2\lambda_n$ and

  $$\sum_{n=1}^\infty 1 - |\alpha_n| = \sum_{n=1}^\infty \frac{2\lambda_n}{\lambda_n + 1} = +\infty,$$

  due to the fact that $\dfrac{2\lambda_n}{\lambda_n + 1} \nrightarrow 0$ when $n \to \infty$.

- If there exists $m \in \mathbb{N}$ such that $\lambda_n \geq 1$ for all $n \geq m$, then $\lambda_n + 1 - |\lambda_n - 1| = 2$ and
$$\sum_{n=1}^{\infty} 1 - |\alpha_n| \geq \sum_{n=m}^{\infty} \frac{2}{\lambda_n + 1} = +\infty.$$

By the Riesz representation theorem, we conclude that $g(z) = 0$, for $z \in \mathbb{D}$. In particular,
$$T(t^k) = \int_I t^k \, \mathrm{d}\mu(t) = f(k) = g\left(\frac{k-1}{k+1}\right) = 0, \qquad k = 1, 2, \ldots,$$
and we conclude the proof. ∎

Now we present the proof of theorem 3.

Proof of theorem 3. To show part (i), it is enough to show that X contains all the functions t^k, for $k = 1, 2, 3, \ldots$, and apply the Weierstrass approximation theorem. Suppose that there exists $k_0 \in \mathbb{N}$ such that $t^{k_0} \notin X$. By the Hahn-Banach theorem, there exists a bounded and linear functional $T\colon C[0,1] \to \mathbb{R}$ such that
$$T(t^{k_0}) \neq 0 \qquad \text{and} \qquad T|_{\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}} \equiv 0.$$
By lemma 4, we conclude that $T(t^{k_0}) = 0$, which contradicts our hypothesis.

In order to prove part (ii), assume that
$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty.$$
Our objective is to give a bounded linear functional $T = \langle \cdot, \mu \rangle \in C[0,1]^*$ such that $T(t^{\lambda_n}) = 0$ for all $n \in \mathbb{N} \cup \{0\}$ $(\lambda_0 = 0)$, but $T(t^{\lambda}) \neq 0$ for $\lambda \notin \{\lambda_n\}_{n \geq 1}$. By the Hahn-Banach theorem, we shall conclude that $t^{\lambda} \notin \overline{\operatorname{span}}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots\}$ for $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$.

To get this, we would need to obtain a complex Borel measure μ on $[0,1]$ such that the analytic function f, given by
$$z \mapsto \int_0^1 t^z \, \mathrm{d}\mu(t),$$
defines a bounded function on the half plane $\mathbb{C}_{-1} := \{z \in \mathbb{C} : \Re(z) > -1\}$, and whose zeros are precisely the sequence $\{\lambda_n\}_{n=1}^{\infty}$. In this case, we shall take $T := \langle \cdot, \mu \rangle$.

We consider the function f given by
$$f(z) := \frac{z}{(2+z)^3} \prod_{n=1}^{\infty} \frac{\lambda_n - z}{2 + \lambda_n + z}, \qquad z \in \mathbb{C} \setminus \{-2 - \lambda_n\}.$$

First, we check that the function f is a meromorphic function whose poles are the set $\{-2-\lambda_n : n \in \mathbb{N}\}$ and whose zero set is $\{\lambda_n\}_{n=1}^{\infty}$. To do this, it is enough to show the uniform convergence of the infinite product on compacts contained in $\mathbb{C} \setminus \{-2 - \lambda_n\}$. This convergence is equivalent to the uniform convergence of the following series:

$$\sum_{n=1}^{\infty} \left| 1 - \frac{\lambda_n - z}{2 + \lambda_n + z} \right| = \sum_{n=1}^{\infty} \left| \frac{2z + 2}{2 + \lambda_n + z} \right|. \tag{2}$$

Fixed K a compact set, there exists $\alpha > 0$ such that $K \subset \mathbb{C}_{-\alpha} = \{z \in \mathbb{C} : \Re(z) > -\alpha\}$. As the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$ is convergent, there exist $N \in \mathbb{N}$ and $C > 0$, which only depends on the compact set K, such that
$$\left| \frac{2z + 2}{2 + \lambda_n + z} \right| \leq \frac{C}{2 + \lambda_n - \alpha},$$
for $n > N$. By the Weierstrass M-test, we obtain the uniform convergence of series (2).

Now we claim that the function f is bounded on $\mathbb{C}_{-1}$. Every factor in the infinite product and the fraction $\frac{z}{2+z}$ are Möbius transform from $\mathbb{C}_{-1}$ into the disc. Finally, as $\frac{1}{(2+z)^2} \leq 1$ for $z \in \mathbb{C}_{-1}$, we conclude that f is bounded on $\mathbb{C}_{-1}$.

Note that $f \in L^1(\{z \in \mathbb{C} : \Re(z) = -1\})$, due to the fact that

$$\int_{\mathbb{R}} |f(-1+\mathrm{i}t)|\,\mathrm{d}t \leq C \int_{\mathbb{R}} \frac{1}{1+t^2}\,\mathrm{d}t = C\pi.$$

By Cauchy's theorem, given $z_0 \in \mathbb{C}_{-1}$, we have that

$$f(z_0) = \int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z,$$

where γ is the path formed by the semicircle with center -1 and radius $R > 1 + |z_0|$, from $-1-\mathrm{i}R$ to $-1+\mathrm{i}R$ and the interval $[-1-\mathrm{i}R, -1+\mathrm{i}R]$. Then, we obtain that

$$f(z_0) = \frac{1}{2\pi}\int_{-R}^{R} \frac{f(-1+\mathrm{i}s)}{1-\mathrm{i}s+z_0}\,\mathrm{d}s + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{f(-1+R\mathrm{e}^{\mathrm{i}\theta})}{-1+R\mathrm{e}^{\mathrm{i}\theta}-z_0} R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta.$$

Since $|f(z)| \leq \left|\frac{z}{(2+z)^3}\right|$, the second summand tends to 0 when $R \to \infty$. We get the equality

$$f(z) = \int_{\mathbb{R}} \frac{f(-1+\mathrm{i}s)}{1-\mathrm{i}s+z}\,\mathrm{d}s, \qquad z \in \mathbb{C}_{-1}.$$

We apply the identity

$$\frac{1}{1-\mathrm{i}s+z} = \int_0^1 t^z \mathrm{e}^{-\mathrm{i}s\log t}\,\mathrm{d}t$$

and Fubini's theorem to get that

$$f(z) = \int_0^1 t^z \left(\frac{1}{2\pi}\int_{\mathbb{R}} f(-1+\mathrm{i}s)\mathrm{e}^{-\mathrm{i}s\log t}\,\mathrm{d}s\right)\mathrm{d}t, \qquad z \in \mathbb{C}_{-1}. \tag{3}$$

We define $g(s) = f(-1+\mathrm{i}s)$. Note that the inner integral in (3) is equal to $\hat{g}(\log t)$, where $\hat{g}$ is the Fourier transform of g. Since $\hat{g}$ is continuous and bounded, we define

$$\mathrm{d}\mu = \frac{1}{2\pi}\hat{g}(\log t)\,\mathrm{d}t$$

to conclude that μ is a complex Borel measure on $[0,1]$. Finally, we obtain the following representation of f:

$$f(z) = \int_0^1 t^z\,\mathrm{d}\mu(t),$$

and the proof is completed. ∎

3. The full Müntz-Szász theorem on $L^2([0,1])$

Now that we have demonstrated the classical Müntz-Szász theorem, it is worth asking if we can extend the result for other functional spaces, such as Lebesgue spaces $L^p([0,1])$, or if it is really necessary that the sequence of exponents $\{\lambda_n\}_{n\in\mathbb{N}}$ is monotone increasing. These questions were partially answered by the mathematicians Borwein and Erdélyi in an article published in 1996 [1] where they proved the Müntz-Szász theorem for spaces $L^1([0,1])$, $L^2([0,1])$ and $C([0,1])$ without the monotonicity assumption. We will present here the proof of these facts following the original article by Borwein and Erdélyi.

We start with the proof for $L^2([0,1])$, since, being a Hilbert space, we have a richer structure to rely on.

Theorem 5. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of different real numbers greater than $-1/2$. Then, the set* $\operatorname{span}\{t^{\lambda_n} : n \geq 0\}$ *is dense in $L^2[0,1]$ if and only if*

$$\sum_{n=0}^{\infty} \frac{2\lambda_n + 1}{(2\lambda_n + 1)^2 + 1} = +\infty.$$

Proof. Our aim is to show that

$$\sum_{n=0}^{\infty} \frac{2\lambda_n + 1}{(2\lambda_n + 1)^2 + 1} = +\infty \iff t^m \in \overline{\operatorname{span}}\{t^{\lambda_n} : n \in \mathbb{N} \cup \{0\}\}, \quad \forall m \in \mathbb{N} \cup \{0\}.$$

We will obtain the result as a consequence of the Weierstrass approximation theorem. Let $m \in \mathbb{N} \cup \{0\}$ be such that $m \notin \{\lambda_n\}_{n=0}^{\infty}$.

Observe $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots, t^{\lambda_n}\}$ is a closed subspace of $L_2([0,1])$, so we can consider the orthogonal projection of t^m onto $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots, t^{\lambda_n}\}$. The expression of this projection is given by $\sum_{i=0}^{n} \langle t^m, t^{\lambda_i} \rangle t^{\lambda_i}$. Observe this function satisfies

$$\min_{a_i \in \mathbb{C}} \left\| t^m - \sum_{i=0}^{n} a_i t^{\lambda_i} \right\|_{L_2([0,1])} = \left\| t^m - \sum_{i=0}^{n} \langle t^m, t^{\lambda_i} \rangle t^{\lambda_i} \right\|_{L_2([0,1])} = \frac{1}{\sqrt{2m+1}} \prod_{i=0}^{n} \left| \frac{m - \lambda_i}{m + \lambda_i + 1} \right|,$$

where the last equality arises from a direct computation of the norm. Then ,

$$t^m \in \overline{\operatorname{span}}\{t^{\lambda_n} : n \in \mathbb{N} \cup \{0\}\} \iff \limsup_{n \to \infty} \prod_{i=0}^{n} \left| \frac{m - \lambda_i}{m + \lambda_i + 1} \right| = 0. \tag{4}$$

We divide into two cases, $\lambda_i > m$ and $\lambda_i < m$, to get

$$\prod_{i=0}^{n} \left| \frac{m - \lambda_i}{m + \lambda_i + 1} \right| = \prod_{i=0, \lambda_i < m}^{n} \left(1 - \frac{2\lambda_i + 1}{m + \lambda_i + 1} \right) \prod_{i=0, m < \lambda_i}^{n} \left(1 - \frac{2m + 1}{m + \lambda_i + 1} \right).$$

Then, the condition (4) is equivalent to one of these two following conditions:

$$\limsup_{n \to \infty} \prod_{i=0, \lambda_i < m}^{n} \left(1 - \frac{2\lambda_i + 1}{m + \lambda_i + 1} \right) = 0 \quad \text{or} \quad \limsup_{n \to \infty} \prod_{i=0, \lambda_i > m}^{n} \left(1 - \frac{2m + 1}{m + \lambda_i + 1} \right) = 0.$$

By Theorem 15.5 from Rudin's book [7], each condition is equivalent to the divergence of one of these two series:

$$\sum_{i=0, \lambda_i < m}^{\infty} \frac{2\lambda_i + 1}{m + \lambda_i + 1} \quad \text{and} \quad \sum_{i=0, \lambda_i > m}^{\infty} \frac{2m + 1}{m + \lambda_i + 1}.$$

By comparison, the divergence of these two series is equivalent to the divergence of

$$\sum_{i=0, \lambda_i < m}^{\infty} (2\lambda_i + 1) \quad \text{and} \quad \sum_{i=0, \lambda_i > m}^{\infty} \left(\frac{1}{2\lambda_i + 1} \right).$$

Finally, the divergence of these two series is equivalent to

$$\sum_{i=0}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = +\infty,$$

and the proof is finished. ∎

4. The full Müntz-Szász theorem on $C([0,1])$

Now we will show the full Müntz-Szász theorem on $C[0,1]$. To do so, we need some preliminary results about Newman's inequality and Chebyshev polynomials.

Theorem 6. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be different positive real numbers. Then, for $p \in \operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots, t^{\lambda_n}\}$, the following inequality holds:*

$$\|t\,p'(t)\|_{C[0,1]} \leq 11\left(\sum_{i=1}^{n} \lambda_i\right)\|p(t)\|_{C[0,1]}.$$

It is interesting to point out that the optimal constant on the Newman inequality is conjectured to be 4, see the paper of Borwein and Erdélyi [2]. A modification of this inequality allows to control the derivative of any polynomial $p \in \operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$. You may find this result in work of Borwein and Erdélyi [1, Theorem 3.4].

Theorem 7. *Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\lambda_n \geq 1$ for $n = 1, 2, \dots$, and $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$. Take $\varepsilon \in (0, 1)$. Then, there exists c which depends uniquely on $\{\lambda_n\}_{n\in\mathbb{N}}$ and ε such that*

$$\|p'\|_{C[0,1-\varepsilon]} \leq c\|p\|_{C[0,1]}$$

for any $p \in \operatorname{span}\{1, t^{\lambda_1}, \dots\}$.

The theory of approximation using Chebyshev polynomials and the following theorem may be found, for example, in Cheney's book [3].

Theorem 8 (Chebyshev polynomials). *Let A be a compact subset contained in $[0, +\infty)$ with at least $n+1$ points, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n different positive real numbers. Then, there exists a unique Chebyshev polynomial T_n such that*

$$T_n(t) = c\left(t^{\lambda_n} - \sum_{i=1}^{n-1} a_i t^{\lambda_i}\right),$$

where the coefficients a_i minimize the norm

$$\left\| t^{\lambda_n} - \sum_{i=1}^{n-1} a_i t^{\lambda_i} \right\|_{C(A)},$$

the constant c is such that $\|T_n\|_{C(A)} = 1$, and $T_n(\max A) > 0$.

The Chebyshev polynomial $T_n \in \operatorname{span}\{t^{\lambda_1}, t^{\lambda_2}, \dots, t^{\lambda_n}\}$ is uniquely characterized by the existence of an alternation set

$$\{t_0 < t_1 < \dots < t_n\} \subset A$$

such that

$$T_n(t_j) = (-1)^{n-j} = (-1)^{n-j}\|T_n\|_{C(A)}, \qquad 0 \leq j \leq n.$$

Now, we present the full Müntz-Szász theorem in $C[0, 1]$.

Theorem 9. *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of different positive real numbers. Then,*

$$\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\}$$

is dense in $C[0, 1]$ if and only if

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = +\infty.$$

Proof. We consider the following four cases depending on the sequence $\{\lambda_n\}$.

1. $\inf_{n\in\mathbb{N}} \lambda_n > 0$.
2. $\lim_{n\to+\infty} \lambda_n = 0$.
3. $\{\lambda_n\} = \{\alpha_n\} \cup \{\beta_n\}$, with $\alpha_n \to 0$ and $\beta_n \to +\infty$.
4. $\{\lambda_n\}$ has a cluster point on $(0, +\infty)$.

Note that every positive sequence $\{\lambda_i\}_{i=1}^{\infty}$ or one of its rearrangements fits in one of these cases.

We consider the first case. We may suppose with no loss of generality that $\lambda_i \geq 1$, for all $i \in \mathbb{N}$. Given $m \in \mathbb{N}$, we have that

$$\begin{aligned}\left|t^m - \sum_{i=1}^{n} a_i t^{\lambda_i}\right| &= \left|\int_0^1 \left(m x^{m-1} - \sum_{i=1}^{n} a_i \lambda_i x^{(\lambda_i - 1)}\right) \mathrm{d}x\right| \leq \int_0^1 \left|m x^{m-1} - \sum_{i=1}^{n} a_i \lambda_i x^{(\lambda_i - 1)}\right| \mathrm{d}x \\ &\leq \left(\int_0^1 \left(m x^{m-1} - \sum_{i=1}^{n} a_i \lambda_i x^{(\lambda_i - 1)}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}\end{aligned}$$

and we get the inequality

$$\min_{a_i \in \mathbb{C}} \left\| t^m - \sum_{i=1}^{n} a_i t^{\lambda_i} \right\|_{C[0,1]} \leq m \left(\min_{b_i \in \mathbb{C}} \left\| t^{m-1} - \sum_{i=1}^{n} b_i t^{\lambda_i - 1} \right\|_{L^2[0,1]} \right). \tag{5}$$

Suppose that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty$. Since $\lambda_i \geq 1$, we also conclude that

$$\sum_{i=1}^{\infty} \frac{2(\lambda_i - 1) + 1}{2((\lambda_i - 1) + 1)^2 + 1} = +\infty.$$

By theorem 5, the set $\operatorname{span}\{1, t^{\lambda_1 - 1}, t^{\lambda_2 - 1}, \ldots\}$ is dense in $L^2[0,1]$, and the inequality (5) shows that, in fact, $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is dense on $C[0,1]$.

Conversely suppose that $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is dense in $C[0,1]$. As the space $C[0,1]$ is dense in $L^2[0,1]$, the set $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is also dense in $L^2[0,1]$, and

$$\sum_{i=1}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = +\infty,$$

by theorem 5. By comparing the sums, we conclude that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = +\infty$.

Now we suppose that the sequence $\{\lambda_i\}_{i \geq 1}$ converges to 0. Then,

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = +\infty \iff \sum_{i=1}^{\infty} \lambda_i = +\infty.$$

If $\sum_{i=1}^{\infty} \lambda_i = +\infty$, then we conclude that

$$\sum_{i=1}^{\infty} \left(1 - \left|\frac{\lambda_i - 1}{\lambda_i + 1}\right|\right) = \infty.$$

We follow the same ideas we used in the proof of lemma 4 to conclude that $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is dense in $C([0,1])$.

On the other hand, if $\eta = \sum_{i=1}^{\infty} \lambda_i < +\infty$, we apply Newman's inequality to get that

$$\|t p'(t)\|_{C[0,1]} \leq 11\eta \|p(t)\|_{C[0,1]}, \tag{6}$$

for any $p \in \operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$. We claim that this last inequality implies that the set $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ is not dense in $C([0,1])$. Indeed, suppose that this set is dense in $C([0,1])$. Given the function $f(t) = \sqrt{1 - t}$ and $m \in \mathbb{N}$, there exists $p \in \operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ such that $\|p - f\|_{C([0,1])} < 1/m^2$. Then,

$$\left|p\left(1 - \frac{1}{m^2}\right) - p(1)\right| \geq \left|f\left(1 - \frac{1}{m^2}\right) - \frac{1}{m^2} - \left(f(1) + \frac{1}{m^2}\right)\right| = \frac{1}{m} - \frac{2}{m^2}.$$

By the mean value theorem, there exists $\xi \in (1 - 1/m^2, 1)$ such that

$$|\xi p'(\xi)| = \xi \frac{\left|p\left(1 - \dfrac{1}{m^2}\right) - p(1)\right|}{\dfrac{1}{m^2}} \geq \frac{m-2}{2},$$

which gives a contradiction with the inequality (6).

Now we consider the third case. We split the sequence $\{\lambda_i\}$ into two sequences $\{\lambda_i : i \in \mathbb{N}\} = \{\alpha_i : i \in \mathbb{N}\} \cup \{\beta_i : i \in \mathbb{N}\}$ such that $\alpha_n \to 0$ and $\beta_n \to +\infty$. Note that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty$ is equivalent to

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \frac{1}{\beta_i} = \infty. \tag{7}$$

If the condition (7) holds, then $\sum \alpha_i = \infty$ or $\sum \frac{1}{\beta_i} = \infty$. Then, we may apply cases 1 and 2 to conclude that $\mathrm{span}\{1, t^{\lambda_1}, t^{\lambda_2}, ...\}$ is dense in $C([0,1])$.

Conversely, if the condition (7) does not hold, then

$$\sum_{i=1}^{\infty} \alpha_i < \infty \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty.$$

Given a sequence $\{w_1, \dots, w_n\}$ of n different positive real numbers, we denote by $T_n\{1, t^{w_1}, \dots, t^{w_n}\}$ the associated Chebychev polynomial in the compact $[0,1]$ given by theorem 8. With this notation, we define

$$\begin{aligned} T_{n,\alpha} &:= T_n\{1, t^{\alpha_1}, \dots, t^{\alpha_n}\}, \\ T_{n,\beta} &:= T_n\{1, t^{\beta_1}, \dots, t^{\beta_n}\}, \\ T_{2n,\alpha,\beta} &:= T_{2n}\{1, t^{\alpha_1}, \dots, t^{\alpha_n}, t^{\beta_1}, \dots, t^{\beta_n}\}. \end{aligned}$$

Now our objective is to count and localize the zeros of these polynomials. It follows from Newman's inequality (theorem 6) and the mean value theorem that for every $\varepsilon > 0$ there exists a $k_1(\varepsilon) \in \mathbb{N}$ depending only on $\{\alpha_n\}_{n=1}^{\infty}$ and ε (and not on n) so that $T_{n,\alpha}$ has at most $k_1(\varepsilon)$ zeros in $[\varepsilon, 1)$ and at least $n - k_1(\varepsilon)$ zeros in $(0, \varepsilon)$.

Analogously, it follows from theorem 7 and the mean value theorem that for every $\varepsilon > 0$ there exists a $k_2(\varepsilon) \in \mathbb{N}$ depending only on $\{\beta_n\}_{n=1}^{\infty}$ and ε (and not on n) so that $T_{n,\beta}$ has at most $k_2(\varepsilon)$ zeros in $(0, 1-\varepsilon]$ and at least $n - k_1(\varepsilon)$ zeros in $(1-\varepsilon, 1)$.

Now, counting the zeros of $T_{n,\alpha} - T_{2n,\alpha,\beta}$ and $T_{n,\beta} - T_{2n,\alpha,\beta}$, we can deduce that for every $\varepsilon > 0$ there exists a $k(\varepsilon) \in \mathbb{N}$ depending only on $\{\lambda_n\}_{n=1}^{\infty}$ and $\varepsilon > 0$ (and not on n) so that $T_{2n,\alpha,\beta}$ has at most $k(\varepsilon)$ zeros in $[\varepsilon, 1-\varepsilon]$.

Take a fixed $\varepsilon = \frac{1}{4}$ and $k := k\left(\frac{1}{4}\right)$. We pick $k+4$ points such that

$$\frac{1}{4} < \eta_0 < \eta_1 < \dots < \eta_{k+3} < \frac{3}{4},$$

and a function $f \in C([0,1])$ such that$f(t) = 0$ for every $t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, and

$$f(\eta_i) := 2(-1)^i, \qquad i = 0, 1, \dots, k+3.$$

Suppose that there exists a polynomial $p \in \mathrm{span}\{1, t^{\lambda_1}, t^{\lambda_2}, ...\}$ such that

$$\|f - p\|_{C([0,1])} < 1.$$

Then, $p - T_{2n,\alpha,\beta}$ has at least $2n+1$ zeros in $(0,1)$. However, for sufficiently large n,

$$p - T_{2n,\alpha,\beta} \in \mathrm{span}\{1, t^{\lambda_1}, \dots, t^{\lambda_{2n}}\},$$

which can have at most $2n$ zeros in $[0, +\infty)$. This contradiction shows that $\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\}$ is not dense in $C([0, 1])$.

Finally we consider the case when the sequence $\{\lambda_n\}$ has a cluster point in $(0, \infty)$. In this case there, exists a subsequence $\{\lambda_{n_k}\}$ such that $\inf_{k\in\mathbb{N}} \lambda_{n_k} > 0$ and $\sum_{k=1}^{\infty} \frac{\lambda_{n_k}}{\lambda_{n_k}^2+1} = \infty$, where we may apply the case 1. ∎

5. The full Müntz-Szász theorem on $L^1([0, 1])$

Now we present the full Müntz-Szász theorem on $L^1([0, 1])$.

Theorem 10. *Suppose that $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of real numbers greater than -1. Then,*

$$\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$$

is dense in $L^1([0, 1])$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i + 1)^2 + 1} = \infty.$$

Proof. Suppose that the set $\operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $L^1([0, 1])$. We fix a non-negative integer m. For $\varepsilon > 0$, we choose $p \in \operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$ such that

$$\|t^m - p\|_{L^1([0,1])} < \varepsilon. \tag{8}$$

We define the function

$$q(t) := \int_0^t p(s)\, \mathrm{d}s \in \operatorname{span}\{t^{\lambda_0+1}, t^{\lambda_1+1}, \dots\}.$$

By the inequality (8), we have that

$$\left\| \frac{t^{m+1}}{m+1} - q \right\|_{C([0,1])} < \varepsilon.$$

We apply the Weierstrass approximation theorem to conclude that the set

$$\operatorname{span}\{1, t^{\lambda_0+1}, t^{\lambda_1+1}, \dots\}$$

is dense in $C([0, 1])$. We apply the full Müntz-Szász theorem in $C([0, 1])$ (theorem 9) to conclude that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i + 1)^2 + 1} = +\infty.$$

Conversely, now we suppose that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i + 1)^2 + 1} = +\infty.$$

By the Hahn-Banach theorem and the Riesz representation theorem, the set $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$ is not dense in $L^1([0, 1])$ if and only if there exists a function $0 \neq h \in L^\infty([0, 1])$ such that

$$\int_0^1 t^{\lambda_i} h(t)\, \mathrm{d}t = 0; \qquad i = 0, 1, \dots .$$

Given this function h, we define

$$f(z) := \int_0^1 t^z h(t)\, \mathrm{d}t,$$

and then

$$g(z) := f\left(\frac{1+z}{1-z} - 1\right).$$

Note that the function g is bounded and analytic in the open unit disc and

$$g\left(\frac{\lambda_n}{\lambda_n+2}\right) = f(\lambda_n) = 0.$$

Note that $\sum_{i=0}^{\infty} \frac{\lambda_i+1}{(\lambda_i+1)^2+1} = +\infty$ implies that

$$\sum_{n=1}^{\infty}\left(1-\left|\frac{\lambda_n}{\lambda_n+2}\right|\right) = +\infty.$$

We consider the Blaschke product [7, Theorem 15.23] to conclude that $g = 0$ on the open unit disc, and $f(z) = 0$ in the half plane $\Re(z) > -1$. In particular, we have that

$$f(n) = \int_0^1 t^n h(t)\,\mathrm{d}t = 0; \qquad n = 0, 1, \ldots$$

We apply the Weierstrass approximation theorem to get

$$\int_0^1 u(t)h(t)\,\mathrm{d}t = 0,$$

for every $u \in C([0,1])$. Finally, we conclude that $h = 0$ and that $\operatorname{span}\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}$ is dense in $L^1([0,1])$. ■

6. The full Müntz-Szász theorem on $L^p([0,1])$ for $1 \le p < \infty$

Once we have shown the full Müntz-Szász theorem on $L^1([0,1])$ and $L^2([0,1])$, it is natural to ask about the full Müntz-Szász theorem on $L^p([0,1])$ for $1 \le p < \infty$. This question was posed by Borwein and Erdélyi [1] and solved by Operstein [6, Theorem 1].

Theorem 11. *Let $1 < p < \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of different real numbers greater than $-1/p$. Then, the set $\operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N}\cup\{0\}\}$ is dense in $L^p[0,1]$ if and only if*

$$\sum_{n=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i+1/p)^2+1} = +\infty. \tag{9}$$

To show this theorem, we need the following lemma.

Lemma 12. *Let $\{\mu_i\}_{i=0}^{\infty}$ be a sequence of positive real numbers such that the set $\operatorname{span}\{t^{\mu_i-1/r} : i \in \mathbb{N}\cup\{0\}\}$ is dense in $L^r([0,1])$. Then, the space $\operatorname{span}\{t^{\mu_i-1/s} : i \in \mathbb{N}\cup\{0\}\}$ is dense in $L^s([0,1])$ for $s > r$, and $\operatorname{span}\{1, t^{\mu_0}, t^{\mu_1}, \ldots\}$ is dense in $C([0,1])$.*

Proof. We consider the spaces $X = L^r([0,1])$, $Y = L^s([0,1])$ and $A = \operatorname{span}\{t^{\mu_i-1/r} : i \in \mathbb{N}\cup\{0\}\}$. Our aim is to define a linear bounded operator J between the spaces X and Y such that $J(X)$ is dense in Y. Note that this fact implies that $J(A)$ is dense in Y. We consider the operator $J\colon L^r([0,1]) \to L^s([0,1])$ defined by

$$(J\varphi)(t) = t^{-(1/r'+1/s)}\int_0^t \varphi(u)\,\mathrm{d}u, \qquad t\in[0,1],\ \varphi\in L^r([0,1]),$$

where $\frac{1}{r}+\frac{1}{r'} = 1$. By the generalized Hardy inequality [5, Theorem 329], this operator J is bounded.

For every $n \in \mathbb{N}$, we define the function $\psi_n(t) \coloneqq (n + 1/r' + 1/s)t^{n+1/s-1/r}$ for $t \in [0,1]$. Note that $\psi_n \in L^r([0,1])$ and $(J\psi_n)(t) = t^n$ for $n \in \mathbb{N}$. By the Weierstrass approximation theorem, we conclude that $J(X)$ is dense in Y and the set $J(A) = \operatorname{span}\{t^{\mu_i-1/s} : i \in \mathbb{N}\cup\{0\}\}$ is dense in $L^s([0,1])$.

To show the second part, we consider the linear bounded operator $J\colon L^r([0,1]) \to C([0,1])$ defined by

$$(J\varphi)(t) = t^{-1/r'}\int_0^t \varphi(u)\,\mathrm{d}u, \qquad t\in(0,1], \qquad (J\varphi)(0) = 0,$$

for $\varphi \in L^r([0,1])$ and $\frac{1}{r}+\frac{1}{r'} = 1$. Using similar ideas to the ones in the first part, we conclude that the set $\operatorname{span}\{1, t^{\mu_0}, t^{\mu_1}, \ldots\}$ is dense in $C([0,1])$. ■

With the help of this lemma and the full Müntz-Szász theorem on $L^1([0,1])$ and $C([0,1])$, we prove theorem 11.

Proof of theorem 11. We take a sequence $\{\lambda_i\}_{i=0}^{\infty}$ satisfying condition (9). Now we consider the sequence $\{v_i\}_{i=0}^{\infty}$, where $v_i = \lambda_i - 1/p'$ for $i \geq 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$. By hypothesis, we have that

$$\sum_{i=0}^{\infty} \frac{v_i + 1}{(v_i + 1)^2 + 1} = \sum_{n=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = +\infty.$$

We apply theorem 10 to conclude that the space

$$\operatorname{span}\{t^{v_i} : i \in \mathbb{N} \cup \{0\}\} = \operatorname{span}\{t^{\lambda_i - 1/q} : i \in \mathbb{N} \cup \{0\}\}$$

is dense in $L^1([0,1])$. We take $\mu_i = \lambda_i + 1/p$ for $i \in \mathbb{N} \cup \{0\}$ and we apply lemma 12 to get that

$$\operatorname{span}\{t^{\mu_i - 1/p} : i \in \mathbb{N} \cup \{0\}\} = \operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N} \cup \{0\}\}$$

is dense in $L^p([0,1])$ for $p > 1$.

Conversely, we suppose that the space $\operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $L^p([0,1])$. We write $\mu_i = \lambda_i + 1/p$, for $i \in \mathbb{N} \cup \{0\}$, to obtain that

$$\operatorname{span}\{t^{\mu_i - 1/p} : i \in \mathbb{N} \cup \{0\}\}$$

is dense in $L^p([0,1])$. By lemma 12, the space $\operatorname{span}\{1, t^{\mu_i} : i \in \mathbb{N} \cup \{0\}\}$ is dense in $C([0,1])$. Now we apply theorem 9 to obtain that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \sum_{i=0}^{\infty} \frac{\mu_i}{\mu_i^2 + 1} = +\infty. \qquad \blacksquare$$

References

[1] Borwein, Peter and Erdélyi, Tamás. "The full Müntz theorem in $C[0,1]$ and $L_1[0,1]$". In: *Journal of the London Mathematical Society. Second Series* 54.1 (1996), pp. 102–110. issn: 0024-6107. https://doi.org/10.1112/jlms/54.1.102.

[2] Borwein, Peter and Erdélyi, Tamás. "Newman's inequality for Müntz polynomials on positive intervals". In: *Journal of Approximation Theory* 85.2 (1996), pp. 132–139. issn: 0021-9045. https://doi.org/10.1006/jath.1996.0034.

[3] Cheney, Elliott Ward. *Introduction to approximation theory.* Reprint of the second (1982) edition. Providence: AMS Chelsea Publishing, 1998. isbn: 978-0-8218-1374-4.

[4] Eceizabarrena, Daniel; Mas Mas, Alejandro; Mengual Bretón, Francisco, and Soria Carro, María. "El teorema de Müntz-Szász sobre la aproximación de funciones continuas". In: *TEMat* 1 (2017), pp. 31–44. issn: 2530-9633. url: https://temat.es/articulo/2017-p31.

[5] Hardy, Godfrey Harold; Littlewood, John Edensor, and Pólya, George. *Inequalities.* 2nd ed. Cambridge: Cambridge University Press, 1952. isbn: 978-0-521-35880-4.

[6] Operstein, Vladimir. "Full Müntz theorem in $L_p[0,1]$". In: *Journal of Approximation Theory* 85.2 (1996), pp. 233–235. issn: 0021-9045. https://doi.org/10.1006/jath.1996.0039.

[7] Rudin, Walter. *Real and complex analysis.* 3rd ed. McGraw-Hill series in higher mathematics. New York: McGraw-Hill, 1987. isbn: 978-0-07-054234-1.

VIII *Escuela-Taller de Análisis Funcional* (2018)

Four different approaches to the fractional Laplacian

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Abstract: In this paper we introduce four different definitions for the fractional Laplacian operator. First, we begin by giving the definition through the Fourier transform, motivated by the problem of finding an inverse operator for the Riesz potential. Next, we introduce a second definition given by a pointwise integral formula with a probabilistic motivation. The last two definitions come from functional analysis and partial differential equations: one is given in terms of the heat semigroup and the other one is given by the extension problem, which allows us to study properties of a nonlocal operator by means of local methods. We prove the equivalence between these four definitions and also give some of the properties of the fractional Laplacian.

Resumen: En este artículo introducimos cuatro definiciones distintas del operador laplaciano fraccionario. En primer lugar comenzamos dando la definición con la transformada de Fourier, que viene motivada por la búsqueda de un operador inverso del potencial de Riesz. A continuación se introduce una segunda definición como operador dado por una fórmula integral puntual a partir de una motivación de naturaleza probabilística. Las dos últimas definiciones provienen del análisis funcional y las ecuaciones diferenciales: una de ellas se da en términos del semigrupo del calor y la otra a partir del conocido como problema de extensión, que permite estudiar las propiedades de un operador no local mediante métodos locales. Se prueba la equivalencia de las cuatro definiciones y se muestran algunas de las propiedades del laplaciano fraccionario.

Keywords: fractional Laplacian, Fourier transform, pointwise formula, semigroup, extension problem.

MSC2010: 26A33, 60G22.

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1. Introduction

Fractional operators, and in particular the fractional Laplacian, are well known from the point of view of functional analysis. However, they also appear in other areas of mathematics. Some of the settings in which this operator arises are the theory of Banach spaces [10], potential theory [8], Lévy proceses [1], the theory of partial differential equations [4], and scattering theory in conformal geometry [6]. Bibliography in this topic is extensive and the above are just some examples.

The goal of this paper is to introduce four of the different definitions of the fractional Laplacian which appear in the literature. There are other definitions that we will not consider here, see for instance the paper of Kwaśnicki [7] for a nice exposition on ten different definitions of the fractional Laplacian.

The function space we are going to work with is the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, which is the space of functions $f \in C^\infty(\mathbb{R}^n)$ satisfying

$$\|f\|_p := \sup_{|\alpha|\le p} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{p/2}\, |\partial^\alpha f(x)| < \infty, \qquad p \in \mathbb{N}\cup\{0\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ denotes a multi-index in $(\mathbb{N}\cup\{0\})^n$ and $\partial^\alpha f$ denotes the derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$. With the metric given by

$$d(f,g) = \sum_{p=0}^{\infty} 2^{-p} \frac{\|f-g\|_p}{1+\|f-g\|_p}, \qquad f,g \in \mathscr{S}(\mathbb{R}^n),$$

the Schwartz space is a Fréchet space.

In section 2 we introduce the fractional Laplacian operator through the Fourier transform, as the inverse operator of the Riesz potential. In section 3 we introduce the fractional Laplacian from a probabilistic motivation, obtaining a pointwise integral formula. Section 4 is devoted to the study of the fractional Laplacian through the heat semigroup. Also in this section, we will prove the equivalence of the previous definitions. Finally, in section 5 we will study the fractional Laplacian as a «Dirichlet-to-Neumann» operator for a harmonic extension problem.

2. First definition: Fourier transform

The first time the fractional Laplacian appeared in the literature is in the paper by M. Riesz [10]. The usual Laplacian, given by $-\Delta f = -\sum_{k=1}^n \frac{\partial^2 f}{\partial x_k^2}$ for $f \in C^2(\mathbb{R}^n)$, can be understood as the inverse operator of the Newton potential I_2, which is defined as

$$I_2 f(x) := c_{n,2} |\cdot|^{-n+2} * f(x) = c_{n,2} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy, \qquad n \ge 3,\ f \in \mathscr{S}(\mathbb{R}^n),$$

where $c_{n,2} = \frac{1}{4\pi^{n/2}} \Gamma\big(\frac{n-2}{2}\big)$.

Riesz generalized the concept of Newton potential by defining the fractional integral operator (or Riesz potential) of order $0 < \alpha < n$, $n \in \mathbb{N}$, as

$$I_\alpha f(x) := c_{n,\alpha} |\cdot|^{-n+\alpha} * f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \qquad f \in \mathscr{S}(\mathbb{R}^n),$$

with $c_{n,\alpha} = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{n/2}} \frac{1}{\Gamma(\frac{\alpha}{2})}$.

In the same way we have that the relation

$$I_2 \circ (-\Delta) = \mathrm{Id} = (-\Delta) \circ I_2 \tag{1}$$

holds in $\mathscr{S}(\mathbb{R}^n)$, Riesz posed the natural question of the existence of an operator $(-\Delta)^{\alpha/2}$ that would satisfy, in $\mathscr{S}(\mathbb{R}^n)$, the analogous relation

$$I_\alpha \circ (-\Delta)^{\alpha/2} = \mathrm{Id} = (-\Delta)^{\alpha/2} \circ I_\alpha. \tag{2}$$

If we understand $(-\Delta)^{\alpha/2}$ as a fractional version of the differential operator $(-\Delta)$, expression (2) somehow represents a fractional version of the fundamental theorem of calculus (thus the name «fractional integral» for I_α).

In order to find an explicit expression for an operator satisfying (2) we will use the Fourier transform. For a function $f \in \mathscr{S}(\mathbb{R}^n)$, its *Fourier transform* $\mathcal{F}[f]$ is defined to be the function

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}\xi \cdot x} \, \mathrm{d}x, \qquad \xi \in \mathbb{R}^n.$$

This defines an invertible operator in $\mathscr{S}(\mathbb{R}^n)$ whose inverse is given by

$$\mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi) \mathrm{e}^{\mathrm{i}x \cdot \xi} \, \mathrm{d}\xi, \qquad x \in \mathbb{R}^n.$$

By using the well-known properties of $\mathcal{F}$ with respect to derivatives, it follows easily that $\mathcal{F}[(-\Delta)f](\xi) = |\xi|^2 \hat{f}(\xi)$. Next, we are going to obtain an analogous expression for the operator $(-\Delta)^{\alpha/2}$ defined above.

As one can see in Stein's book [11, Ch. V], the following identity

$$c_{n,\alpha} \mathcal{F}(|\cdot|^{-n+\alpha})(\xi) = |\xi|^{-\alpha}, \qquad \xi \in \mathbb{R}^n,$$

holds for each $n \in \mathbb{N}$ and each $0 < \alpha < n$. Hence, as the Fourier transform takes the convolution of two functions to the product of their Fourier transforms (convolution theorem), we have that

$$\mathcal{F}[I_\alpha f](\xi) = \mathcal{F}[c_{n,\alpha} |\cdot|^{-n+\alpha} * f](\xi) = c_{n,\alpha} \mathcal{F}(|\cdot|^{-n+\alpha}) \hat{f}(\xi) = |\xi|^{-\alpha} \hat{f}(\xi).$$

Thus, by taking into account that we want to define $(-\Delta)^{\alpha/2}$ as the inverse of I_α, and by writing $\alpha/2 \mapsto s$, we can define the operator $(-\Delta)^s$ as follows.

Definition 1. Given $0 < s < 1$, we define the fractional Laplacian as the operator $(-\Delta)^s$ satisfying

$$\mathcal{F}[(-\Delta)^s f](\xi) = |\xi|^{2s} \hat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \ f \in \mathscr{S}(\mathbb{R}^n).$$ ◂

With this definition, we have

$$\mathcal{F}[I_{2s} \circ (-\Delta)^s f](\xi) = |\xi|^{-2s} \mathcal{F}[(-\Delta)^s f](\xi) = |\xi|^{-2s} |\xi|^{2s} \hat{f}(\xi) = \hat{f}(\xi),$$

so by taking the Fourier transform we have that, indeed, $(-\Delta)^s$ and I_{2s} are inverse operators in $\mathscr{S}(\mathbb{R}^n)$.

It is a well-known fact that the usual Laplacian satisfies

$$\Delta(u \circ T) = \Delta u \circ T, \qquad T \in \mathbb{O}(n), \tag{3}$$

i. e., the Laplacian commutes with elements of the orthogonal group. The same property can be easily obtained for the fractional Laplacian $(-\Delta)^s$ thanks to the invariance of the Fourier transform with respect to these transformations. Indeed, since $\mathcal{F}(f \circ T) = \mathcal{F}(f) \circ T$ and $\mathcal{F}^{-1}(f \circ T) = \mathcal{F}^{-1}(f) \circ T$ for every $T \in \mathbb{O}(n)$ and every $f \in \mathscr{S}(\mathbb{R}^n)$, we can write, for each $T \in \mathbb{O}(n)$ and each $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}(-\Delta)^s(u \circ T)(x) &= \mathcal{F}^{-1}[|\cdot|^{2s} \mathcal{F}(u \circ T)(\cdot)](x) = \mathcal{F}^{-1}[|\cdot|^{2s} \mathcal{F}(u) \circ T(\cdot)](x) \\ &= \mathcal{F}^{-1}[|T(\cdot)|^{2s} \mathcal{F}(u) \circ T(\cdot)](x) = \mathcal{F}^{-1}[|\cdot|^{2s} \mathcal{F}(u)(\cdot)] \circ T(x) = (-\Delta)^s u \circ T(x),\end{aligned}$$

where we have used that $|x| = |T(x)| = |T^{-1}(x)|$ for each $x \in \mathbb{R}^n$. This in particular proves that, if f has radial symmetry (*i. e.* if $f \circ T = f$ for every $T \in \mathbb{O}(n)$), then $(-\Delta)^s f$ also has radial symmetry. Note that, for the Fourier transform, the invariance with respect to elements of the orthogonal group and the fact that $\mathcal{F}^{-1} f(x) = \mathcal{F} f(-x)$ imply that the Fourier transform and the inverse Fourier transform of a function with radial symmetry coincide.

3. Second definition: pointwise formula

In this section we will describe a probabilistic situation in which the fractional Laplacian appears in a natural way. In this situation, the operator represents the rate of change in time of the probability of a particle to be in a particular position at a particular moment if it moves according to a certain process. This situation is depicted in Bucur and Valdinoci's book [2].

We are going to discretize the movement of the particle in such a way that $\tau > 0$ is the discrete step in time and $h > 0$ is the discrete space step. We will use the time step $\tau = h^{2s}$ for a fixed space step h. Let us denote by $u(x,t)$ the probability of finding the particle in position x at time t.

Let us define, for each $0 < s < 1$, the following probability on $\mathbb{N}$. For each subset I of natural numbers, we define

$$P(I) = c_s \sum_{k\in I} \frac{1}{k^{1+2s}},$$

where $c_s^{-1} := \sum_{k\in\mathbb{N}} \frac{1}{k^{1+2s}}$.

The particle under study moves according the following probability law: in each time step τ, the particle chooses a direction $v \in \mathbb{S}^{n-1}$ randomly according to a uniform distribution on the unit sphere $\mathbb{S}^{n-1}$, and a natural number $k \in \mathbb{N}$ according to the probability law P depicted above, and then it performs a translation by the vector khv. Note that, in this motion, large jumps are allowed, but their probability is very low.

According to this, a particle in position x_0 at time t, after a time step τ (*i. e.*, in time $t+\tau$), will be placed in position $x_0 + hkv$ for some $k \in \mathbb{N}$ and some $v \in \mathbb{S}^{n-1}$. Then, given $x \in \mathbb{R}^n$ and $t, \tau > 0$, the probability that the particle is in position x after a time step τ from the initial time t, $u(x, t+\tau)$, is the sum of the probabilities of finding the particle in position $x + hkv$ in the previous time for some $k \in \mathbb{N}$ and some $v \in \mathbb{S}^{n-1}$ multiplied by the probability of having chosen that direction v and that natural number k, *i. e.*,

$$u(x,t+\tau) = \frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv,t)}{k^{1+2s}}\, \mathrm{d}\sigma(v),$$

where σ is the surface measure of the sphere.

On one hand we can write

$$u(x,t+\tau) - u(x,t) = \frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv,t)-u(x,t)}{k^{1+2s}}\, \mathrm{d}\sigma(v),$$

and, on the other hand, by the radial symmetry of the process,

$$u(x,t+\tau) - u(x,t) = \frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x-hkv,t)-u(x,t)}{k^{1+2s}}\, \mathrm{d}\sigma(v).$$

If we add the above and divide by 2,

$$u(x,t+\tau) - u(x,t) = \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv,t)+u(x-hkv,t)-2u(x,t)}{k^{1+2s}}\, \mathrm{d}\sigma(v).$$

Now, dividing by $\tau = h^{2s}$ on both sides of the above inequality, we arrive at

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{h}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{N}} \int_{\mathbb{S}^{n-1}} \frac{u(x+hkv,t)+u(x-hkv,t)-2u(x,t)}{(hk)^{1+2s}}\, \mathrm{d}\sigma(v).$$

Here we recognize a Riemann sum. By writing $hk \mapsto r$, and taking into account that $\tau = h^{2s}$, we can take the (formal) limit when h goes to 0 on both sides to obtain

$$\begin{aligned}
\partial_t u(x,t) &= \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{u(x+rv,t)+u(x-rv,t)-2u(x,t)}{r^{1+2s}}\, \mathrm{d}\sigma(v)\, \mathrm{d}r \\
&= \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \int_0^\infty r^{n-1} \int_{\mathbb{S}^{n-1}} \frac{u(x+rv,t)+u(x-rv,t)-2u(x,t)}{r^{n+2s}}\, \mathrm{d}\sigma(v)\, \mathrm{d}r \\
&= \frac{1}{2}\frac{c_s}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{R}^n} \frac{u(x+y,t)-2u(x,t)+u(x-y,t)}{|y|^{n+2s}}\, \mathrm{d}y,
\end{aligned}$$

where we have used polar coordinates.

Thus, the operator L_1^s given by

$$L_1^s u(x) := -\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{n+2s}} \, \mathrm{d}y,$$

where $c_{n,s}$ is a positive constant (which we will choose later), depicts (up to a constant) the rate of change in time of the probability of the particle being in a certain position in a certain moment according to the laws we have just described. As we will see in section 4 below, this operator L_1^s coincides with the fractional Laplacian $(-\Delta)^s$, so we have got a probabilistic interpretation for this operator. Note that, according to the obtained expression, L_1^s is a nonlocal operator, as in order to obtain its value at a point, we need to know the value of the original function in the whole space.

Even though the above computations are just formal, the definition of L_1^s makes sense as, for functions in $\mathscr{S}(\mathbb{R}^n)$, the integral there converges. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\quad = \int_{|y|\leq 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y + \int_{|y|\geq 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\quad = I + II.
\end{aligned}$$

Since $u \in \mathscr{S}(\mathbb{R}^n)$, by using the Taylor expansion, we know that, if $|y| \leq 1$, then $2u(x) - u(x+y) - u(x-y) = -2\langle \nabla^2 u(x) y, y\rangle + o(|y|^2)$ for each $x \in \mathbb{R}^n$, so

$$\begin{aligned}
I &= \int_{|y|\leq 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&= \int_{|y|\leq 1} \frac{|2\langle \nabla^2 u(x) y, y\rangle + o(|y|^2)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\leq \int_{|y|\leq 1} \frac{2|\nabla^2 u(x)||y|^2 + |o(|y|^2)|}{|y|^{n+2s}} \, \mathrm{d}y \\
&\leq C_x \int_{|y|\leq 1} \frac{1}{|y|^{n-2(1-s)}} \, \mathrm{d}y < \infty,
\end{aligned} \tag{4}$$

as $0 < s < 1$.

The boundedness of II is simpler:

$$II = \int_{|y|\geq 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} \, \mathrm{d}y \leq 4\|u\|_{L_\infty(\mathbb{R}^n)} \int_{|y|\geq 1} \frac{1}{|y|^{n+2s}} \, \mathrm{d}y < \infty.$$

Remark 2. Observe that in the above argument, we only use that u is a bounded $C^2(\mathbb{R}^n)$ function. Thus, L_1^s makes sense in a bigger class than $\mathscr{S}(\mathbb{R}^n)$. ◂

This pointwise formula allows us to prove in a very simple way the behavior of this operator with respect to translations $\tau_h f(x) = f(x+h)$, $h \in \mathbb{R}^n$; dilations $\Delta_\lambda f(x) = f(\lambda x)$, and transformations of the orthogonal group $\mathbb{O}(n)$ (*i. e.*, the isometries of $\mathbb{R}^n$ which fix the origin, as, for instance, rotations or reflections).

Proposition 3. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. For each $h \in \mathbb{R}^n$ and each $\lambda > 0$, we have that*

$$L_1^s(\tau_h u) = \tau_h(L_1^s u) \quad \text{and} \quad L_1^s(\Delta_\lambda u) = \lambda^{2s} \Delta_\lambda [L_1^s u]. \tag{5}$$

Proof. The behavior of L_1^s with respect to translations is easily obtained by direct computations using the definition of the operator. The behavior with respect to dilations is straightforward to obtain as well.

Let us see the proof of the behavior with respect to dilations in order to illustrate the simplicity of these computations. Let us fix $x \in \mathbb{R}^n$. By using the change of variables $z = \lambda y$, we have

$$\begin{aligned}
L_1^s(\Delta_\lambda u)(x) &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2\Delta_\lambda u(x) - \Delta_\lambda u(x+y) - \Delta_\lambda u(x-y)}{|y|^{n+2s}} \, \mathrm{d}y \\
&= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(\lambda x) - u(\lambda x + \lambda y) - u(\lambda x - \lambda y)}{|y|^{n+2s}} \, \mathrm{d}y \\
&= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(\lambda x) - u(\lambda x + z) - u(\lambda x - z)}{\left(\frac{|z|}{\lambda}\right)^{n+2s}} \, \frac{\mathrm{d}z}{\lambda^n} \\
&= \lambda^{2s} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(\lambda x) - u(\lambda x + z) - u(\lambda x - z)}{|z|^{n+2s}} \, \mathrm{d}z \\
&= \lambda^{2s} L_1^s u(\lambda x) = \lambda^{2s} \Delta_\lambda [L_1^s u](x). \qquad \blacksquare
\end{aligned}$$

In the following two sections we will give two new definitions of the fractional Laplacian and we will prove that they are equivalent to the definition of L_1^s and the one of $(-\Delta)^s$. First, we will rewrite L_1^s.

Theorem 4. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. For each $x \in \mathbb{R}^n$ we have*

$$L_1^s u(x) = c_{n,s} \, \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y, \tag{6}$$

where PV *is the Cauchy principal value,* i. e.*,*

$$\mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y \tag{7}$$

and $c_{n,s}$ is a constant that depends on the dimension n and the order of the fractional Laplacian s.

Proof. Let $x \in \mathbb{R}^n$. As the integral defining L_1^s converges, we can rewrite it as follows:

$$\begin{aligned}
L_1^s u(x) &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \, \mathrm{d}y \\
&= \frac{c_{n,s}}{2} \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \, \mathrm{d}y \\
&= \frac{c_{n,s}}{2} \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \, \mathrm{d}y + \frac{c_{n,s}}{2} \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} \frac{u(x) - u(x-y)}{|y|^{n+2s}} \, \mathrm{d}y.
\end{aligned}$$

If we use the changes of variables $z = x + y$ and $z = x - y$, respectively, we can write the previous sum as

$$\frac{c_{n,s}}{2} \lim_{\epsilon \to 0^+} \int_{|z-x|>\epsilon} \frac{u(x) - u(z)}{|z-x|^{n+2s}} \, \mathrm{d}z + \frac{c_{n,s}}{2} \lim_{\epsilon \to 0^+} \int_{|x-z|>\epsilon} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, \mathrm{d}z.$$

By grouping all the terms in one integral, we finally obtain the desired result:

$$L_1^s u(x) = c_{n,s} \lim_{\epsilon \to 0^+} \int_{|x-z|>\epsilon} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, \mathrm{d}z. \qquad \blacksquare$$

Remark 5. Note that when we first defined the operator L_1^s, we proved that this definition made sense in the Schwartz class by using that, in the Taylor expansion of $u(x) - u(x+y) - u(x-y)$ used in (4), the linear term disappears, while in the formulation of theorem 4 we do not have this property at hand; hence we need to plug in the principal value. ◂

4. Third definition: the heat semigroup. Equivalence of definitions

In this section we will give a new definition of the fractional Laplacian in terms of the heat semigroup and then we will prove that this and the ones introduced above through the Fourier transform and the pointwise formula are equivalent and define the same operator.

Let us take a positive second order differential operator $L = L_x$ acting on functions in the spatial variable x defined on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Let us consider the problem

$$\begin{cases} v_t(x,t) + Lv(x,t) = 0, & \text{for } (x,t) \in \Omega \times (0,\infty), \\ v(x,0) = u(x), & \text{for } x \in \Omega. \end{cases}$$

Inspired by the form of solutions of linear first order differential equations, given an initial data u for the previous problem, we define the operator e^{-tL} by

$$\mathrm{e}^{-tL}u(x) = v(x,t), \qquad x \in \Omega, t \geq 0, \tag{8}$$

where v is the solution to the previous problem corresponding to the initial data u.

If we now think of the family $\{\mathrm{e}^{-tL},\ t \geq 0\}$, it turns out that it is a *semigroup of class* (C_0) [14, Def. 1, Ch. 9]. Indeed, if v is the solution to the previous problem with initial data u and we consider $t_2 \in (0,\infty)$, then the function $v^{t_2}(x,t) := v(x,t+t_2)$ satisfies the differential equation and also verifies $v^{t_2}(x,0) = v(x,t_2) = \mathrm{e}^{-t_2L}u(x)$. Then,

$$\mathrm{e}^{-t_1L}(\mathrm{e}^{-t_2L}u(x)) = v^{t_2}(x,t_1) = v(x,t_1+t_2) = \mathrm{e}^{-(t_1+t_2)L}u(x), \qquad x \in \Omega,\ t_1,t_2 \in (0,\infty).$$

If we take $L = -\Delta$ acting in the spatial variable x, we have defined the operator $\mathrm{e}^{t\Delta}$ in the way we just depicted in (8). In this case, a more specific expression can be given by solving the initial value problem

$$\begin{cases} v_t(x,t) = \Delta v(x,t), & \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) = u(x), & \text{for } x \in \mathbb{R}^n. \end{cases}$$

By applying the Fourier transform with respect to the variable x, we can rewrite the previous problem as

$$\begin{cases} \hat{v}_t(\xi,t) = |\xi|^2\hat{v}(\xi,t), & \text{for } (\xi,t) \in \mathbb{R}^n \times (0,\infty), \\ \hat{v}(\xi,0) = \hat{u}(\xi), & \text{for } \xi \in \mathbb{R}^n. \end{cases} \tag{9}$$

The resulting problem is a Cauchy problem associated to a homogeneous linear first order differential equation with initial value $\hat{u}$. Its solution is $\hat{v}(\xi,t) = \mathrm{e}^{-t|\xi|^2}\hat{u}(\xi)$. By applying the inverse Fourier transform,

$$\begin{aligned} \mathrm{e}^{t\Delta}u(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-t|\xi|^2}\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{-t|\xi|^2}\left(\int_{\mathbb{R}^n} u(z)\mathrm{e}^{-\mathrm{i}\xi\cdot z}\,\mathrm{d}z\right)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} u(z)\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{-t|\xi|^2}\mathrm{e}^{\mathrm{i}(x-z)\cdot\xi}\,\mathrm{d}\xi\right)\mathrm{d}z = \int_{\mathbb{R}^n} u(z)W_t(x-z)\,\mathrm{d}z, \end{aligned}$$

where

$$W_t(x) = \frac{1}{(4\pi t)^{n/2}}\mathrm{e}^{-\frac{|x|^2}{4t}}$$

is the Gauss kernel.

Once we got an explicit expression for $\mathrm{e}^{t\Delta}$, and inspired by the following numerical identity,

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (\mathrm{e}^{-t\lambda} - 1)\,\frac{\mathrm{d}t}{t^{1+s}},$$

which is valid for any $0 < s < 1$ and $\lambda > 0$, we can give the following alternative definition for the fractional Laplacian.

Definition 6. For $0 < s < 1$ and $u \in \mathscr{S}(\mathbb{R}^n)$, we define the operator L_2^s as

$$L_2^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (\mathrm{e}^{t\Delta}u(x) - u(x))\,\frac{\mathrm{d}t}{t^{1+s}}.$$ ◂

This definition can be justified by means of the spectral theorem, and can be found for instance in Stinga's thesis [13].

At this point, we prove the equivalence of the operators L_1^s and L_2^s to the fractional Laplacian $(-\Delta)^s$ which we defined in section 2. The following result can be found in Stinga's thesis [13, Lemma 2.1].

Theorem 7. *Let* $0 < s < 1$ *and* $u \in \mathscr{S}(\mathbb{R}^n)$*. Then,* $(-\Delta)^s u = L_1^s u = L_2^s u$*, i. e.,*

$$(-\Delta)^s u(x) = -\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{u(x+y)-2u(x)+u(x-y)}{|y|^{n+2s}}\,\mathrm{d}y = \frac{1}{\Gamma(-s)}\int_0^\infty (\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}}, \qquad x\in\mathbb{R}^n,$$

where

$$c_{n,s} = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)}.$$

Proof. We will prove that the last expression coincides with the other ones. Let us see first the equivalence between $(-\Delta)^s u$ and $L_2^s u$. By the Fourier inversion theorem,

$$\mathrm{e}^{t\Delta}u(x) - u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}(\mathrm{e}^{-t|\xi|^2}-1)\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi.$$

By using this and the change of variables $w = t|\xi|^2$, we obtain

$$\begin{aligned}\int_0^\infty |\mathrm{e}^{t\Delta}u(x)-u(x)|\,\frac{\mathrm{d}t}{t^{1+s}} &\le C_n\int_0^\infty\int_{\mathbb{R}^n}|\mathrm{e}^{-t|\xi|^2}-1||\hat{u}(\xi)|\,\mathrm{d}\xi\,\frac{\mathrm{d}t}{t^{1+s}}\\ &= C_n\int_{\mathbb{R}^n}\int_0^\infty|\mathrm{e}^{-w}-1|\,\frac{\mathrm{d}w}{w^{1+s}}|\xi|^{2s}|\hat{u}(\xi)|\,\mathrm{d}\xi\\ &= C_{n,s}\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{u}(\xi)|\,\mathrm{d}\xi < \infty,\end{aligned}$$

as $u \in \mathscr{S}(\mathbb{R}^n)$. Hence, by Fubini's theorem

$$\begin{aligned}\frac{1}{\Gamma(-s)}\int_0^\infty(\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}} &= \frac{1}{\Gamma(-s)}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_0^\infty(\mathrm{e}^{-t|\xi|^2}-1)\,\frac{\mathrm{d}t}{t^{1+s}}\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi\\ &= \frac{1}{\Gamma(-s)}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_0^\infty(\mathrm{e}^{-w}-1)\,\frac{\mathrm{d}w}{w^{1+s}}|\xi|^{2s}\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi = \mathcal{F}^{-1}\left(|\cdot|^{2s}\hat{u}\right)(x).\end{aligned}$$

Now, we see that $L_1^s u$ and $L_2^s u$ coincide for a suitable choice of $c_{n,s}$. More precisely, we will prove that

$$\frac{1}{\Gamma(-s)}\int_0^\infty(\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}} = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)}\mathrm{PV}\int_{\mathbb{R}^n}\frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z, \qquad x\in\mathbb{R}^n.$$

Let $\varepsilon > 0$. By using the fact that $\|W_t(x-\cdot)\|_{L^1(\mathbb{R}^n)} = 1$ for every $x \in \mathbb{R}^n$ and Fubini's theorem,

$$\begin{aligned}\int_0^\infty(\mathrm{e}^{t\Delta}u(x)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}} &= \int_0^\infty\int_{\mathbb{R}^n}W_t(x-z)(u(z)-u(x))\,\mathrm{d}z\,\frac{\mathrm{d}t}{t^{1+s}}\\ &= \int_{\mathbb{R}^n}\int_0^\infty W_t(x-z)(u(z)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}z = I_\varepsilon + II_\varepsilon.\end{aligned}$$

On one hand,

$$\begin{aligned}I_\varepsilon &:= \int_{|x-z|>\varepsilon}\int_0^\infty(4\pi t)^{-n/2}\mathrm{e}^{-\frac{|x-z|^2}{4t}}(u(z)-u(x))\,\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}z\\ &= \int_{|x-z|>\varepsilon}(u(z)-u(x))\int_0^\infty(4\pi t)^{-n/2}\mathrm{e}^{-\frac{|x-z|^2}{4t}}\,\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}z\\ &= \int_{|x-z|>\varepsilon}(u(x)-u(z))\frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}}\frac{1}{|x-z|^{n+2s}}\,\mathrm{d}z,\end{aligned}$$

where the change of variables $r = \frac{|x-z|^2}{4t}$ is used. Note that, as u is bounded, I_ε is absolutely convergent for any $\varepsilon > 0$. On the other hand, by using polar coordinates,

$$\begin{aligned} II_\varepsilon &:= \int_0^\infty \int_{|x-z|<\varepsilon} W_t(x-z)(u(z)-u(x))\,\mathrm{d}z\,\frac{\mathrm{d}t}{t^{1+s}} \\ &= \int_0^\infty (4\pi t)^{-n/2} \int_0^\varepsilon \mathrm{e}^{-\frac{r^2}{4t}} r^{n-1} \int_{|z'|=1} (u(x+rz')-u(x))\,\mathrm{d}S(z')\,\mathrm{d}r\,\frac{\mathrm{d}t}{t^{1+s}}. \end{aligned}$$

Now, by Taylor's theorem and by using the symmetry of the sphere, we can write

$$\int_{|z'|=1} (u(x+rz')-u(x))\,\mathrm{d}S(z') = K_n r^2 \Delta u(x) + O(r^3),$$

with K_n some constant that we will specify later. Indeed, by Taylor's theorem we can write, for each $z' \in \mathbb{S}^{n-1}$,

$$u(x+rz') = u(x) + r\langle \nabla u(x), z'\rangle + \frac{r^2}{2}\langle \nabla^2 u(x) z', z'\rangle + O(r^3),$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. Taking into account the symmetry of the sphere, we know that the integral of an odd function over $\mathbb{S}^{n-1}$ is 0: this means, in our formula above, that the first order terms integrate zero over the sphere. Likewise, when we examine closely the cross derivative terms of $\nabla^2 u(x)$, they also accompany an odd function in the formula above and so they disappear when we take the integral. This computation can be easily done by taking polar coordinates. It can also be noticed that what we are left with is a multiple of $r\Delta u(x)$, as we wrote above.

Hence,

$$|II_\varepsilon| \le K_{n,\Delta u(x)} \int_0^\varepsilon r^{n+1} \int_0^\infty \frac{\mathrm{e}^{-\frac{r^2}{4t}}}{t^{n/2+s}}\,\frac{\mathrm{d}t}{t} = K_{n,\Delta u(x)} \int_0^\varepsilon r^{n+1} K_{n,s} r^{-n-2s}\,\mathrm{d}r = K_{n,\Delta u(x),s}\,\varepsilon^{2(1-s)}.$$

This proves that $II_\varepsilon \to 0$ when $\varepsilon \to 0$, so

$$\int_0^\infty (\mathrm{e}^{t\Delta}u(x) - u(x))\,\frac{\mathrm{d}t}{t^{1+s}} = \lim_{\varepsilon\to 0} I_\varepsilon + II_\varepsilon = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z. \qquad \blacksquare$$

With computations very similar to the ones we just did, we can prove that, whenever $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we have that $(-\Delta)^s u(x)$ converges to $u(x)$ when $s \to 0^+$, for every $x \in \mathbb{R}^n$. Actually, for a given $x \in \mathbb{R}^n$, to prove the aforementioned convergence, we would only need that u belongs to $C^2(B(x)) \cap L^\infty(\mathbb{R}^n)$, with $B(x)$ the ball of center x and radius 1 [13, Prop. 2.3]. Note also that, when $u \in \mathscr{S}(\mathbb{R}^n)$, the pointwise convergence is obvious by the definition via Fourier transform, in the same way that it is obvious that $(-\Delta)^s \to (-\Delta)$ when $s \to 1^-$.

5. Fourth definition: the extension problem

When one works with nonlocal operators as $(-\Delta)^s$, one of the principal difficulties which appears is the fact that they do not act on functions in the same way that differential operators do, but they are defined by integral formulas. As a consequence, we do not have some of the properties that local operators have. From the point of view of the tools to study these operators, it is desirable to have some procedure which allows us to connect a nonlocal problem with a local one at hand. The bibliography in the topic of differentiable problems is extensive, and then the set of techniques is very rich. With this motivation, we present the trace relation and the extension problem.

Caffarelli and Silvestre [3] introduced a method which allows to transform nonlocal problems in $\mathbb{R}^n$ into other ones in which some differential operator in $\mathbb{R}^{n+1}_+$ appears. The method is described as follows: given $0 < s < 1$ and $u \in \mathscr{S}(\mathbb{R}^n)$, we want to study the solution of the system

$$\begin{cases} L_{1-2s}U(x,y) := \mathrm{div}_{x,y}(y^{1-2s}\nabla_{x,y}U) = 0, \quad x \in \mathbb{R}^n_+, y > 0, \\ U(x,0) = u(x), \\ U(x,y) \to 0 \text{ when } y \to \infty. \end{cases} \tag{10}$$

By using the definition of the divergence, the system in (10) can be rewritten as

$$\begin{cases} -\Delta_x U(x,y) = \left(\partial_{yy} + \frac{1-2s}{y}\partial_y\right) U(x,y), & x \in \mathbb{R}^n_+, y > 0, \\ U(x,0) = u(x), \\ U(x,y) \to 0 \text{ when } y \to \infty, \end{cases} \tag{11}$$

and the solution of (11) is given by the following result.

Theorem 8 (extension theorem). *The solution U of the extension problem (11) is given by the convolution*

$$U(x,y) = (P_s(\,\cdot\,,y) * u)(x) = \int_{\mathbb{R}^n} P_s(x-z,y)u(z)\,\mathrm{d}z, \tag{12}$$

where

$$P_s(x,y) = \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{(n+2s)/2}}$$

is the generalized Poisson kernel *for the extension problem in the semispace $\mathbb{R}^{n+1}_+$. Moreover, for U defined as in (12) one has*

$$(-\Delta)^s u(x) = -\frac{2^{s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y), \tag{13}$$

which is what we call the trace relation.

Remark 9. The extension theorem provides an interesting relation between the operators $(-\Delta)^s$ and ∂_y. This relation allows to obtain properties of the nonlocal operator from the properties of the local one. ◂

Proof of theorem 8. By taking the partial Fourier transform in the variable x in (11), we get the system

$$\begin{cases} \partial_{yy}\hat{U}(\xi,y) + \frac{1-2s}{y}\partial_y \hat{U}(\xi,y) - |\xi|^2\hat{U}(\xi,y) = 0, & (\xi,y) \in \mathbb{R}^{n+1}_+, \\ \hat{U}(\xi,0) = \hat{u}(\xi), \quad \hat{U}(\xi,y) \to 0 \text{ when } y \to \infty, & \xi \in \mathbb{R}^n. \end{cases} \tag{14}$$

Now, if we fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and write $Y(y) = Y_\xi(y) := \hat{U}(\xi,y)$, the previous problem can be written as

$$\begin{cases} y^2Y''(y) + (1-2s)yY'(y) - |\xi|^2y^2Y(y) = 0, & y \in \mathbb{R}_+, \\ Y(0) = \hat{u}(\xi), \quad Y(y) \to 0 \text{ when } y \to \infty. \end{cases} \tag{15}$$

The equation in the above problem, under some adjustment of the parameters, is known in the literature as the generalised modified Bessel equation (see for instance Lebedev's book [9]), and is given by

$$y^2Y'' + (1-2\alpha)yY'(y) + [\beta^2\gamma^2y^{2\gamma} + (\alpha - \nu^2\gamma^2)]Y(y) = 0, \tag{16}$$

where

$$\alpha = s, \quad \gamma = 1, \quad \nu = s, \quad \beta = |\xi|.$$

It is well known (see Lebedev's book [9]) that (16) has two linear independent solutions given by

$$u_1(y) = y^s I_s(|\xi|y) \qquad \text{and} \qquad u_2(y) = y^s K_s(|\xi|y),$$

where I_s and K_s are Bessel functions of second and third kind, respectively:

$$\begin{aligned} I_r(z) &= \left(\frac{z}{2}\right)^r \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{\Gamma(k+1)\Gamma(k+s+1)}, && |z| < \infty, |\arg(z)| < \pi; \\ K_r(z) &= \frac{\pi}{2}\frac{I_{-r}(z) - I_r(z)}{\sin \pi r}, && |\arg(z)| < \pi. \end{aligned} \tag{17}$$

Thus, for each $\xi \neq 0$, the solution of (15) is

$$\hat{U}(\xi,y) = Ay^sI_s(|\xi|y) + By^sK_s(|\xi|y), \qquad (\xi,y) \in \mathbb{R}^{n+1}_+. \tag{18}$$

From the expressions in (17) one can obtain (see [9, formulas (5.11.9) and (5.11.10), p. 123]) the following asymptotics for $z \to \infty$:

$$I_s(z) \approx e^z (2\pi z)^{-1/2}, \qquad K_s(z) \approx \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}, \tag{19}$$

where we use the symbol $A \approx B$ to say that there exist constants $c, C > 0$ such that $cA \leq B \leq CA$. Thus, the function $I_s(z)$ diverges, while $K_s(z)$ takes finite values for z sufficiently large. From here, we can deduce that the condition that $\hat{U}(\xi, y) \to 0$ when $y \to \infty$ implies that, in (18), we must have $A = 0$. Therefore,

$$\hat{U}(\xi, y) = B y^s K_s(|\xi| y), \qquad (\xi, y) \in \mathbb{R}^{n+1}_+. \tag{20}$$

On the other hand, it can be proved that, when $z \to 0^+$ in (17),

$$I_s(z) \approx \frac{1}{\Gamma(s+1)} \left(\frac{z}{2}\right)^s \qquad \text{and} \qquad I_{-s}(z) \approx \frac{1}{\Gamma(1-s)} \left(\frac{z}{2}\right)^{-s} \tag{21}$$

(c.f. [9, formula (5.7.1), p. 108]). Hence, by imposing the condition $\hat{U}(\xi, 0) = \hat{u}(\xi)$, we can fix B in the following way:

$$B y^s K_s(|\xi| y) = B \frac{\pi}{2} \frac{y^s I_{-s}(|\xi| y) - y^s I_s(|\xi| y)}{\sin \pi s} \overset{y \to 0^+}{\to} \frac{B \pi 2^{s-1}}{\Gamma(1-s) \sin \pi s} |\xi|^{-s} = B 2^{s-1} \Gamma(s) |\xi|^{-s}, \tag{22}$$

where in the last equality we have used the property of the Γ function that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad z \notin \mathbb{Z}. \tag{23}$$

Since $\hat{U}(\xi, 0) = \hat{u}(\xi)$, we have that

$$\hat{U}(\xi, y) = \frac{|\xi|^s \hat{u}(\xi)}{2^{s-1}\Gamma(s)} y^s K_s(|\xi| y). \tag{24}$$

Suppose that U is given by the convolution of u with a kernel $P_s(x, y)$ and let us see the explicit expression for this kernel. With this assumption, from the properties of the Fourier transform with respect to the convolution of functions, we deduce that $\hat{U}(\xi, y) = \hat{P}_s(\xi, y)\hat{u}(\xi)$, so

$$P_s(x, y) = \mathcal{F}^{-1}\left(\frac{\hat{U}(\cdot, y)}{\hat{u}(\cdot)}\right)(x) = \mathcal{F}^{-1}\left(\frac{|\cdot|^s}{2^{s-1}\Gamma(s)} y^s K_s(|\cdot| y)\right)(x).$$

Now, as the function $\frac{|\cdot|^s}{2^{s-1}\Gamma(s)} y^s K_s(|\cdot| y)$ is a function with spherical symmetry, we know that its Fourier transform coincides with its inverse Fourier transform, so finding an expression for the kernel $P_s(x, y)$ is equivalent to computing the Fourier transform (in the case in which the Fourier transform is applied to radial functions, it is called the Hankel transform) of the function $\frac{|\cdot|^s}{2^{s-1}\Gamma(s)} y^s K_s(|\cdot| y)$.

For this, we use the fact that the Hankel transform of a radial function $f(\cdot) = f_0(|\cdot|)$ is (see Stein and Weiss's book [12, Ch. IV, Th. 3.3])

$$\mathcal{F}(f_0)(r) = \frac{1}{(2\pi)^{\frac{n}{2}} r^{\frac{n-2}{2}}} \int_0^\infty f_0(s) J_{\frac{n-2}{2}}(rs) s^{\frac{n}{2}} \, ds.$$

Here, $J_{\frac{n-2}{2}}$ denotes the Bessel function, defined for k a real number greater than 1/2 by letting

$$J_k(t) := \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma(\frac{1}{2})} \int_{-1}^1 e^{its}(1-s^2)^{(2k-1)/2} \, ds.$$

Thus, by applying this to our function and by using [5, formula 3 in 6.576, p. 684], we get

$$\begin{aligned}
\mathcal{F}\left(\frac{|\cdot|^s}{2^{s-1}\Gamma(s)} y^s K_s(|\cdot| y)\right)(x) &= \mathcal{F}\left(\frac{|\cdot|^s}{2^{s-1}\Gamma(s)} y^s K_s(|\cdot| y)\right)(|x|) \\
&= \frac{y^s}{2^{s-1}\Gamma(s)(2\pi)^{\frac{n}{2}} |x|^{\frac{n-2}{2}}} \int_0^\infty |\xi|^{\frac{n}{2}+s} K_s(|\xi| y) J_{\frac{n-2}{2}}(|x| s) \, d|\xi| \\
&= \frac{\Gamma(\frac{n}{2}+s)}{\pi^{n/2}} \frac{y^{2s}}{(y^2+|x|^2)^{(n+2s)/2}}.
\end{aligned}$$

Only (13) is left, *i. e.*, we just have to check the equality

$$(-\Delta)^s u(x) = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y).$$

Let us recall that the Fourier transform of the function $(-\Delta)^s u(x)$ is $|\xi|^{2s}\hat{u}(\xi)$. Then, (13) is equivalent to seeing that

$$|\xi|^{2s}\hat{u}(\xi) = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\frac{\partial \hat{U}}{\partial y}(\xi,y). \tag{25}$$

By using the following identities for the Bessel function of third kind (c.f. [9, formula (5.7.9), p. 110)],

$$K_s'(z) = \frac{s}{z}K_s(z) - K_{s+1}(z) \tag{26}$$

and

$$\frac{2s}{z}K_s(z) - K_{s+1}(z) = -K_{s-1}(z) = -K_{1-s}(z), \tag{27}$$

together with (23), we get

$$y^{1-2s}\partial_y\hat{U}(\xi,y) = \frac{|\xi|^{s+1}\hat{u}(\xi)}{2^{s-1}\Gamma(s)}y^{1-s}\left(\frac{2s}{y}K_s(|\xi|y) - K_{s+1}(|\xi|y)\right) = -\frac{|\xi|^{s+1}\hat{u}(\xi)}{2^{s-1}\Gamma(s)}y^{1-s}K_{1-s}(|\xi|y).$$

In view of the behaviour of the Bessel function of third kind K_s, we have that

$$\lim_{y\to 0^+} y^{1-s}K_{1-s}(|\xi|y) = \frac{\Gamma(1-s)|\xi|^{s-1}}{2^s},$$

so we get

$$\lim_{y\to 0^+} y^{1-2s}\partial_y\hat{U}(\xi,y) = -\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}|\xi|^{2s}\hat{u}(\xi).$$

This proves (13), thus finishing the proof. ■

Alternative proof of (13). By using the following property of the generalised Poisson kernel,

$$\int_{\mathbb{R}^n} P_s(x,y)\,\mathrm{d}x = 1, \qquad y>0, \tag{28}$$

we can obtain an alternative proof where we do not need to use (26) and (27).

Let $u \in \mathscr{S}(\mathbb{R}^n)$ and consider a solution

$$U(x,y) = (P_s(\,\cdot\,,y) * u)(x)$$

of the extension problem (12). Observe that, by using (28), we can write

$$U(x,y) = \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\int_{\mathbb{R}^n}\frac{(u(z)-u(x))y^{2s}}{(y^2+|x-z|^2)^{(n+2s)/2}}\,\mathrm{d}z + u(x).$$

By differentiating the two sides with respect to y, we obtain

$$y^{1-2s}\partial_y U(x,y) = 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\int_{\mathbb{R}^n}\frac{u(z)-u(x)}{(y^2+|z-x|^2)^{(n+2s)/2}}\,\mathrm{d}z + O(y^2).$$

If we let $y\to 0^+$ and we use the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y) &= 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\mathrm{PV}\int_{\mathbb{R}^n}\frac{u(z)-u(x)}{(|z-x|^2)^{(n+2s)/2}}\,\mathrm{d}z \\ &= -2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}c_{n,s}^{-1}(-\Delta)^s u(x),\end{aligned} \tag{29}$$

where in the second equality we have used the definition of fractional Laplacian (6).

Finally, recall that

$$c_{n,s} = \frac{s2^{2s}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)}, \tag{30}$$

so, by substituting the expression of (30) in (29) we obtain

$$\lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y) = -2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}c_{n,s}^{-1}(-\Delta)^s u(x) = -\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}(-\Delta)^s u(x),$$

which completes the alternative proof. ∎

References

[1] Bañuelos, Rodrigo and Bogdan, Krzysztof. "Lévy processes and Fourier multipliers". In: *Journal of Functional Analysis* 250.1 (2007), pp. 197–213. issn: 0022-1236. https://doi.org/10.1016/j.jfa.2007.05.013.

[2] Bucur, Claudia and Valdinoci, Enrico. *Nonlocal diffusion and applications*. Vol. 20. Lecture Notes of the Unione Matematica Italiana. Cham: Springer, 2016. https://doi.org/10.1007/978-3-319-28739-3.

[3] Caffarelli, Luis and Silvestre, Luis. "An extension problem related to the fractional Laplacian". In: *Communications in Partial Differential Equations* 32.7-9 (2007), pp. 1245–1260. issn: 0360-5302. https://doi.org/10.1080/03605300600987306.

[4] Garofalo, Nicola. "Fractional thoughts". In: *New developments in the analysis of nonlocal operators*. AMS Special Session on New Developments in the Analysis of Nonlocal Operators (University of St. Thomas, Minneapolis, 2016). Contemporary Mathematics 723. Providence: American Mathematical Society, 2019, pp. 1–135. https://doi.org/10.1090/conm/723/14569.

[5] Gradshteyn, I. S. and Ryzhik, I. M. *Table of integrals, series, and products*. Ed. by Zwillinger, Daniel and Moll, Victor. 8th ed. Amsterdam: Academic Press, 2014. https://doi.org/10.1016/C2010-0-64839-5.

[6] Graham, C. Robin and Zworski, Maciej. "Scattering matrix in conformal geometry". In: *Inventiones Mathematicae* 152.1 (2003), pp. 89–118. issn: 0020-9910. https://doi.org/10.1007/s00222-002-0268-1.

[7] Kwaśnicki, Mateusz. "Ten equivalent definitions of the fractional Laplace operator". In: *Fractional Calculus and Applied Analysis* 20.1 (2017), pp. 7–51. issn: 1311-0454. https://doi.org/10.1515/fca-2017-0002.

[8] Landkof, Naum S. *Foundations of modern potential theory*. Trans. Russian by Doohovskoy, Alexander P. Die Grundlehren der mathematischen Wissenschaften 180. Berlin, Heidelberg: Springer-Verlag, 1972. isbn: 978-3-642-65185-4.

[9] Lebedev, Nikolaĭ N. *Special functions and their applications*. Trans. Russian by Silverman, Richard A. Revised edition, unabridged and corrected republication. New York: Dover Publications, 1972. isbn: 978-0-486-60624-8.

[10] Riesz, Marcel. "Intégrales de Riemann-Liouville et potentiels". In: *Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum* 9.1 (1938), pp. 1–42.

[11] Stein, Elias M. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series. Princeton: Princeton University Press, 1970. isbn: 978-0-691-08079-6.

[12] Stein, Elias M. and Weiss, Guido. *Introduction to Fourier analysis on Euclidean spaces*. Princeton Mathematical Series 32. Princeton: Princeton University Press, 1971. isbn: 978-0-691-08078-9.

[13] Stinga, Pablo Raúl. *Fractional powers of second order partial differential operators: extension problem and regularity theory*. PhD thesis. Universidad Autónoma de Madrid, 2010. url: http://hdl.handle.net/10486/4839.

[14] Yosida, Kōsaku. *Functional analysis*. Classics in Mathematics 123. Reprint of the sixth (1980) edition. Berlin, Heidelberg: Springer, 1995. https://doi.org/10.1007/978-3-642-61859-8.

IX *Escuela-Taller de Análisis Funcional* (2019)

Holomorphic functional calculus for sectorial operators

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Abstract: This article is about holomorphic functional calculus on Hilbert spaces. It is an expository paper where we survey the main results in this setting and we present some examples. We define the natural functional calculus for sectorial operators and give an example of a sectorial operator that does not admit a functional calculus.

Resumen: Este artículo trata sobre el cálculo funcional holomorfo en espacios de Hilbert. Es un artículo expositorio donde tratamos los resultados principales y presentamos algunos ejemplos. Definimos el cálculo funcional natural para operadores sectoriales y damos un ejemplo de un operador sectorial que no admite un cálculo funcional.

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1. Introduction

In this expository paper we develop the main ideas about holomorphic functional calculus for sectorial operators on Hilbert spaces. Of course, we cannot be exhaustive and make a complete presentation of the theory. We present the sketch of some proofs, and the complete demonstration of some results. Our purpose is to show some of the usual procedures and tools in this context. The interested reader can consult, for instance, the references listed at the end.

The functional calculus that we study was formalized in the late 80's and 90's mainly by McIntosh [1, 4, 17, 18]. The motivation was in Kato's square root problem and the operator-method approach to evolution equations of Grisvard and Da Prato. Many examples within the class of operators under consideration can be found among the partial differential operators (elliptic differential operators, Schrödinger operators with singular potentials, Stokes operators...).

The purpose of a functional calculus is to give a meaning to $f(T)$, where f is a complex function defined on a subset of $\mathbb{C}$ and T is an operator defined on a Hilbert space.

For example, in a finite dimensional Hilbert space, we can find exponentials and logarithms of matrices through the theory of linear systems of differential equations. These operations can be regarded as a special case of a functional calculus for operators in a finite dimensional setting.

Another example is the well-known equality

$$\Delta f = \mathcal{F}^{-1}\big(|y|^2\mathcal{F}(f)\big), \tag{1}$$

where Δ represents the Laplace operator, $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$. This equality holds provided that f satisfies certain regularity and decay conditions. From (1) it follows that, if P is a polynomial, we can compute $P(\Delta)$ using the following expression:

$$P(\Delta)f = \mathcal{F}^{-1}(P(|y|^2)\mathcal{F}(f)). \tag{2}$$

Now, if we want to define $m(\Delta)$ for a general complex function m, (2) suggests the following definition:

$$m(\Delta)f = \mathcal{F}^{-1}(m(|y|^2)\mathcal{F}(f)).$$

This is the first step to construct a functional calculus for the Laplace operator. In a second step, we need to specify the Hilbert space H where f belongs to and the class ξ of admissible functions m in this functional calculus. Actually, (1) can be seen as a spectral representation for the operator Δ. This fact allows to extend these ideas and to define a functional calculus for operators on Hilbert spaces by using spectral representations.

The abstract method of defining a functional calculus follows ideas from Haase [8], and it is roughly done in the following way. If T is an operator in a Hilbert space H, we consider a class ξ of functions defined on the spectrum $\sigma(T)$ of T and a mapping $\Phi : \xi \to B(H)$, where $B(H)$ represents the space of bounded and linear operators on H. Φ is actually a method to assign an operator $f(T)$ defined by $\Phi(f)$. In a first approach, the class ξ is formed by smooth enough functions. Later on, this class of functions will be enlarged by including less regular functions. In this second step, $f(T)$ may be unbounded if T is not in $B(H)$. Since the purpose of functional calculus is to make computations, it is convenient that ξ is an algebra and Φ is an algebra homomorphism. Furthermore, the mapping Φ should be somehow connected to the operator T (note that we write $f(T)$). One of such relations should be that $\Phi((\lambda - z)^{-1}) = R(\lambda, T)$, where $\lambda \in \rho(T)$ and $R(\lambda, T) = (\lambda I - T)^{-1}$ is the resolvent operator. Haase [8] presented an abstract approach to the construction of functional calculus.

This paper is structured as follows. In section 2 we present the holomorphic functional calculus. The definition and the main properties of the sectorial operators on Hilbert spaces are presented in section 3. The holomorphic function algebras, basic for functional calculus, are studied in section 4. Section 5 is focused on the definitions and properties of the holomorphic functional calculus for sectorial operators. Finally, in section 6 we discuss an example: a sectorial operator in a separable Hilbert space that does not have a bounded holomorphic functional calculus.

The theory developed in this paper is concerned with operators in Hilbert spaces and holomorphic functions. The standard definitions and properties about these topics that are used in the sequel can be found in the monographs of Rudin [19, 20]. A detailed study about the holomorphic functional calculus for sectorial operators appears, for instance, in [7, 14, 15, 24], where most of the results in this paper are included.

2. Holomorphic functional calculus

Let H be a Hilbert space. In this section, we assume that $T\colon H \to H$ is a bounded linear operator.

Let f be a holomorphic function defined on $\mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, where R is the radius of convergence of f. Let us also suppose $R > \|T\|$, where $\|T\| = \sup_{z\in H\setminus\{0\}} \frac{\|Tz\|_H}{\|z\|_H}$ is the operator norm of T.

A natural way of defining $f(T)$ is the following one. It is clear that the series defined by

$$f(T) = \sum_{n=0}^{\infty} a_n T^n$$

is convergent in $B(H)$ and $f(T) \in B(H)$. Moreover, if $P_{\|T\|}$ denotes the space of holomorphic functions on $\{z \in \mathbb{C} : |z| < R\}$ for some $R > \|T\|$, the mapping

$$\begin{aligned} \Psi\colon P_{\|T\|} &\longrightarrow B(H) \\ f &\longmapsto f(T) \end{aligned}$$

is an algebra homomorphism. Notice that this way, the definition of $f(T)$ coincides with the natural definition whenever f is a polynomial.

There is also a natural way of defining $f(T)$ when f is a rational function. Let p, q be polynomials such that q does not vanish in $\sigma(T)$. Then, $r = p/q$ is a rational function with no poles in $\sigma(T)$. Write $q(z) = \prod_{j=1}^{n}(\alpha_j - z)$. Since $\alpha_j \notin \sigma(T)$, we have that the resolvent operator $R(\alpha_j, T) = (\alpha_j I - T)^{-1}$ is a bounded operator on H. Therefore, one can define $r(T)$ as

$$r(T) = p(T)\prod_{j=1}^{n} R(\alpha_j, T).$$

Moreover, $r(T) \in B(H)$ and the mapping

$$\begin{aligned} \Psi\colon R_{\sigma(T)} &\longrightarrow B(H) \\ r &\longmapsto r(T) \end{aligned}$$

is and algebra homomorphism. Here $R_{\sigma(T)}$ denotes the spaces of rational functions without poles in $\sigma(T)$.

Our objective is to define a functional calculus extending the two above particular cases (power series with a radius of convergence greater than $\|T\|$ and rational functions without poles in $\sigma(T)$). In order to do this, we consider holomorphic functions in some neighbourhood of the spectrum of $T \in B(H)$.

Let T be a bounded operator in H and $\Omega \subset \mathbb{C}$ an open set containing $\sigma(T)$. For each connected component Ω_i of Ω, we consider a closed contour γ_i in Ω_i around $\sigma(T) \cap \Omega_i$. We let γ be the union of all the γ_i. In particular, γ is a finite collection of smooth closed paths that is contained in $\Omega \setminus \sigma(T)$, see figure 1. Then, we have

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}z}{z-\alpha} = \begin{cases} 1, & \alpha \in \sigma(T), \\ 0, & \alpha \notin \Omega. \end{cases}$$

Suppose that f is a holomorphic function in Ω; in short, $f \in \mathcal{H}(\Omega)$. Motivated by the Cauchy integral formula, we define

$$f(T) = \frac{1}{2\pi\mathrm{i}}\int_\gamma f(z)R(z,T)\,\mathrm{d}z,$$

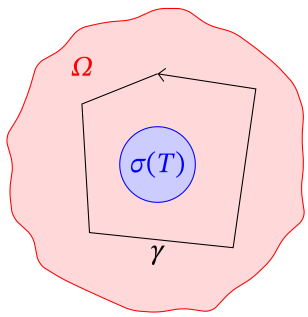

Figure 1: The contour γ around $\sigma(T)$.

where $R(z,T) = (zI - T)^{-1}$ is the resolvent operator. The integral is understood in the $B(H)$-Böchner sense [3]. Since $f \in H(\Omega)$ and γ is contained in $\Omega \setminus \sigma(T)$, the Cauchy integral theorem and Hahn-Banach theorem imply that the integral defining $f(T)$ does not depend on the contour γ satisfying the above conditions. Moreover, $f(T) \in B(H)$.

Theorem 1. *Let $T \in B(H)$. Assume that Ω is an open set in $\mathbb{C}$ containing $\sigma(T)$. Then, the mapping*

$$\begin{aligned} \Phi\colon \mathcal{H}(\Omega) &\longrightarrow B(H) \\ f &\longmapsto f(T) \end{aligned}$$

satisfies the following properties:

(1) *Φ is an algebra homomorphism.*

(2) *$\Phi(p) = p(T)$, for every polynomial p.*

(3) *If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}(\Omega)$, $f \in \mathcal{H}(\Omega)$ and $f_n \to f$, as $n \to \infty$, uniformly on every compact subset of Ω, then $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$ in $B(H)$.*

Moreover, Φ is the unique mapping from $H(\Omega)$ into $B(H)$ satisfying the properties (1), (2) and (3).

This holomorphic functional calculus is also called the Dunford–Riesz functional calculus. Note that the two previous examples are special cases of the Dunford–Riesz functional calculus. Indeed, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with radius of convergence greater than $\|T\|$, by taking $\Omega = B(0,R)$, we have that $\Phi(f) = \sum_{n=0}^{\infty} a_n T^n$. Also, if $r \in R_{\sigma(T)} \cap \mathcal{H}(\Omega)$ with $\sigma(T) \subset \Omega$, and r is a rational function without poles in $\sigma(T)$, then $\Phi(r) = p(T)\prod_{i=1}^{n}(\alpha_i I - T)^{-1}$. Here, we are assuming $r(z) = \frac{p(z)}{\prod_{i=1}^{n}(\alpha_i - z)}$, where $p(z)$ is a polynomial over $\mathbb{C}$.

The spectral mapping theorem holds for Φ, that is, if Ω is an open set in $\mathbb{C}$ containing $\sigma(T)$ and $f \in H(\Omega)$, then $f(\sigma(T)) = \sigma(f(T))$ [20, Theorem 10.28(b)].

We are going to give an idea on how to extend the holomorphic functional calculus to closed operators. Let T be a closed operator and Ω an open set such that $\sigma(T) \subset \Omega$ and the resolvent set $\rho(T) \cap (\mathbb{C} \setminus \Omega) \neq \varnothing$. We choose $\alpha \in \rho(T) \cap (\mathbb{C} \setminus \Omega)$ and we consider the function

$$R_\alpha(z) = \begin{cases} \frac{1}{z-\alpha} & \text{if } z \in \mathbb{C} \setminus \{\alpha\}, \\ \infty & \text{if } z = \alpha, \\ 0 & \text{if } z = \infty. \end{cases}$$

It is clear that $R_\alpha(\Omega) = W$ is an open set in $\mathbb{C} \setminus \{0\}$. Also, we have that $-R_\alpha(\sigma(T)) = \sigma((\alpha I - T)^{-1})$. Since $\sigma(T) \subset \Omega$, $\sigma((\alpha I - T)^{-1}) \subseteq -W$. We define

$$f(T) := (f \circ (-R_\alpha^{-1}))((\alpha I - T)^{-1}),$$

where we use the holomorphic functional calculus defined in theorem 1 on the right hand side.

3. Sectorial operators

In this section we introduce sectorial operators and we discuss their main properties. These operators may not be defined on the whole Hilbert space H but on a dense subspace that we call domain of T (and denote by $D(T)$). Moreover, they may be unbounded. For a (possibly unbounded) linear operator $T\colon D(T) \to H$, we define its spectrum $\sigma(T)$ as the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \mathrm{Id}$ has a bounded inverse.

If $0 < \omega \leq \pi$, by S_ω we denote the open sector, that is symmetric with respect the positive real axis and with opening angle ω, that is,

$$S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\mathrm{Arg}(z)| < \omega\}.$$

Also, we define $S_0 = (0, \infty)$. $\overline{S_\omega}$ denotes the closure of S_ω, for every $0 \leq \omega \leq \pi$.

Definition 2. Let $T\colon D(T) \to H$ be a linear operator. We say that T is a **sectorial operator of angle $\omega \in [0,\pi)$** (in short, $T \in \mathrm{Sect}(\omega)$) when the following two properties hold:

1) $\sigma(T) \subset \overline{S_w}$,
2) for every $\omega < \alpha < \pi$,

$$M(T,\alpha) := \sup\{\|z R(z,T)\| : z \notin \overline{S_\alpha}\} < \infty.$$ ◂

Note that if $T \in \mathrm{Sect}(\omega)$ for some $\omega \in [0,\pi)$, then $(-\infty, 0) \subseteq \rho(T)$ and T is a closed operator. We name sectoriality angle ω_T of the operator T to the number

$$\omega_T = \min\{0 \leq \omega < \pi : T \in \mathrm{Sect}(\omega)\}.$$

We now give a few examples of sectorial operators.

Example 3. If $T \in B(H)$ is self-adjoint and positive, then $T \in \mathrm{Sect}(0)$. ◂

Example 4. Let $0 < \omega < \frac{\pi}{2}$. A family $\{T(z) : z \in S_\omega\} \subseteq B(H)$ is said to be a holomorphic semigroup when

1) $T(z_1)T(z_2) = T(z_1 + z_2)$, $z_1, z_2 \in S_w$,
2) the mapping

$$\begin{aligned} \Psi\colon\ S_\omega &\longrightarrow B(H) \\ z &\longmapsto T(z) \end{aligned}$$

is holomorphic.

The infinitesimal generator of $\{T(z)\}$ is the operator A defined by

$$Ax = \lim_{t\to 0^+} \frac{T(t) - I}{t} x, \quad x \in D(A),$$

where the domain $D(A)$ of A is formed by all those $x \in H$ such that the limit $\lim_{t\to 0^+} \frac{T(t)x - x}{t}$ exists. Then, the operator $-A$ is in $\mathrm{Sect}(\frac{\pi}{2} - \omega)$. ◂

Example 5. We consider on $\mathbb{C}^2$ the usual inner product defined by

$$(u,v)\cdot(a,b) = u\bar{a} + v\bar{b}, \quad u,v,a,b \in \mathbb{C}.$$

We define the space $\ell^2(\mathbb{C}^2)$ of sequences $z = \{z_n = (u_n\,,\, v_n)\}_{n=1}^\infty$ in $\mathbb{C}^2$ such that

$$\|z\|_2 = \|\{z_n\}_{n=1}^\infty\|_2 := \left(\sum_{n=1}^\infty (|u_n|^2 + |v_n|^2)\right)^{1/2} < \infty.$$

The vector space $\ell^2(\mathbb{C}^2)$ is actually a Hilbert space with the natural operations and the inner product of $\ell^2(\mathbb{C}^2)$ is defined by

$$z \cdot y = \{z_n\}_{n=1}^\infty \cdot \{y_n\}_{n=1}^\infty = \sum_{n=1}^\infty z_n \cdot \bar{y_n}, \quad z, y \in \ell^2(\mathbb{C}^2).$$

We define the operator T on $\ell^2(\mathbb{C}^2)$ as follows: for $z = \{(u_n, v_n)\}_{n=1}^{\infty}$, we define

$$T(z) = \left\{ \begin{pmatrix} 2^{-n} & 1 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n=1}^{\infty}.$$

Therefore, $T \in B(\ell^2(\mathbb{C}^2))$. Note that T is not a sectorial operator for any angle $0 \leq \omega < \pi$. Let $\varepsilon < 0$. Let $z = \{(u_n, v_n)\}_n \in \ell^2(\mathbb{C}^2)$. We have that

$$(\varepsilon I - T)(z) = \left\{ \begin{pmatrix} \varepsilon - 2^{-n} & -1 \\ 0 & \varepsilon - 2^{-n} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n=1}^{\infty}.$$

And therefore, the resolvent operator has the following expression:

$$(\varepsilon I - T)^{-1}(z) = \left\{ \begin{pmatrix} (\varepsilon - 2^{-n})^{-1} u_n + (\varepsilon - 2^{-n})^{-2} v_n \\ (\varepsilon - 2^{-n})^{-1} v_n \end{pmatrix} \right\}_{n=1}^{\infty}, \quad z = \{(u_n, v_n)\}_n \in \ell^2(\mathbb{C}^2),$$

and

$$\|(\varepsilon I - T)^{-1}\| \geq \sup_{n \geq 1} (\varepsilon - 2^{-n})^{-2} = \varepsilon^{-2}.$$

Hence, $T \notin \mathrm{Sect}(\omega)$, for any $\omega \in [0, \pi)$. ◂

Example 6. The following differential operators are sectorial:

1) $Tf = -f''$, $f \in D(T) = W^{2,2}(\mathbb{R})$.
2) $Tf = -f''$, $f \in D(T) = \{f \in W^{2,2}(0, 1) : f(0) = f(1) = 0\}$.
3) $Tf = -f''$, $f \in D(T) = \{f \in W^{2,2}(0, 1) : f'(0) = f'(1) = 0\}$.
4) Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2 boundary and

$$Tu = -\Delta u, \; u \in D(T) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Here, by W we denote the usual Sobolev spaces [20, Chapter 8.8]. ◂

Now, we present some of the main properties of sectorial operators.

Proposition 7. *Let H be a Hilbert space and $T \in \mathrm{Sect}(\omega)$, where $0 \leq \omega \leq \pi$. The following statements hold.*

1) *If T is injective, then $T^{-1} \in \mathrm{Sect}(\omega)$.*
2) *Let $x \in H$. We have that $x \in \overline{D(T)}$ if and only if $\lim_{t\to\infty} t^n(t+T)^{-n}x = x$.*
3) *Let $x \in H$. Then, $x \in \overline{R(T)}$ if and only if $\lim_{t\to 0} T^n(t+T)^{-n}x = x$.*
4) *$\overline{D(T)} = H$ and $H = N(T) \oplus \overline{R(T)}$. T is injective if and only if $R(T)$ is dense in H.*
5) *$T^* \in \mathrm{Sect}(\omega)$ and $\omega_T = \omega_{T^*}$.*

Proof.

1) Suppose that T is injective. We have that $T^{-1} : R(T) \subset H \to D(T) \subset H$. If $\lambda \neq 0$, $\lambda \in \sigma(T)$ if and only if $\frac{1}{\lambda} \in \sigma(T^{-1})$. Moreover $\lambda(\lambda + T^{-1})^{-1} = I - \frac{1}{\lambda}(\frac{1}{\lambda} + T)^{-1}$ provided that $\frac{-1}{\lambda} \in \rho(T)$. Then, $\sigma(T^{-1}) \subset \overline{S_\omega}$ and, for every $\alpha \in (\omega, \pi)$, $M(T^{-1}, \alpha) < \infty$.
2) Since $(t+T)^{-n}x \in D(A)$, for every $t > 0$, we deduce that $x \in \overline{D(A)}$ provided that $x = \lim_{t\to+\infty} t^n(t+T)^{-n}x$. Suppose that $x \in D(T)$. We can write

$$x = t(t+T)^{-1}x + \frac{1}{t}(t(t+T)^{-1})Tx.$$

By iteration of this equality we get

$$x = (t(t+T)^{-1})^n x + \frac{1}{t}\sum_{k=1}^{n}(t(t+T)^{-1})^k Tx.$$

Since $\sup_{t>0}\|t(t+T)^{-1}\| < \infty$, it follows that $\lim_{t\to+\infty}(t(t+T)^{-1})^n x = x$. By using again that $\sup_{t>0}\|t(t+T)^{-1}\| < \infty$, we deduce that $\lim_{t\to+\infty}(t(t+T)^{-1})^n y = y$, for every $y \in \overline{D(T)}$.

3) This property can be proved in a similar way as the previous one.

4) Let $x \in H$. Since $\sup_{n\in\mathbb{N}} \|n(n+T)^{-1}n\| < \infty$, there exists an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and $y \in H$ such that $\phi(n)(\phi(n)+T)^{-1}x \to y$, as $n \to \infty$, in the weak topology of H. That is,

$$\lim_{n\to\infty} \langle \phi(n)(\phi(n)+T)^{-1}x\,,\, z\rangle = \langle y\,,\, z\rangle, \quad z \in H.$$

This implies that $T(\phi(n)+T)^{-1}x \to x-y$, as $n \to \infty$, in the weak topology of H. Now, T is a closed operator. This means that $G(T)$ is a closed subspace of $H \times H$. Hence, $G(T)$ is also weakly closed in $H \times H$. Then, $x = y$. Since the closure of $D(T)$ in the weak topology of H coincides with the closure of $D(T)$ in H, we conclude that $x \in \overline{D(T)}$.

We now prove that $H = N(T) \oplus \overline{R(T)}$. Note that, according to 2), if $x \in N(T) \cap \overline{R(T)}$, then

$$0 = Tx = \lim_{t\to 0^+} (t+T)^{-1}Tx = \lim_{t\to 0^+} T(t+T)^{-1}x = x.$$

Hence, $N(T) \cap \overline{R(T)} = \{0\}$. Let $x \in H$. Since $\sup_{t>0} \|t(t+T)^{-1}x\| < \infty$, there exists a decreasing sequence $(t_n)_{n=1}^{\infty} \subset (0,\infty)$ such that $t_n \to 0$, and $t_n(t_n+T)^{-1}x \to y$, as $n \to \infty$, in the weak topology of H. On the other hand,

$$t_nT(t_n+T)^{-1}x = t_n(x - t_n(t_n+T)^{-1}) \to 0, \quad \text{as } n \to \infty,$$

in H. By recalling that $G(T)$ is a weakly closed subspace of H, it follows that $Ty = 0$. Moreover, $T(t_n+T)^{-1}x \to x-y$, in the weak topology of H. Hence, $x-y$ is in the weak closure of $R(T)$ that coincides with the closure of $R(T)$ in H. We conclude that $x \in N(T) + \overline{R(T)}$.

5) This property follows by taking into account that $p(T^*) = \{\bar{\lambda} : \lambda \in p(T)\}$ and that $R(\lambda, T)^* = R(\bar{\lambda}\,, T^*)$, for every $\lambda \in p(T)$. ∎

Before stating the last properties of sectorial operators, we give some definitions.

Definition 8. A collection of operators $\{T_i\}_{i\in I}$ is said to be **uniformly sectorial of angle** $\omega \in [0,\pi)$ when $T_i \in \mathrm{Sect}(\omega)$, $i \in I$, and, for every $\alpha \in (\omega,\pi)$, $\sup_{i\in I} M(T_i,\alpha) < \infty$. ◂

Definition 9. Suppose that the sequence $\{T_n\}_{n\leq 1}^{\infty}$ is uniformly sectorial of angle ω. We say that $\{T_n\}_{n\in\mathbb{N}}$ is a **sectorial approximation** on S_ω for an operator T when there exists $\lambda \notin \overline{S_\omega}$ such that $\lambda \in \rho(T)$ and $R(\lambda, T_n) \to R(\lambda, T)$, as $n \to \infty$, in $B(H)$. In this case we write $T_n \to T,\ (S_\omega)$. ◂

Note that if $\{T_n\}_{n\in\mathbb{N}}$ is a sectorial aproximation on S_ω for T, then $T \in \mathrm{Sect}(\omega)$ and, for every $\lambda \notin \overline{S_\omega}$, $R(\lambda, T_n) \to R(\lambda, T)$, in $B(H)$.

In the next proposition we present some properties concerning sectorial convergence.

Proposition 10. *Suppose that the sequence $\{T_n\}_{n=1}^{\infty}$ uniformly sectorial of angle ω.*

1) *If $T_n \to T,\ (S_\omega)$ and T_n, T are injective, then $T_n^{-1} \to T^{-1},\ (S_\omega)$.*
2) *If $T_n \to T(S_\omega)$ and $T \in B(H)$, then there exists $n_0 \in \mathbb{N}$ such that $T_n \in B(H)$ when $n \geq n_0$ and $T_n \to T$, as $n \to \infty$ in $B(H)$.*
3) *If $\{T_n\}_{n=1}^{\infty} \subset B(H)$, $T \in B(H)$ and $T_n \to T$, as $n \to \infty$ in $B(H)$, then $T_n \to T,\ (S_w)$.*

Proof.

1) Follows from proposition 7.1).
2) Assume that $T_n \to T,\ (S_\omega)$, and $T \in B(H)$. Then, $-R(-1, T_n) = (I+T_n)^{-1} \to (I+T)^{-1}$ in $B(H)$. The set of invertible operators in $B(H)$ is open in $B(H)$. Also if A_n, $n \in \mathbb{N}$, and A are in $B(H)$, they are invertible and $A_n \to A$, as $n \to \infty$ in $B(H)$, then $A_n^{-1} \to A^{-1}$ as $n \to \infty$ in $B(H)$. Thus 2) is proved.
3) Assume that $\{T_n\}_{n=1}^{\infty} \subset B(H)$, $T \in B(H)$ and $T_n \to T$ as $n \to \infty$ in $B(H)$, $I+T$ is invertible and $(I+T_n)^{-1} \to (I+T)^{-1}$, as $n \to \infty$ in $B(H)$. ∎

4. Spaces of holomorphic functions

In this section we present the definitions and the main properties of the spaces of holomorphic functions which will be used to define holomorphic functional calculus in the next section. We recall that $\mathcal{H}(\Omega)$ denotes the space of holomorphic functions on Ω.

Let $\varphi \in (0, \pi]$. We say that a function $f \in \mathcal{H}(S_\varphi)$ is in the Dunford–Riesz class on S_φ, shortly $f \in \mathcal{DR}(S_\varphi)$, when

i) $f \in H^\infty(S_\varphi)$, that is, f is bounded on S_φ;

ii) there exists $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $|z| \to \infty$ with $z \in S_\varphi$;

iii) there exists $\beta > 0$ such that $f(z) = O(|z|^\beta)$, as $|z| \to 0$ with $z \in S_\varphi$.

If there is no possible confusion about the sector, we may write $\mathcal{DR}$ instead of $\mathcal{DR}(S_\varphi)$. Note that, if $f \in \mathcal{DR}(S_\varphi)$, then the function $g(z) = f(1/z)$, $z \in S_\varphi$, is also in $\mathcal{DR}(S_\varphi)$. In short, a holomorphic function is in $\mathcal{DR}$ when it is bounded and tends to zero at the origin and at infinity sufficiently fast. Some examples of functions in $\mathcal{DR}(S_\varphi)$ are the following:

a) $f_1(z) = \frac{z}{1+z^2}$, $z \in S_\varphi$, $0 < \varphi < \frac{\pi}{2}$.

b) $f_2(z) = z\mathrm{e}^{-z}$, $z \in S_\varphi$, $0 < \varphi \leq \frac{\pi}{2}$.

c) $f_3(z) = \sqrt{z}\mathrm{e}^{-\sqrt{z}}$, $z \in S_\varphi$, $0 < \varphi \leq \pi$. Here, if $z = r\mathrm{e}^{\mathrm{i}\theta}$, with $r > 0$ and $\theta \in [-\pi, \pi)$, then $\sqrt{z} = \sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}$.

In the following proposition, equivalent definitions of the functions in $\mathcal{D}R(S_\varphi)$ are stated. The proof of these characterizations is straightforward from the definition.

Proposition 11. *Let $\varphi \in (0, \pi]$. Suppose that $f \in \mathcal{H}(S_\varphi)$. The following properties are equivalent:*

1) $f \in \mathcal{DR}(S_\varphi)$.

2) *There exist $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \min\{|z|^s, |z|^{-s}\}$, $z \in S_\varphi$.*

3) *There exist $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C\frac{|z|^s}{1+|z|^{2s}}$, $z \in S_\varphi$.*

We now define another space of functions, namely $\mathcal{DR}_0$. A function $f \in \mathcal{H}(S_\varphi)$ is said to be in $\mathcal{DR}_0(S_\varphi)$ when

1) $f \in H^\infty(S_\varphi)$;

2) there exist $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $|z| \to \infty$ with $z \in S_\varphi$;

3) there exist $r > 0$ and $F \in \mathcal{H}(B(0, r))$ such that $F(z) = f(z)$, $z \in S_\varphi \cap B(0, r)$.

In other words, a function in $\mathcal{DR}_0$ decays at infinity and is holomorphic in a neighborhood of the origin.

Suppose that $f \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$, that is $f = g + h$ with $g \in \mathcal{DR}(S_\varphi)$ and $h \in \mathcal{DR}_0(S_\varphi)$. Since h can be extended as a holomorphic function to a neighborhood of the origin, there exists $c \in \mathbb{C}$ such that $h(z) = c + O(|z|)$, as $z \to 0$ with $z \in S_\varphi$. On the other hand, if $f \in \mathcal{H}(S_\varphi)$ and $c \in \mathbb{C}$, we can write

$$f(z) = \frac{c}{1+z} + \frac{f(z) - c}{1+z} + \frac{z}{1+z}f(z), \quad z \in S_\varphi.$$

By taking these properties in mind, we can establish the following proposition.

Proposition 12. *Let $\varphi \in (0, \pi]$. Assume that $f \in \mathcal{H}(S_\varphi)$. The following assumptions are equivalent:*

1) $f \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$.

2) *$f \in H^\infty$ and it satisfies that*

    a) *there exists $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $|z| \to \infty$ with $z \in S_\varphi$;*

    b) *there exist $\beta > 0$ and $c \in \mathbb{C}$ such that $f(z) = c + O(|z|^\beta)$, as $|z| \to 0$ with $z \in S_\varphi$.*

Note that the function $f(z) = 1$ is not in $\mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$.

We introduce the last space of functions. By $\mathcal{A}(S_\varphi)$ we denote the function space formed by all those $f \in \mathcal{H}(S_\varphi)$ for which there exists $n \in \mathbb{N}$ such that $f(z)(1+z)^{-n} \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$. The functions in $\mathcal{A}(S_\varphi)$ can be characterized as shown in the next proposition.

Proposition 13. *Let $\varphi \in (0,\pi]$. Suppose that $f \in \mathcal{H}(S_\varphi)$. The following properties are equivalent:*

1) $f \in \mathcal{A}(S_\varphi)$.
2) *f has the following properties:*
    a) *$f \in \mathcal{H}_c(S_\varphi)$, that is, for every $0 < r < R < \infty$, f is bounded on $S_\varphi \cap \{z \in \mathbb{C} : r \leq |z| \leq R\}$;*
    b) *There exists $\alpha < 0$ such that $f(z) = O(|z|^\alpha)$, as $z \to \infty$ with $z \in S_\varphi$;*
    c) *There exist $\beta > 0$ and $c \in \mathbb{C}$ such that $f(z) = c + O(|z|^\beta)$, as $z \to 0$ with $z \in S_\varphi$.*
3) *There exist $c \in \mathbb{C}$, $n \in \mathbb{N}$ and $F \in \mathcal{DR}(S_\varphi)$ such that*

$$f(z) = c + (1+z)^n F(z), \quad z \in S_\varphi.$$

If, in addition, f is bounded, 3) holds with $n = 1$.

If $\omega \in [0,\pi)$ and $\mathcal{M}$ represents either $\mathcal{DR}$, $\mathcal{DR}_0$, or $\mathcal{A}$ we define

$$\mathcal{M}[S_\omega] = \bigcup_{\omega<\varphi\leq\pi} \mathcal{M}(S_\varphi).$$

5. Holomorphic functional calculus for sectorial operators

In this section we develop a holomorphic functional calculus for sectorial operators and functions in the spaces defined in the previous section. We will be able to give a meaning to the expression $f(T)$ when T is a sectorial operator and f is in the function spaces from the previous section. The basic tool is the Cauchy integral formula. We begin by defining the contours of the integrals.

Let $\varphi \in (0,\pi)$ and $\delta > 0$. We define the path $\Gamma_\varphi := \Gamma_\varphi^+ + \Gamma_\varphi^-$, where

$$\Gamma_\varphi^+(t) = -t\mathrm{e}^{\mathrm{i}\varphi}, \quad t \in (-\infty, 0], \quad \text{and} \quad \Gamma_\varphi^-(t) = t\mathrm{e}^{-\mathrm{i}\varphi}, \quad t \in (0,\infty).$$

Thus, Γ_φ is the boudnary of the sector S_φ oriented in the positive sense. We also consider the path $\Gamma_{\varphi,\delta} = \Gamma_{\varphi,\delta}^+ + \Gamma_{\varphi,\delta}^0 + \Gamma_{\varphi,\delta}^-$, where

$$\Gamma_{\varphi,\delta}^+(t) = -t\mathrm{e}^{\mathrm{i}\varphi},\ t \in (-\infty,-\delta];\ \Gamma_{\varphi,\delta}^0(\theta) = \delta\mathrm{e}^{\mathrm{i}\theta},\ \theta \in (\varphi, 2\pi-\varphi];\ \Gamma_{\varphi,\delta}^-(t) = t\mathrm{e}^{\mathrm{i}\varphi},\ t \in (\delta,\infty).$$

See figure 2. Note that $\Gamma_{\varphi,\delta}$ is the boundary of $S_\varphi \cup B(0,\delta)$ positively oriented. We can think that the paths Γ_φ and $\Gamma_{\varphi,\delta}$ go around the sector S_ω when $0 < \omega < \varphi$.

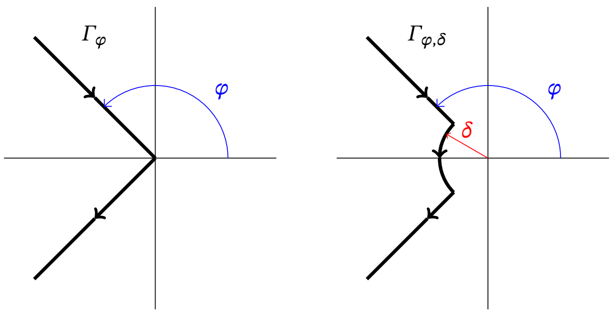

Figure 2: The contours Γ_φ and $\Gamma_{\varphi,\delta}$, for $\varphi = 3\pi/4$.

In the sequel of this section we assume that $T \in \mathrm{Sect}(\omega)$, where $\omega \in (0,\pi]$. Our objective is to define the operator $f(T)$ for every $f \in \mathcal{A}[S_\omega]$. Note that we cannot assure that $\sigma(T) \subset S_\varphi$ when $\varphi \in (\omega,\pi]$.

We first define $f(T)$ when $f \in \mathcal{DR}(S_\varphi)$ and $\varphi \in (\omega,\pi]$.

Proposition 14. *Let $\varphi \in (\omega, \pi]$. Suppose that $f \in \mathcal{DR}(S_\varphi)$. If $\omega < \lambda < \varphi$, the $B(H)$-Bochner integral*

$$\int_{\Gamma_\lambda} f(z)R(z,T)\,\mathrm{d}z$$

converges absolutely (with the $B(H)$ norm). Moreover, if $\omega < \lambda, \lambda' < \varphi$, then

$$\int_{\Gamma_\lambda} f(z)R(z,T)\,\mathrm{d}z = \int_{\Gamma_{\lambda'}} f(z)R(z,T)\,\mathrm{d}z.$$

Proof. The convergence of the integral can be shown using proposition 11.2) and splitting the integral near zero and near infinity. The second statement follows from the Hahn-Banach theorem and the Cauchy integral theorem. ∎

Following proposition 14, we define the following functional calculus. If $\varphi \in (\omega, \pi]$ and $f \in \mathcal{DR}(S_\varphi)$, we define $f(T)$ by

$$f(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z)R(z,T)\,\mathrm{d}z,$$

where $\lambda \in (\omega, \varphi)$. Thus $f(T) \in B(H)$ when $f \in \mathcal{DR}(S_\varphi)$ and $\varphi \in (\omega, \pi]$.

Assume now that $f \in \mathcal{DR}_0(S_\varphi)$. We cannot ensure that the integral in proposition 14 converges if $f(0) \neq 0$. We choose $\delta > 0$ so that f can be holomorphically extended to $B(0,\delta) \setminus S_\varphi$. Proceeding similarly we can define

$$f(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} f(z)R(z,T)\,\mathrm{d}z,$$

where δ, λ are as above, since one can check that the integral is absolutely convergent. Thus, $f(T) \in B(H)$. Note that if $f(0) = 0$, then $f \in \mathcal{DR} \cap \mathcal{DR}_0$ and both definitions for $f(T)$ coincide, by the Cauchy integral theorem. If $f = g + h$ with $g \in \mathcal{DR}$ and $h \in \mathcal{DR}_0$, we define $f(T) = g(T) + h(T)$. This definition does not depend on the choice of g and h. Some properties of the functional calculus we have defined are contained in the following proposition.

Proposition 15. *Let $\varphi \in (\omega, \pi]$.*

1) *Suppose that $x \in N(T)$ and $f = g + h$, where $g \in \mathcal{DR}(S_\varphi)$ and $h \in \mathcal{DR}_0(S_\varphi)$. Then, $f(T)(x) = h(0)x$.*
2) *The mapping $\Psi\colon \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi) \to B(H)$ defined by $\Psi(f) = f(T)$ is a homomorphism of algebras.*
3) *If $f_\mu(z) = \frac{1}{\mu - z}$, where $\mu \notin \overline{S_\varphi}$, then $f_\mu \in \mathcal{DR}_0(S_\varphi)$ and $f_\mu(T) = R(\mu, T)$.*
4) *Suppose that $A \in C(H)$ commutes with $R(\lambda, T)$ for every $\lambda \in \rho(T)$. Then, $Af(T) = f(T)A$ for every $f \in \mathcal{DR} + \mathcal{DR}_0$.*

Proof.

1) Since $x \in N(T)$, we can write, for any $z \in \rho(T)$,

$$zR(z,T)x = zR(z,T)x - R(z,T)Tx = \big(zR(z,T) - TR(z,T)\big)x = (z - T)R(z,T)x = x.$$

Then, by taking $\lambda \in (\omega, \varphi)$ and $\delta > 0$ such that h can be holomorphically extended to $B(0,\delta) \setminus S_\varphi$,

$$\begin{aligned} f(T)x &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} g(z)R(z,T)x\,\mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} h(z)R(z,T)x\,\mathrm{d}z \\ &= \left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \frac{g(z)}{z}\,\mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} \frac{h(z)}{z}\,\mathrm{d}z \right) x. \end{aligned}$$

By using the Cauchy integral theorem, since $g \in \mathcal{DR}$ and $h \in \mathcal{DR}_0$, we conclude that

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \frac{g(z)}{z}\,\mathrm{d}z = 0 \qquad \text{and} \qquad \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} \frac{h(z)}{z}\,\mathrm{d}z = h(0).$$

The first statement is proved.

2) All the properties of an algebra homomorphism are straightforward to prove except for

$$(fF)(T) = f(T) \circ F(T),$$

for every $f, F \in \mathcal{DR} + \mathcal{DR}_0$. The proof is easy but fairly long; it can be done by using some properties of the resolvent operator and the Fubini and Cauchy integral theorems.

3) Let $\mu \notin \overline{S_\varphi}$. It is clear that $f_\mu \in \mathcal{DR}_0(S_\varphi)$. We have that

$$f_\mu(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda,\delta}} \frac{R(z,T)}{\mu - z} \, \mathrm{d}z,$$

where $\lambda \in (\omega, \varphi)$ and $\delta < |\mu|$. The function $\Psi(z) = R(z,T)$ is holomorphic in $\rho(T)$ taking values in $B(H)$. We finish by using the Hahn-Banach and Cauchy integral theorems.

4) Suppose that $f \in \mathcal{DR}(S_\varphi)$. In the general case we could proceed in a similar way. We have that

$$f(T)x = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z) R(z,T) x \, \mathrm{d}z, \qquad x \in H,$$

where $\lambda \in (\omega, \varphi)$. Let $x \in D(A)$. It follows that

$$f(T)Ax = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z) R(z,T) Ax \, \mathrm{d}z.$$

The last $B(H)$-Bochner integral is absolutely convergent in the $H-$norm. The standard properties of the $B(H)$-Bochner integral lead to $f(T)Ax = Af(T)x$. ■

Finally, we define $f(T)$ for every $f \in \mathcal{A}(S_\varphi)$. Let $f \in \mathcal{A}(S_\varphi)$ with $\varphi \in (\omega, \pi]$. There exists $n \in \mathbb{N}$ such that $f(z)(1+z)^n \in \mathcal{DR} + \mathcal{DR}_0$. Therefore, we can define the operator $f_n(T)$ by

$$f_n(T) = (I+T)^n \left(\frac{f}{p^n}\right)(T),$$

where $p(z) = 1 + z$. We make two small remarks about this definition. Note firstly that, since $(I+T)^{-n} \in B(H)$, $(I+T)^n$ is a closed operator. Then, the operator $f_n(T)$ is closed because $(f/p^n)(T) \in B(H)$. Secondly, one can prove that the definition of f_n does not really depend on n, as long as $fp^{-n} \in \mathcal{DR} + \mathcal{DR}_0$. The proof is not difficult but it is a bit technical, so we omit it. Therefore, we define $f(T) = f_n(T)$ for any admissible n.

Some properties of this functional calculus are summarized in the following proposition. For the sake of conciseness, we omit the proof, which is easy but tedious.

Proposition 16. *Let $\varphi \in (\omega, \pi]$, and $f \in \mathcal{A}(S_\varphi)$.*

i) *If $T \in B(H)$, then $f(T) \in B(H)$.*
ii) *If $S \in B(H)$ and S commutes with T, then S commutes with $f(T)$.*
iii) *Suppose that $g \in \mathcal{A}(S_\varphi)$. We have that*

$$f(T) + g(T) \subset (f+g)(T) \quad \text{and} \quad f(T)g(T) \subset (fg)(T).$$

Furthermore, $D\big((fg)(T)\big) \cap D\big(g(T)\big) = D\big(f(T)g(T)\big)$.

We cannot say that the natural functional calculus commutes with the sum and the product of functions (see proposition 16.iii)). But if $f, g \in \mathcal{A}(S_\varphi)$, we can see that

$$f(T) + g(T) = (f+g)(T),$$
$$f(T)g(T) = (fg)(T),$$

provided that $g(T) \in B(H)$. For instance, if $f(z) = c + (1+z)^n g(z)$, $z \in S_\varphi$, where $c \in \mathbb{C}$, $n \in \mathbb{N}$, and $g \in DR(S_\varphi)$, then $f(T) = c + (1+T)^n g(T)$.

If $0 < \varphi \leq \pi$ and $f \in H(S_\varphi)$, we define the function f^* by

$$f^*(z) = \overline{f(\overline{z})}, \quad z \in S_\varphi.$$

Thus, $f^* \in \mathcal{H}(S_\varphi)$, where f^* is named the conjugate of the function f. It is clear that the function spaces that we have considered in section 4 are invariant with respect to conjugation.

In the following proposition we show how the natural functional calculus acts on adjoints.

Proposition 17. *Let $\varphi \in (\omega, \pi]$ and $f \in \mathcal{A}(S_\varphi)$. Then, $f(T^*) = (f^*(T))^*$.*

Proof. Suppose firstly that $f \in \mathcal{DR}(S_\varphi)$. We have that

$$f^*(T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f^*(z) R(z, T)\, \mathrm{d}z \in B(H).$$

Here $\omega < \lambda < \varphi$. For every $x, y \in H$,

$$\begin{aligned}
\langle f^*(T)x, y\rangle &= \Big\langle \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \overline{f(\overline{z})} R(z, T)x\, \mathrm{d}z, y \Big\rangle \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \overline{f(\overline{z})} \langle R(z, T)x, y\rangle\, \mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} \overline{f(\overline{z})} \langle x, R(\overline{z}, T^*)y\rangle\, \mathrm{d}z \\
&= \Big\langle x, \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\lambda} f(z) R(z, T^*)y\, \mathrm{d}z \Big\rangle \\
&= \langle x, f(T^*)y\rangle.
\end{aligned}$$

We have taken into account that all the Böchner integrals that appear are absolutely convergent. Then, $(f^*(T))^* = f(T^*)$.

If $f \in \mathcal{DR}_0(S_\varphi)$ we can proceed in a similar way. Assume now that $f \in \mathcal{A}(S_\varphi)$. There exist $n \in \mathbb{N}$, $g \in \mathcal{DR}(S_\varphi)$ and $h \in \mathcal{DR}_0(S_\varphi)$ such that $\frac{f(z)}{(1+z)^n} = g(z) + h(z)$, $z \in S_\varphi$. Then,

$$f(T^*) = (1 + T^*)^n F(T^*),$$

where $F = g + h$. Here $(F^*(T))^* = (g^*(T))^* + (h^*(T))^* \in B(H)$. Moreover, we can write

$$\begin{aligned}
(1 + T^*)^n &= (-1)^n (1 + T^*)^n [R(-1, T))^n]^* [(1 + T)^n]^* \\
&= (-1)^n (1 + T^*) R(-1, T^*)^n [(1 + T)^n]^* = [(1 + T)^n]^*.
\end{aligned}$$

We get

$$f(T^*) = [(1 + T)^n]^* (F^*(T))^* = (F^*(T)(1 + T)^n)^*.$$

According to proposition 15.4), for every $x \in D(T^n) = D((1 + T)^n)$, we have

$$F^*(T)(1 + T)^n x = (1 + T)^n F^*(T)x,$$

or, in other words, $D(T^n) \subset D(f^*(T))$. Since $\overline{D(T)} = \mathcal{H}$, using proposition 7.2), we deduce that $(t + (t + T)^{-1})^n x \to x$, as $t \to \infty$, and $f^*(T)(t+(t+T)^{-1})^n x = (t(t+T)^{-1})^n f^*(T)x \to f^*(T)x$, as $t \to \infty$. Furthermore, $(t + (t + T)^{-1})^n x \in D(T^n)$, for every $t > 0$.

This fact allows us to conclude that the closure $\overline{f^*(T)_{|D(T^n)}} = f^*(T)$ can be written as

$$f(T^*) = (f^*(T)_{|D(T^n)})^* = (\overline{f^*(T)_{|D(T^n)}})^* = (f^*(T))^*. \qquad \blacksquare$$

We can find in the literature extensions of the natural functional calculus. There exists a trade-off between operators and function spaces. When the operator is better we can define a functional calculus for a wider class of functions and vice-versa. We finish this section with the following definition.

Definition 18. Let $\varphi \in (\omega, \pi]$. Suppose that $\mathcal{F}$ is an algebra contained in $\mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$. The natural functional calculus on $\mathcal{F}$ for T is said to be **bounded** when $f(T) \in B(H)$, for every $f \in \mathcal{F}$, and there exists $C > 0$ such that

$$\|f(T)\| \leq C\|f\|_\infty, \qquad \forall f \in \mathcal{F}.$$

◂

6. An operator without a bounded H^∞-calculus

McIntosh and Yagi [18] presented the first example of a sectorial operator without a bounded H^∞-calculus. We are going to give an example that can be found in the work of Le Merdy [11] (see also Haase's thesis [7, p. 49]).

Suppose that H is a separable Hilbert space. We can think for instance $H = \ell^2(\mathbb{N})$. We say that a sequence $\{e_n\}_{n=1}^\infty$ in H is a conditional basis in H when the following two properties hold:

i) For every $x \in H$ there exists a unique sequence $(\mu_n)_{n=1}^\infty \subset \mathbb{C}$ such that $x = \sum_{n=1}^\infty \mu_n e_n$.
ii) There exist a sequence $(\mu_n)_{n=1}^\infty \subset \mathbb{C}$ of complex numbers and a sequence $(\theta_n)_{n=1}^\infty \subset \{-1, 1\}$ of signs such that $\sum_{n=1}^\infty \mu_n e_n$ converges in H, but $\sum_{n=1}^\infty \theta_n \mu_n e_n$ does not converge in H.

According to Lindenstrauss and Tzafriri [16, Proposition 2.b.11], there exists a conditional basis $\{e_n\}_{n=1}^\infty$ of H. We may assume that $\|e_n\| = 1$ for all $n \in \mathbb{N}$. We define the n-th projection operator P_n in the standard way:

$$\begin{aligned} P_n :\ & H \longrightarrow H \\ x = \sum_{k=1}^\infty \mu_k e_k \longmapsto & P_n(x) = \sum_{k=1}^n \mu_k e_k. \end{aligned}$$

For every $n \in \mathbb{N}$, $P_n \in B(H)$. Also, one can prove that the sequence $\{P_n\}_{n=1}^\infty$ is bounded in $B(H)$. The constant $M_D = \sup_{n\in\mathbb{N}} \|P_n\|$ is known as the constant basis for $\{e_n\}_{n=1}^\infty$. This constant plays an important role in the theory of basis, but we will only use that M_D is finite.

Let $a = (a_n)_{n=1}^\infty$ be a complex sequence. We define the multiplier operator associated to a as follows: if $x = \sum_{n=1}^\infty \mu_n e_n \in H$ is such that $\sum_{n=1}^\infty a_n \mu_n e_n$ converges in H, then we set

$$T_a x = \sum_{n=1}^\infty a_n \mu_n e_n.$$

These multipliers operators were studied by Venni [23]. Note that the domain

$$D(T_a) = \left\{ \sum_{n=1}^\infty \mu_n e_n \,:\, \sum_{n=1}^\infty a_n \mu_n e_n \in H \right\}$$

is dense in H because the dense set $\left\{ \sum_{n=1}^M \mu_n e_n \in H \,:\, \mu_n \in \mathbb{C}, M \in \mathbb{N} \right\}$ is contained in $D(T_a)$.

Proposition 19. *The multiplier operator T_a is a closed operator.*

Proof. Suppose that $\{x_k\}_{k=1}^\infty \subset D(T_a)$ is such that $x_k \to y$ and $T_a x_k \to z$ as $k \to \infty$, for some $y, z \in H$. Then,

$$\begin{aligned} P_1(x_k) &\xrightarrow{k\to\infty} P_1 y, \\ P_1(T_a x_k) = a_1 P_1 x_k &\xrightarrow{k\to\infty} P_1 z, \end{aligned}$$

and, in general, for every $n \in \mathbb{N}$

$$\begin{aligned} P_{n+1}(x_k) - P_n(x_k) &\xrightarrow{k\to\infty} P_{n+1}(y) - P_n(y), \\ P_{n+1}(T_a x_k) - P_n(T_a x_k) &\xrightarrow{k\to\infty} P_{n+1}(z) - P_n(z). \end{aligned}$$

We deduce that $a_1 P_1(y) = P_1(z)$ and, for every $n \in \mathbb{N}$, $a_{n+1}(P_{n+1}(y) - P_n(y)) = P_{n+1}(z) - P_n(z)$. Hence $y \in D(T_a)$ and $T_a y = z$. ■

Let $(a_n)_{n=1}^\infty$ be a sequence of complex numbers. We define

$$\|a\| = \limsup_{n\to\infty} |a_n| + \sum_{n=1}^{\infty} |a_{n+1} - a_n|.$$

This quantity can be infinite. When $\sum_{n=1}^\infty |a_{n+1} - a_n| < \infty$ we say that $(a_n)_{n=1}^\infty$ has bounded variation, and in this case $(a_n)_{n=1}^\infty$ is convergent.

Proposition 20. *Let $a = (a_k)$ be a complex sequence and T_a the associated multiplier operator. If $\|a\| < \infty$, then $T_a \in B(H)$ and $\|T_a\| \le M_0\|a\|$.*

Proof. Suppose that $x = \sum_{k=1}^\infty \mu_k e_k \in H$ with $(\mu_k)_{k=1}^\infty \subset \mathbb{C}$. For $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} a_k\mu_k e_k = \sum_{k=1}^{n} a_k(P_k - P_{k-1})x + a_1P_1x = \sum_{k=1}^{n-1}(a_k - a_{k+1})P_kx + a_nP_nx.$$

Since $\|a\| < \infty$, $\{a_nP_nx\}_{n=1}^\infty$ converges and $\{\sum_{k=1}^\infty (a_k - a_{k+1})P_kx\}_{k=1}^\infty$ is absolutely convergent in norm. Hence, $\sum_{k=1}^\infty a_k\mu_k e_k$ converges and

$$\left\|\sum_{k=1}^{\infty} a_k\mu_k e_k\right\| \le M_0\|a\|\|x\|.$$ ∎

Let $\lambda \in \mathbb{C}$ such that $\lambda \ne a_n$, $n \in \mathbb{N}$. Then, the multiplier operator associated to the sequence $\lambda - a = \{\lambda - a_n\}_{n=1}^\infty$ is $T_{\lambda - a} = \lambda I - T_a$ and it is injective. Moreover, $(\lambda I - T_a)^{-1}$ is the multiplier operator associeated to the sequence $\{\frac{1}{\lambda - a_n}\}_{n=1}^\infty$.

Proposition 21. *Let $a = (a_n)_{n=1}^\infty$ be an increasing sequence of real numbers satisfying that $a_1 > 0$ and $\lim_{n\to\infty} a_n = \infty$. Then, the associated multiplier operator T_a is sectorial of angle 0.*

Proof. Let $\lambda \in \mathbb{C} \setminus [a_1, \infty)$. We define $a(\lambda) = (\frac{1}{\lambda - a_n})_{n=1}^\infty$. We can write

$$\|a(\lambda)\| = \sum_{n=1}^{\infty}\left|\frac{1}{\lambda - a_{n+1}} - \frac{1}{\lambda - a_n}\right| = \sum_{n=1}^{\infty}\left|\int_{a_n}^{a_{n+1}} \frac{\mathrm{d}t}{(\lambda - t)^2}\right| \le \int_{a_1}^{\infty} \frac{\mathrm{d}t}{|\lambda - t|^2}.$$

Hence, $\lambda \in \rho(T_a)$. We conclude that $\sigma(T_a) \subset [a_1, \infty)$. Also, if $\lambda = re^{\mathrm{i}\theta}$, with $r > 0$ and $\nu \in [-\pi, \pi)$ we get

$$\|\lambda(\lambda I - T_a)^{-1}\| \le M_0|\lambda|\|a(\lambda)\| \le M_0\int_0^\infty \frac{r\,\mathrm{d}t}{re^{\mathrm{i}\theta} - t|^2} = M_0\int_0^\infty \frac{\mathrm{d}u}{|e^{\mathrm{i}\theta} - u|^2}.$$

If $\varphi \in (0, \frac{\pi}{2})$ we have that

$$\int_0^\infty \frac{\mathrm{d}u}{|e^{\mathrm{i}\theta} - u|^2} = \int_0^\infty \frac{\mathrm{d}u}{(\cos\theta - u)^2 + \sin^2\theta}$$

is bounded by

$$\int_0^\infty \frac{\mathrm{d}u}{u^2 + \sin^2\varphi}, \qquad \text{for } \theta \in [\pi/2, \pi - \varphi] \cup [\varphi - \pi, -\pi/2];$$

$$\int_0^\infty \frac{\mathrm{d}u}{(\cos\varphi + u)^2}, \qquad \text{for } \theta \in (\pi - \varphi, \pi) \cup [-\pi, \varphi - \pi];$$

$$\int_0^{2\cos\theta} \frac{\mathrm{d}u}{\sin^2\varphi} + \int_{2\cos\theta}^\infty \frac{\mathrm{d}u}{(\cos\theta - u)^2 + \sin^2\varphi}$$
$$\le \frac{2\cos\theta}{\sin^2\varphi} + \int_0^\infty \frac{\mathrm{d}u}{\frac{1}{4}u^2 + \sin^2\varphi}, \qquad \text{for } \theta \in [\varphi, \pi/2] \cup [-\pi/2, -\varphi].$$

Hence, $\|\lambda(\lambda I - T_a)^{-1}\| \le C_\varphi$, $\lambda \notin \overline{S_\varphi}$. Thus, we prove that $T_a \in \mathrm{Sect}(0)$. ∎

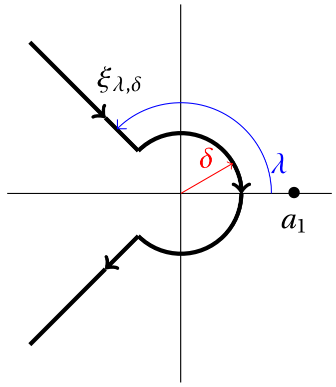

Figure 3: The contour $\xi_{\lambda,\delta}$ and a_1.

Finally, let us show that if $a = (2^n)_{n=1}^{\infty}$, then the operator T_a does not have a bounded functional calculus. Let $f \in \mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$ where $\varphi \in (0, \pi]$. There exists $m \in \mathbb{N}$ such that $f(z)(1+z)^{-m} = g(z) \in \mathcal{DR} + \mathcal{DR}_0$. Since $\sigma(T_a) \subset [a_1, +\infty)$ we can write

$$g(T_a) = \frac{1}{2\pi \mathrm{i}} \int_{\xi_{\lambda,\delta}} g(z) R(z, T_a) \,\mathrm{d}z,$$

where the path $\xi_{\lambda,\delta} = \xi_{\lambda,\delta}^{+} + \xi_{\lambda,\delta}^{0} + \xi_{\lambda,\delta}^{-}$, with $0 < \lambda < \varphi$, $0 < \delta < a_1$ and

$$\xi_{\lambda,\delta}^{+}(t) = -t\mathrm{e}^{\mathrm{i}\lambda}, \; t \in (-\infty, -\delta], \; \xi_{\lambda,\delta}^{-}(t) = t\mathrm{e}^{-\mathrm{i}\lambda}, \; t \in [\delta, \infty), \; \xi_{\lambda,\delta}^{0}(t) = \delta \mathrm{e}^{-\mathrm{i}\theta}, \; \theta \in (-\lambda, +\lambda).$$

See figure 3. By using the Hahn-Banach and Cauchy integral theorems we deduce that

$$g(T_a)(e_n) = \frac{1}{2\pi \mathrm{i}} \int_{\xi_{\lambda,\delta}} g(z) R(z, T_a) e_n \,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\xi_{\lambda,\delta}} \frac{g(z)}{a_n - z} \,\mathrm{d}z \, e_n = g(a_n) \, e_n.$$

Then, if $(\mu_k)_{k=1}^{n} \subset \mathbb{C}$ with $n \in \mathbb{N}$,

$$f(T_a)\left(\sum_{k=1}^{n} \mu_k e_k\right) = \sum_{k=1}^{n} \mu_k (1+T)^m g(a_k) \, e_k = \sum_{k=1}^{n} \mu_k f(a_k) \, e_k.$$

If the natural functional calculus on $\mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$ for T_a were bounded, then we would have $f(T_a) \in B(H)$ and, hence, the series $\sum_{k=1}^{\infty} \mu_k f(a_k) e_k$ would converge in H provided that $\sum_{k=1}^{\infty} \mu_k e_k \in H$.

Now, take the sequence $a_n = 2^n$, $n \in \mathbb{N}$. Since $\{e_n\}$ is a conditional basis, there exist two sequences $(\mu_n)_{n=1}^{\infty} \subset \mathbb{C}$ and $(\theta_n)_{n=1}^{\infty} \subset \{-1, 1\}$ such that $\sum_{n=1}^{\infty} \mu_n e_n$ converges in H and $\sum_{n=1}^{\infty} \theta_n \mu_n e_n$ does not converge in H. According to Garnett [6, Theorem 1.1, VII.1], there exists $f \in H^\infty(S_{\pi/2}) \cap \mathcal{A}(S_{\pi/2})$ such that $f(2^n) = \theta_n$, $n \in \mathbb{N}$. We conclude that, if T_a is the multiplier operator associated with $\{a_n\}_{n=1}^{\infty}$, $f(T_a) \notin B(H)$. Hence, the natural functional calculus on $\mathcal{A}(S_\varphi) \cap H^\infty(S_\varphi)$ for T_a is not bounded.

Multiplier operators like those we have just studied have also been considered to obtain examples related to certain operator theoretical problems [2, 9, 10, 12, 21, 23].

References

[1] Auscher, Pascal; McIntosh, Alan, and Nahmod, Andrea. “Holomorphic functional calculi of operators, quadratic estimates and interpolation”. In: *Indiana University Mathematics Journal* 46.2 (1997), pp. 375–403. ISSN: 0022-2518. https://doi.org/10.1512/iumj.1997.46.1180.

[2] Baillon, Jean Baptiste and Clément, Philippe. “Examples of unbounded imaginary powers of operators”. In: *Journal of Functional Analysis* 100.2 (1991), pp. 419–434. ISSN: 0022-1236. https://doi.org/10.1016/0022-1236(91)90119-P.

[3] Bochner, Salomon. "Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind". In: *Fundamenta Mathematicae* 20.1 (1933), pp. 262–176. URL: http://eudml.org/doc/212635.

[4] Cowling, Michael; Doust, Ian; McIntosh, Alan, and Yagi, Atsushi. "Banach space operators with a bounded H^∞ functional calculus". In: *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics* 60.1 (1996), pp. 51–89. ISSN: 0263-6115. https://doi.org/10.1017/S1446788700037393.

[5] Dunford, Nelson and Schwartz, Jacob T. *Linear operators.* Part 1. Wiley Classics Library. New York: John Wiley & Sons, Inc., 1988. ISBN: 978-0-471-60848-6.

[6] Garnett, John B. *Bounded analytic functions.* 1st ed. Graduate Texts in Mathematics 236. New York: Springer, 2007. https://doi.org/10.1007/0-387-49763-3.

[7] Haase, Markus. *The functional calculus for sectorial operators and similarity methods.* PhD thesis. Fakultät für Mathematik, Universität Ulm, 2003.

[8] Haase, Markus. "A general framework for holomorphic functional calculi". In: *Proceedings of the Edinburgh Mathematical Society* 48.2 (2005), pp. 423–444. ISSN: 0013-0915. https://doi.org/10.1017/S0013091504000513.

[9] Kalton, Nigel. "Applications of Banach space theory to sectorial operators". In: *Recent Progress in Functional Analysis.* International Functional Analysis Meeting on the Occasion of the 70th Birthday of Professor Manuel Valdivia (Valencia, 2000). North-Holland Mathematics Studies 189. Elsevier, 2001, pp. 61–74. https://doi.org/10.1016/s0304-0208(01)80036-0.

[10] Lancien, Gilles. "Counterexamples concerning sectorial operators". In: *Archiv der Mathematik* 71.5 (1998), pp. 388–398. ISSN: 0003-889X. https://doi.org/10.1007/s000130050282.

[11] Le Merdy, Christian. "H^∞-functional calculus and applications to maximal regularity". In: *Semi-groupes d'opérateurs et calcul fonctionnel* (Besançon, 1998). Publications mathématiques de l'UFR Sciences et techniques de Besançon 16. Besançon: Université de Franche-Comté, 1999, pp. 41–77.

[12] Le Merdy, Christian. "A bounded compact semigroup on Hilbert space not similar to a contraction one". In: *Semigroups of operators: theory and applications* (Newport Beach, 1998). Progress in Nonlinear Differential Equations and Their Applications 42. Basel: Birkhäuser, 2000, pp. 213–216.

[13] Le Merdy, Christian. "The Weiss conjecture for bounded analytic semigroups". In: *Journal of the London Mathematical Society. Second Series* 67.3 (2003), pp. 715–738. ISSN: 0024-6107. https://doi.org/10.1112/S002461070200399X.

[14] Le Merdy, Christian. "On square functions associated to sectorial operators". In: *Bulletin de la Société Mathématique de France* 132.1 (2004), pp. 137–156. ISSN: 0037-9484. https://doi.org/10.24033/bsmf.2462.

[15] Le Merdy, Christian. "Square functions, bounded analytic semigroups, and applications". In: *Perspectives in operator theory.* Banach Center Publications 75. Warsaw: Institute of Mathematics, Polish Academy of Sciences, 2007, pp. 191–220. https://doi.org/10.4064/bc75-0-12.

[16] Lindenstrauss, Joram and Tzafriri, Lior. *Classical Banach spaces. I.* Sequence spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete 92. Berlin, New York: Springer, 1977. ISBN: 978-3-540-08072-5.

[17] McIntosh, Alan. "Operators which have an H_∞ functional calculus". In: *Miniconference on operator theory and partial differential equations* (North Ryde, 1986). Proceedings of the Centre for Mathematical Analysis 14. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University. Canberra, 1986, pp. 210–231. URL: https://projecteuclid.org/euclid.pcma/1416336602.

[18] McIntosh, Alan and Yagi, Atsushi. "Operators of type ω without a bounded H_∞ functional calculus". In: *Miniconference on Operators in Analysis* (Sydney, 1990). Proceedings of the Centre for Mathematical Analysis 24. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University. Canberra, 1990, pp. 159–172. URL: https://projecteuclid.org/euclid.pcma/1416335070.

[19] Rudin, Walter. *Real and complex analysis.* 3rd ed. McGraw-Hill series in higher mathematics. New York: McGraw-Hill, 1987. ISBN: 978-0-07-054234-1.

[20] RUDIN, Walter. *Functional analysis*. 2nd ed. International Series in Pure and Applied Mathematics. New York: McGraw-Hill, 1991. ISBN: 978-0-07-054236-5.

[21] SIMARD, Arnaud. “Counterexamples concerning powers of sectorial operators on a Hilbert space”. In: *Bulletin of the Australian Mathematical Society* 60.3 (1999), pp. 459–468. ISSN: 0004-9727. https://doi.org/10.1017/S0004972700036613.

[22] STEIN, Elias M. *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*. Annals of Mathematics Studies 63. Princeton: Princeton University Press, 1970. ISBN: 978-0-691-08067-3. URL: http://www.jstor.org/stable/j.ctt1bgzbcx.

[23] VENNI, Alberto. “A counterexample concerning imaginary powers of linear operators”. In: *Functional Analysis and Related Topics*. International Conference in Memory of Professor Kôsaku Yosida (Kyoto, 1991). Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1993, pp. 381–387. https://doi.org/10.1007/BFb0085493.

[24] WEIS, Lutz. “The H^∞-holomorphic functional calculus for sectorial operators — a survey”. In: *Partial Differential Equations and Functional Analysis: The Philippe Clément Festschrift*. Ed. by Koelink, Erik; Neerven, Jan van; Pagter, Ben de; Sweers, Guido; Luger, Annemarie, and Woracek, Harald. Operator Theory: Advances and Applications 168. Basel: Birkhäuser, 2006, pp. 263–294. https://doi.org/10.1007/3-7643-7601-5_16.

IX *Escuela-Taller de Análisis Funcional* (2019)

Geometry of polynomial spaces and polynomial inequalities

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Abstract: This paper summarises the contents of the course on geometry of polynomial spaces and polynomial inequalities delivered at the 9th workshop of Functional Analysis organised by the Spanish Functional Analysis Network in Bilbao between the 3rd and the 8th of March, 2019, *in memoriam* of Prof. Bernardo Cascales. We first survey the most relevant results needed to understand polynomials in normed spaces. Then, we provide a few examples of polynomial spaces whose extreme points are fully described, and a couple of applications of the so-called Krein-Milman approach to obtain several sharp polynomial inequalities.

Resumen: Este artículo resume el contenido del curso sobre geometría de espacios polinomiales y desigualdades polinomiales que tuvo lugar en la IX Escuela-Taller de Análisis Funcional organizada por la Red de Análisis Funcional y Aplicaciones en Bilbao entre el 3 y el 8 de marzo de 2019, *in memoriam* al profesor Bernardo Cascales. Repasamos los resultados más relevantes para entender los polinomios en espacios normados. Después proporcionamos algunos ejemplos de espacios polinomiales cuyos puntos extremos están completamente descritos y un par de aplicaciones del llamado método de Klein-Milman para obtener desigualdades polinomiales óptimas.

Keywords: Bernstein and Markov inequalities, unconditional constants, polarizations constants, polynomial inequalities, homogeneous polynomials, extreme points.

MSC2010: 46G25, 46B28, 41A44.

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1. Introduction

This paper contains three main blocks. The first one is devoted to introducing polynomials in normed spaces. Although polynomials in a finite number of variables are well known to all undergraduate students, it is not so clear what polynomials in infinitely many variables are. In this block we will present all definitions and basic results required to understand polynomials in an arbitrary normed space, including the polarisation formula and polarisation constants. The second block deals with the geometry of polynomial spaces. The reader is referred to Dineen's book [15] for a modern monograph on polynomials on normed spaces. The characterisation of the extreme points of polynomial spaces is a question that has called the interest of a significant number of researchers in the last decades [1, 5, 10, 11, 18, 22, 24, 28–31, 33, 34]. This problem conveys a tremendous difficulty in most infinite (or even finite) dimensional polynomial spaces of interest, but in very specific cases, a complete and explicit description of the extreme points can be given. We will focus on a number of these particular examples, providing the reader not only with the extreme points of several 3-dimensional polynomial spaces, but also with a formula to calculate the polynomial norm, a parametrisation of the unit sphere and nice pictures of the unit balls of those spaces. Finally, a third section contains the applications of the geometrical results of the second block. In order to understand the applications, a precise introduction to several well-known polynomial inequalities will be provided. It is possible to find a vast diversity of applications in the literature, and therefore we will make a (restrictive) selection consisting of two types of polynomial inequalities, namely, the polynomial Bohnenblust-Hille inequality and the Bernstein-Markov type inequalities.

The arrangement described in the previous paragraph respects the structure of the course on geometry of polynomial spaces and applications delivered during the 9th workshop of Functional Analysis organized by the Spanish Functional Analysis Network in Bilbao between the 3rd and the 8th of March, 2019, *in memoriam* of Prof. Bernardo Cascales.

2. Polynomials in normed spaces

In this section we present the essential definitions and results needed to understand polynomials in normed spaces. We begin by recalling a number of basic concepts and definitions related to polynomials in a finite number of variables. In order to handle monomials in $\mathbb{K}^n$, we introduce the following notation. An n-dimensional multiindex is an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N} \cup \{0\}$ for all $i = 1, \dots, n$. If $x = (x_1, \dots, x_n) \in \mathbb{K}^n$, then $|\alpha|$, $\alpha!$ and x^α represent, respectively,

$$\alpha_1 + \dots + \alpha_n, \qquad \alpha_1! \cdots \alpha_n! \qquad \text{and} \qquad x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

With the above notation, a polynomial in $\mathbb{K}^m$ of degree at most n is a linear combination of monomials of the form x^α with $x \in \mathbb{K}^m$ and $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $|\alpha| \le n$. Hence, a polynomial P in m variables (real or complex) of degree at most n has the form

$$P(x) = \sum_{\substack{\alpha \in (\mathbb{N}\cup\{0\})^m \\ |\alpha| \le n}} a_\alpha x^\alpha, \qquad \text{for all } x \in \mathbb{K}^m,$$

with $a_\alpha \in \mathbb{K}$. Accordingly, a polynomial P in m variables is homogeneous of degree n if

$$P(x) = \sum_{\substack{\alpha \in (\mathbb{N}\cup\{0\})^m \\ |\alpha| = n}} a_\alpha x^\alpha, \qquad \text{for all } x \in \mathbb{K}^m,$$

with $a_\alpha \in \mathbb{K}$. Our first objective is to recall how to extend the above well-known definition of polynomial and homogeneous polynomial to arbitrary linear spaces.

Let E be a linear space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). From now on, for each $n \in \mathbb{N}$, $\mathcal{L}_a(^nE)$ denotes the space of all n-linear forms on E. Recall that L is n-linear if it is linear in every coordinate. As usual, $E^n := E \times \overset{n)}{\cdots} \times E$. Also, we consider the diagonal mapping

$$\begin{aligned} \Delta_n : \ & E \to E^n \\ & x \mapsto (x, \dots, x). \end{aligned}$$

Definition 1 (homogeneous polynomials). If E is a linear space over $\mathbb{K}$ and $n \in \mathbb{N}$, we say that P is an **n-homogeneous polynomial** if there exists $L \in \mathcal{L}_a(^nE)$ with $P = L \circ \Delta_n$. Equivalently, we write $P = \hat{L}$. The space of all n-homogeneous polynomials on E is denoted by $\mathcal{P}_a(^nE)$. We say that P is a polynomial of degree at most n on E if $P = P_n + \dots + P_1 + P_0$, where $P_k \in \mathcal{P}_a(^kE)$ for $k = 1, \dots, n$ and P_0 is a constant. ◂

Observe that, if $P \in \mathcal{P}_a(^nE)$, then $P(\lambda x) = \lambda^n P(x)$ for all $x \in E$ and every $\lambda \in \mathbb{K}$. This property is also satisfied by all homogeneous polynomials in a finite number of variables. On the other hand, the n-linear form that defines a given n-homogeneous polynomial is not uniquely determined. Let us see this with an example. First, notice that for all bilinear forms L on $\mathbb{K}^n$, where $n \in \mathbb{N}$, there exists an $n \times n$ matrix with entries in $\mathbb{K}$ such that $L(z, w) = zAw^{\top}$ for all $z, w \in \mathbb{K}^n$. Here $w^{\top}$ means, as usual, the transpose of w. Hence, all 2-homogeneous polynomials on $\mathbb{K}^n$ are of the form

$$P(z) = L(z, z) = \sum_{i,j=1}^{n} a_{ij} z_i z_j,$$

where $A = (a_{ij})_{i,j=1}^n$. If we consider now $B = \frac{1}{2}(A + A^{\top})$, then $zAz^{\top} = zBz^{\top}$ for every $z \in \mathbb{K}^n$. The latter means that the two bilinear forms determined by the matrices A and B define the polynomial P.

The previous example motivates the definition of symmetric multilinear forms.

Definition 2 (symmetric multilinear forms). Let E be a linear space over $\mathbb{K}$ and let $n \in \mathbb{N}$. An n-linear form L is **symmetric** if

$$L(x_1, \dots, x_n) = L\left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right)$$

for all $(x_1, \dots, x_n) \in E^n$ and every permutation σ of $\{1, \dots, n\}$. The space of all symmetric n-linear forms on E is denoted by $\mathcal{L}_a^s(^nE)$. ◂

Remark 3. Let us consider the mapping s from $\mathcal{L}_a(^nE)$ onto $\mathcal{L}_a^s(^nE)$ given by

$$s(L)(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} L\left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right),$$

where S_n is the group of all permutations of $\{1, \dots, n\}$. Then, s is a projection. Furthermore, L and $s(L)$ define the same homogeneous polynomial. Therefore, if $P \in \mathcal{P}_a(^nE)$, it is always possible to choose $L \in \mathcal{L}_a^s(^nE)$ such that $\hat{L} = P$. ◂

Now, using the so-called multinomial formula, it is possible to see that definition 1 does extend the concept of homogeneous polynomial from a finite number of variables to arbitrary linear spaces.

Proposition 4 (multinomial formula). *Let E be a real or complex linear space, $P \in \mathcal{P}_a(^nE)$, $x_1, \dots, x_k \in E$ and $a_1, \dots, a_k \in \mathbb{K}$. Then,*

$$P\left(\sum_{i=1}^{k} a_i x_i\right) = \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m| = n}} \frac{n!}{m!} a_1^{m_1} \cdots a_k^{m_k} L\left(x_1^{m_1}, \dots, x_k^{m_k}\right),$$

where $L \in \mathcal{L}_a^s(^nE)$ satisfies $\hat{L} = P$ and

$$L\left(x_1^{m_1}, \dots, x_k^{m_k}\right) := L\Big(\overbrace{x_1, \dots, x_1}^{m_1}, \dots, \overbrace{x_k, \dots, x_k}^{m_k}\Big).$$

Proof. Let $m = (m_1, \dots, m_k) \in \{0, 1, \dots, k\}^k$ with $|m| = n$ and define

$$A_m = \left\{(i_1, \dots, i_k) \in \{0, 1, \dots, k\}^k \text{ such that 1 appears } m_1 \text{ times}, \dots, \ k \text{ appears } m_k \text{ times}\right\}$$

and

$$a_m = \sum_{(i_1, \dots, i_k) \in A_m} L(x_{i_1}, \dots, x_{i_k}).$$

Then, using linearity,

$$P(a_1 x_1 + \dots + a_k x_k) = L((a_1 x_1 + \dots + a_k x_k)^n) = \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m| = n}} a_1^{m_1} \cdots a_k^{m_k} a_m.$$

Observe that, due to the symmetry of L, we have

$$a_m = L(x_1^{m_1}, \dots, x_k^{m_k}) \cdot \#(A_m),$$

where $\#(A_m)$ denotes the cardinality of A_m. Using elementary combinatorics we arrive at $\#(A_m) = \frac{n!}{m_1! \cdots m_k!}$, which concludes the proof. ∎

The importance of the multinomial formula in this context relies on the fact that it can be used to extend the classical definition of polynomials in several variables. Indeed, let E be a real or complex linear space and F a finite dimensional subspace of E. Let $\{e_1, \dots, e_k\}$ be a Hammel basis for F and $x = x_1e_1 + \dots + x_ke_k \in F$. Then, using the multinomial formula,

$$\begin{aligned} P(x) &= P(x_1e_1 + \cdots + x_ke_k) \\ &= \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m|=n}} \frac{n!}{m!} x_1^{m_1} \cdots x_k^{m_k} L\left(e_1^{m_1}, \dots, e_k^{m_k}\right) = \sum_{\substack{m \in (\mathbb{N} \cup \{0\})^k \\ |m|=n}} a_m x^m, \end{aligned}$$

which shows that the restriction of an n-homogeneous polynomial in the sense of definition 1 to a k-dimensional space is an n-homogeneous polynomial in k variables.

2.1. The polarization formula

If E is a linear space and $P \in \mathcal{P}_a(^nE)$, we have seen in remark 3 that there exists $L \in \mathcal{L}_a^s(^nE)$ such that $P = \hat{L}$. We can go further and prove that this symmetric n-linear form that determines P is unique, and we call it polar of P. The polarization formula does not only prove the uniqueness of the symmetric n-linear form that defines a given n-homogeneous polynomial: additionally, it provides an explicit expression of the polar in terms of the polynomial it defines. There are many forms of the polarization formula. We have chosen one taken from Dineen's book [15] that uses Rademacher functions.

Theorem 5 (polarization formula). *Let E be a real or complex linear space, $P \in \mathcal{P}_a(^nE)$, $L \in \mathcal{L}_a^s(^nE)$ and assume that $\hat{L} = P$. If $(x_1, \dots, x_n) \in E^n$, then*

$$L(x_1, \dots, x_n) = \frac{1}{n!} \int_0^1 r_1(t) \cdots r_n(t) P(r_1(t) x_1 + \dots + r_n(t) x_n) \, \mathrm{d}t, \tag{1}$$

where r_j is the j-th Rademacher function, defined by

$$r_j(t) = \operatorname{sign}(\sin(2^j \pi t)) \qquad \text{for all } 1 \le j \le n.$$

Proof. Let $m = (m_1, \dots, m_n) \in (\mathbb{N} \cup \{0\})^n$. Recall that the Rademacher functions satisfy

$$\int_0^1 r_1^{m_1+1}(t) \cdots r_n^{m_n+1}(t) \, \mathrm{d}t = \int_0^1 r_1^{m_1+1}(t) \, \mathrm{d}t \cdots \int_0^1 r_n^{m_n+1}(t) \, \mathrm{d}t \tag{2}$$

and

$$\int_0^1 r_j^{m_j+1}(t) \, \mathrm{d}t = \begin{cases} 1 & \text{if } m_j + 1 \text{ is even,} \\ 0 & \text{if } m_j + 1 \text{ is odd.} \end{cases} \tag{3}$$

If we assume $|m| = n$, the product $\int_0^1 r_1^{m_1+1}(t) \, \mathrm{d}t \cdots \int_0^1 r_n^{m_n+1}(t) \, \mathrm{d}t$ vanishes unless $m_1 = \dots = m_n = 1$, in which case its value is 1. Now, using the multinomial formula,

$$\begin{aligned}
&\int_0^1 r_1(t)\cdots r_n(t)\,P\left(r_1(t)\,x_1+\ldots+r_n(t)\,x_n\right)\mathrm{d}t\\
&\quad=\int_0^1 r_1(t)\cdots r_n(t)\,L\left(r_1(t)\,x_1+\ldots+r_n(t)\,x_n,\ldots,r_1(t)\,x_1+\ldots+r_n(t)\,x_n\right)\mathrm{d}t\\
&\quad=\int_0^1 r_1(t)\cdots r_n(t)\sum_{\substack{m\in(\mathbb{N}\cup\{0\})^n\\|m|=n}}\frac{n!}{m!}r_1^{m_1}(t)\cdots r_n^{m_n}(t)\,L\left(x_1,\ldots,x_n\right)\mathrm{d}t\\
&\quad=\sum_{\substack{m\in(\mathbb{N}\cup\{0\})^n\\|m|=n}}\frac{n!}{m!}\int_0^1 r_1^{m_1+1}(t)\cdots r_n^{m_n+1}(t)\,\mathrm{d}t L\left(x_1,\ldots,x_n\right)=n!L(x_1,\ldots,x_n).
\end{aligned}$$

■

Remark 6. Let $P\in\mathcal{P}_a(^nE)$ and $L\in\mathcal{L}_a^s(^nE)$ be the polar of P. It has already been mentioned that there are many forms of the polarization formula. To see this we just need to replace the Rademacher functions by any set of functions satisfying the identities (2) and (3). For instance, we may consider any set of n independent and orthonormal random variables $r_1,\ldots,r_n$ on $[0,1]$ taking values on $\mathbb{K}$ and

$$\Psi=\bar{r}_1\cdots\bar{r}_n\cdot P\left(\sum_{i=1}^n r_i x_i\right).$$

Proceeding as in the proof of theorem 5, the expectancy of Ψ would be given by

$$\mathbb{E}[\Psi]=n!L(x_1,\ldots,x_n),$$

that is,

$$L(x_1,\ldots,x_n)=\frac{1}{n!}\mathbb{E}[\Psi],$$

which is a much more general way to express the polarization formula (1). Thus, if $r_1,\ldots,r_n$ are n independent Bernouilli random variables taking the value -1 with probability $1/2$ and 1 with probability $1/2$, we would have

$$L\left(x_1,\ldots,x_n\right)=\frac{1}{2^n n!}\sum_{\varepsilon_i=\pm1}\varepsilon_1\cdots\varepsilon_n P\left(\sum_{i=1}^n\varepsilon_i x_i\right). \tag{4}$$

This is a very convenient form to put down the polarization formula. ◂

2.2. Continuity in polynomial spaces

All polynomials in finitely many variables are continuous. However, this is far from being true when polynomials on an infinite dimensional normed space are considered. Actually, continuity fails to be universal even for linear forms on an infinite dimensional normed space.

Let $(E,\|\cdot\|)$ be a normed space over $\mathbb{K}$. We represent the space of continuous n-homogeneous polynomials, the space of continuous n-linear forms and the space of continuous symmetric n-linear forms, respectively, by $\mathcal{P}(^nE)$, $\mathcal{L}(^nE)$ and $\mathcal{L}^s(^nE)$. Also, for $P\in\mathcal{P}(^nE)$ and $L\in\mathcal{L}^s(^nE)$, we define

$$\begin{aligned}
\|P\|&=\sup\left\{|P(x)|:\|x\|\le1\right\},\\
\|L\|&=\sup\left\{|L(x_1,\ldots,x_n)|:\|(x_1,\ldots,x_n)\|\le1\right\},
\end{aligned}$$

where

$$\|(x_1,\ldots,x_n)\|=\sup\left\{\|x_i\|:1\le i\le n\right\}.$$

These definitions are intended to introduce a norm in $\mathcal{P}(^nE)$ and $\mathcal{L}(^nE)$. However, at this stage we do not even know whether $\|P\|$ or $\|L\|$ are finite. As a matter of fact, the finiteness of $\|P\|$ or $\|L\|$ characterizes the continuity of P or L, as we will see later. First we present a fundamental result in the theory of polynomials in normed spaces also known as the polarization inequality. The proof provided below is taken from Dineen's book [15].

Theorem 7 (polarization inequality). *Let E be a normed space. Then,*

$$\|P\| \leq \|L\| \leq \frac{n^n}{n!}\|P\|$$

for every $L \in \mathcal{L}^s(^nE)$ and $P \in \mathcal{P}(^nE)$ such that L is the polar of P.

Proof. The first inequality is trivial since P is a restriction of L. To prove the second inequality, we use the polarization formula (4):

$$\begin{aligned}
\|L\| &= \sup\{|L(x_1,\dots,x_n)| : \|(x_1,\dots,x_n)\| \leq 1\} \\
&= \sup\left\{\left|\frac{1}{2^n n!}\sum_{\varepsilon_i=\pm 1}\varepsilon_1\cdots\varepsilon_n P\left(\sum_{i=1}^n \varepsilon_i x_i\right)\right| : \|(x_1,\dots,x_n)\| \leq 1\right\} \\
&\leq \frac{1}{2^n n!}\sum_{\varepsilon_i=\pm 1}\sup\left\{\left|P\left(\sum_{i=1}^n \varepsilon_i x_i\right)\right| : \|(x_1,\dots,x_n)\| \leq 1\right\} \\
&= \frac{n^n}{2^n n!}\sum_{\varepsilon_i=\pm 1}\sup\left\{\left|P\left(\frac{1}{n}\sum_{i=1}^n \varepsilon_i x_i\right)\right| : \|(x_1,\dots,x_n)\| \leq 1\right\} \\
&\leq \frac{n^n}{n!}\|P\|.
\end{aligned}$$

■

As is well known, boundedness is a characteristic property of continuous linear forms on any normed space. A similar result holds for homogeneous polynomials, as we are about to see. The proof of the following result is inspired in Dineen's book [15].

Theorem 8. *Let E be a normed space over $\mathbb{K}$ and $P \in \mathcal{P}_a(^nE)$. Then, the following are equivalent:*

(i) *P is continuous in E.*

(ii) *P is continuous at 0.*

(iii) *P is bounded over B_E, the closed unit ball of E.*

Proof. That (i) implies (ii) is trivial. We prove now that (ii) implies (iii). By continuity at 0, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|P(x)| < \varepsilon$ whenever $\|x\| < 2\delta$. Therefore, if $x \in \mathsf{B}_E$ we have

$$|P(x)| = \left|P\left(\delta\cdot\frac{x}{\delta}\right)\right| = \frac{1}{\delta^n}|P(\delta x)| < \frac{\varepsilon}{\delta^n}.$$

Thus, P is bounded on B_E.

Finally, we show that (iii) implies (i). Choose an arbitrary x_0 in E and take $x \in E$ with $\|x - x_0\| \leq 1$. In particular, $\|x\| \leq 1 + \|x_0\|$. Then, by Newton's binomial formula and Martin's theorem, we have

$$\begin{aligned}
|P(x) - P(x_0)| &= |P((x - x_0) + x_0) - P(x_0)| \\
&= \left|\sum_{j=0}^{n-1}\binom{n}{j}L\left(x_0^j, (x - x_0)^{n-j}\right)\right| \\
&\leq \sum_{j=0}^{n-1}\binom{n}{j}\|L\|\,\|x_0\|^j\,\|x - x_0\|^{n-j} \\
&\leq \frac{n^n}{n!}\|P\|\,\|x - x_0\|\sum_{j=0}^{n-1}\binom{n}{j}\|x_0\|^j \\
&\leq \frac{n^n}{n!}(1 + \|x_0\|)^n\,\|P\|\,\|x - x_0\|.
\end{aligned}$$

Recall that $\|P\|$ is finite since P is bounded on B_E. Without loss of generality, we may also assume that $\|P\| > 0$. Hence, for an arbitrary $\varepsilon > 0$ we can take

$$\delta = \min\left\{1, \frac{n!}{n^n(1 + \|x_0\|)^n\|P\|}\right\} > 0$$

and we have that $|P(x) - P(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$, that is, P is continuous at x_0. ■

Remark 9. Let E be a real or complex normed space. Theorems 5 and 8 show two relevant facts:

1. If $P \in \mathcal{P}_a({}^nE)$ and $L \in \mathcal{L}_a^s({}^nE)$ is its polar, then P is bounded (continuous) if and only if L is bounded (continuous).
2. The spaces $\mathcal{P}({}^nE)$ and $\mathcal{L}^s({}^nE)$ are topologically isomorphic and $\mathcal{L}^s({}^nE) \ni L \mapsto \hat{L} \in \mathcal{P}({}^nE)$ is a natural isomorphism whose inverse is provided by the polarization formula. ◂

Using the axiom of choice it is easy to construct non-bounded (and therefore non-continuous) polynomials.

Example 10. Let E be any normed space of dimension $\mathfrak{c}$ (here $\mathfrak{c}$ is the continuum, or the cardinality of $\mathbb{R}$). Let $\mathcal{B} = \{e_x : x \in \mathbb{R}\}$ be a Hammel basis of normalized vectors of E indexed in $\mathbb{R}$ and define $L \in \mathcal{L}_a^s({}^nE)$ on $\mathcal{B}$ by $L(e_{x_1}, \dots, e_{x_n}) = x_1 \cdots x_n$. On the rest of E, L is defined by linearity. Then, the n-homogeneous polynomial induced by L is not bounded on B_E, since $\mathcal{B} \subset \mathsf{B}_E$, but

$$\lim_{x\to\infty} P(e_x) = \lim_{x\to\infty} x^n = \infty. \qquad ◂$$

In general, for any normed space E, the algebraic size, measured in terms of dimension, of the set of non-bounded n-homogeneous polynomials (respectively non-bounded symmetric n-linear forms) is maximal. Consider the sets $\mathcal{NBL}^s({}^nE)$ and $\mathcal{NBP}({}^nE)$ of, respectively, all the non-bounded symmetric n-linear forms and all the non-bounded n-homogeneous polynomials on E. Then, Gámez-Merino, Muñoz-Fernández, Pellegrino, and Seoane-Sepúlveda [16] proved in 2012 the following.

Theorem 11. *If $n \in \mathbb{N}$ and E is a normed space of infinite dimension λ, then the sets $\mathcal{NBL}^s({}^nE) \cup \{0\}$ and $\mathcal{NBP}({}^nE) \cup \{0\}$ contain a 2^λ-dimensional subspace. We say then that the sets $\mathcal{NBL}^s({}^nE)$ and $\mathcal{NBP}({}^nE)$ are 2^λ-lineable.*

2.3. Polarization constants

If E is a real or complex normed space $P \in \mathcal{P}({}^nE)$ and $L \in \mathcal{L}^s({}^nE)$ is the polar of P, according to Martin's theorem (theorem 7),

$$\|L\| \leq \frac{n^n}{n!}\|P\|.$$

The constant $\frac{n^n}{n!}$ cannot be replaced by a smaller constant in general since equality can be attained for the space $E = \ell_1^n$ and the polynomial $\Phi_n(x_1, \dots, x_n) := x_1 \cdots x_n$ and its polar. However, $\frac{n^n}{n!}$ can indeed be replaced by a smaller estimate for specific spaces. This motivates the following definition.

Definition 12 (polarization constants). If E is a normed space over $\mathbb{K}$, we define the ***n*-th polarization constant** of E as

$$\mathbb{K}(n; E) := \inf\{K > 0 : \|L\| \leq K\|P\|, \ \forall P \in \mathcal{P}({}^nE) \text{ and } \hat{L} = P\}. \qquad ◂$$

A somewhat more general concept than that of polarization constant arises from the following result by Harris [19].

Theorem 13. *Let E be a complex normed space and $n_1, \dots, n_k \in \mathbb{N}$. If $n = n_1 + \cdots + n_k$ and $L \in \mathcal{L}^s({}^nE)$, then*

$$\sup\{|L(x_1^{n_1}, \dots, x_k^{n_k})| : \|x_i\| = 1, \ 1 \leq i \leq k\} \leq \frac{n_1! \cdots n_k! \, n^n}{n_1^{n_1} \cdots n_k^{n_k} n!} \|\hat{L}\|.$$

A similar result with a different constant can be proved when E is a real normed space. All this serves as a motivation for the following definition.

Definition 14 (generalized polarization constants). Let E be a normed space over $\mathbb{K}$ and $n, n_1, \dots, n_k \in \mathbb{N}$ with $n = n_1 + \cdots + n_k$. Then, we define the **generalized polarization constant** $\mathbb{K}(n_1, \dots, n_k; E)$ as

$$\mathbb{K}(n_1, \dots, n_k; E) := \inf\{c : |L(x_1^{n_1}, \dots, x_k^{n_k})| \leq c\|\hat{L}\|, \ \forall L \in \mathcal{L}^s({}^nE), \ \|x_i\| = 1\}. \qquad ◂$$

From theorem 7 we deduce that

$$1 \leq \mathbb{K}(n; E) \leq \frac{n^n}{n!}$$

for any normed space E over $\mathbb{K}$, but the exact value of $\mathbb{K}(n; E)$ for many choices of E has remained as an unresolved problem until today. The calculation of the exact value of the polarization constants of specific spaces seems to be a challenging problem and yet, significant progress has been made. Of particular interest are the works of Sarantopoulos [37] and Kirwan, Sarantopoulos, and Tonge [23], where the spaces satisfying $\mathbb{K}(n; E) = \frac{n^n}{n!}$ are studied. At the other end of the scale, according to an old result, if E is an Euclidean space over $\mathbb{K}$, then $\mathbb{K}(n; E) = 1$ [2, 21, 38] (see Dineen's book [15] for a modern exposition). Furthermore, Benítez and Sarantopoulos [4] proved that $\mathbb{R}(n; E) = 1$ implies that E is a real Euclidean space. However, $\mathbb{C}(n; E) = 1$ does not necessarily imply that E is a complex Euclidean space. The value of $\mathbb{K}(n; \ell_p)$ is known for some choices of p (see for instance [36]), but most of the polarization constants of the classical spaces are still unknown nowadays. For a complete account on polarization constants, we recommend Rodríguez-Vidanes's work [35].

The use of the Krein-Milman approach (which will be described right after theorem 15) in combination with a description of the extreme points of certain polynomial spaces may produce good results in the difficult task of calculating polarization constants. The next section is devoted to the study of the geometry of certain polynomial spaces.

3. Geometry of some 3-dimensional polynomial spaces

Let E be a finite dimensional normed space. Recall that $C \subset E$ is a convex body if it is a compact convex set with nonempty interior. A point $e \in C$ is an extreme point of C if it is not an interior point of any segment contained in C. We use the notation $\operatorname{ext}(C)$ to represent the set of all the extreme points of C. According to the Krein-Milman theorem (or its finite dimensional version proved by Minkowski in 3-dimensional spaces and by Steinitz for any dimension), the set of the extreme points of a convex body C in the finite dimensional normed space E determines C. The precise formulation of this result is the following.

Theorem 15 (Minkowski-Steinitz). *If E is a finite dimensional normed space and $C \subset E$ is a convex body, then*

(i) $\operatorname{ext}(C) \neq \emptyset$.

(ii) $C = \operatorname{co}(\operatorname{ext}(C))$.

Note that $\operatorname{co}(A)$ *is the convex hull of the set* A.

This result has been used in a large variety of settings to optimize convex functions. In fact, the result that allows the optimization is the following:

> If $C \subset E$ is a convex body in the real finite dimensional normed space E and $f : C \to \mathbb{R}$ is a convex function that attains its maximum in C, then there is a point $e \in \operatorname{ext}(C)$ such that $f(e) = \min\{f(x) : x \in C\}$.

We will address to this result as the Krein-Milman approach from now on. A combination of the Krein-Milman approach and an exhaustive description of the extreme points of the unit ball of a polynomial space provides, in many cases, sharp polynomial inequalities. The previous argument motivates the study of the geometry of polynomial spaces. As a matter of fact, many publications have dealt with this question in the past. Konheim and Rivlin [24], as late as in 1966, characterised the extreme points of the space of real polynomials of degree not exceeding n, namely $\mathcal{P}_n(\mathbb{R})$, endowed with the norm

$$\|P\| = \sup\{|P(x)| : x \in [-1, 1]\}.$$

Unfortunately, Konheim and Rivlin's results do not provide an explicit representation of the extreme points of $\mathcal{P}_n(\mathbb{R})$. Choi, Kim, and Ki [11], on the one hand, and Choi and Kim [10] on the other, characterised the extreme points of $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_\infty^2)$. Grecu [18] extended those results to $\mathcal{P}(^2\ell_p^2)$ for arbitrary $p \geq 1$.

Aron and Klimek [1] characterised the extreme points of the space of the real quadratic polynomials on $[-1, 1]$ and the unit disk in $\mathbb{C}$. Muñoz-Fernández and Seoane-Sepúlveda [33] studied the extreme points of the real trinomials on $[-1, 1]$, whereas Neuwirth [34] did the same thing on the unit disk in $\mathbb{C}$. Kim [22] studied polynomials on an hexagon or an octagon. Special attention has also been given to polynomials on non-balanced convex bodies. Thus, Muñoz-Fernández, Révész, and Seoane-Sepúlveda [31] studied the geometry of the space $\mathcal{P}(^2\Delta)$ of the 2-homogeneous polynomials on the simplex Δ (the triangle of vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$). Milev and Naidenov [28, 29] studied the extreme points of the space of polynomials (homogeneous or not) of degree at most 2 on Δ. Gámez-Merino, Muñoz-Fernández, Sánchez, and Seoane-Sepúlveda [17] studied the geometry of the space $\mathcal{P}(^2\square)$ of the 2-homogeneous polynomials on the unit square $\square = [0, 1]^2$. Muñoz-Fernández, Pellegrino, Seoane-Sepúlveda, and Weber [30] characterised the extreme points of the space $\mathcal{P}(^2D(\alpha, \beta))$ of the 2-homogeneous polynomials on the circular sectors $D(\alpha, \beta) = \{re^{i\theta} : r \in [0, 1],\ \theta \in [\alpha, \beta]\}$ for $\beta - \alpha = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and $\beta - \alpha \geq \pi$. These results have been generalised in a recent work by Bernal-González, Muñoz-Fernández, Rodríguez-Vidanes, and Seoane-Sepúlveda [5] for an arbitrary length of the interval $[\alpha, \beta]$.

In the rest of this section we will provide a few illustrative examples representing a tiny fraction of the results mentioned above about geometry of polynomial spaces.

3.1. The geometry of $\mathcal{P}_2(\mathbb{R})$

Here we consider the space $\mathcal{P}_2(\mathbb{R})$ of the real polynomials $ax^2 + bx + c$ of degree not greater than 2, endowed with he norm

$$\|P\|_{\mathcal{P}_2(\mathbb{R})} = \sup\{|P(x)| : |x| \leq 1\}.$$

Elementary calculus shows that

$$\|(a, b, c)\|_{\mathcal{P}_2(\mathbb{R})} = \begin{cases} \left|\frac{b^2}{4a} - c\right| & \text{if } a \neq 0,\ \left|\frac{b}{2a}\right| < 1 \text{ and } \frac{c}{a} + 1 < \frac{1}{2}\left(\left|\frac{b}{2a}\right| - 1\right)^2, \\ |a + c| + |b| & \text{otherwise.} \end{cases}$$

The geometry of this 3-dimensional space was investigated by Aron and Klimek [1] (see also the work of Muñoz-Fernández and Seoane-Sepúlveda [33]). The conclusions extracted from these papers are summarised in the following result. From now on, if E is a normed space, B_E and S_E stand for the closed unit ball and the unit sphere of E, respectively. Also, graph(f) stands for the graph of the function f.

Theorem 16. *Define* $\Gamma(a) = 2(\sqrt{2a} - a)$ *and*

$$U = \left\{(a, b) \in \mathbb{R}^2 : a < 0 \text{ and } |b| \leq \min\{|a|, \Gamma(|a|)\}\right\},$$
$$V = \left\{(a, b) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1] : |b| \geq |a|\right\},$$
$$W = \left\{(a, b) \in \mathbb{R}^2 : a > 0 \text{ and } |b| \leq \min\{|a|, \Gamma(|a|)\}\right\}.$$

If for every $(a, b) \in \mathbb{R}^2$ *we define*

$$f_+(a, b) = 1 - a - |b|, \qquad f_-(a, b) = -f_+(-a, b),$$

and for every $(a, b) \in \mathbb{R}^2$ *with* $a \neq 0$ *we define*

$$g_+(a, b) = \frac{b^2}{4a} - 1, \qquad g_-(a, b) = -g_+(-a, b),$$

then

(a) $\mathsf{S}_{\mathcal{P}_2(\mathbb{R})} = \text{graph}\left(f_+|_{W \cup V}\right) \cup \text{graph}\left(f_-|_{U \cup V}\right) \cup \text{graph}\left(g_+|_W\right) \cup \text{graph}\left(g_-|_U\right)$.

(b) $\text{ext}(\mathsf{B}_{\mathcal{P}_2(\mathbb{R})}) = \{\pm(t, \pm\Gamma(t), 1 - t - \Gamma(t)) : t \in [1/2, 2]\} \cup \{\pm(0, 0, 1)\}$.

We show a picture of $\mathsf{S}_{\mathcal{P}_2(\mathbb{R})}$ in figure 1.

For fixed $m > n$ in $\mathbb{N}$, the geometry of the space $\{ax^m + bx^n + c : a, b, c \in \mathbb{R}\}$ endowed with the sup norm on $[-1, 1]$ has been studied by Muñoz-Fernández and Seoane-Sepúlveda [33] for all possible choices of m, n. The results depend strongly on whether m and n are even or odd.

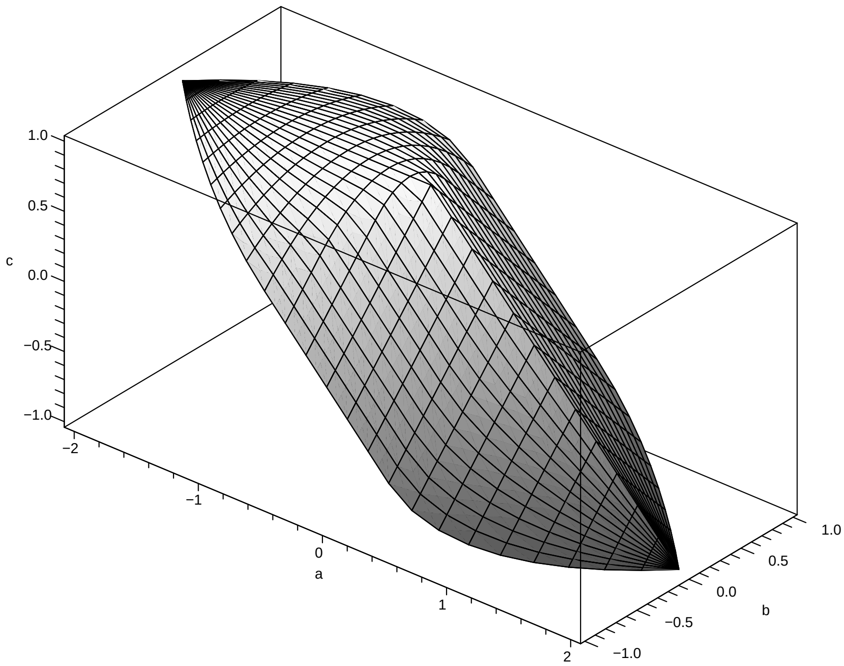

Figure 1: Unit sphere of $\mathcal{P}_2(\mathbb{R})$.

3.2. The geometry of $\mathcal{P}(^2\Delta)$

Recall first that $\mathcal{P}(^2\Delta)$ is the space of polynomials $P(x,y) = ax^2 + by^2 + cxy$ endowed with the norm defined by

$$\|P\|_\Delta = \sup\{|P(\mathbf{x})| : \mathbf{x} \in \Delta\},$$

where Δ represents the region enclosed by the triangle in $\mathbb{R}^2$ of vertices $(0,0)$, $(0,1)$ and $(1,0)$ (or simplex, for short). All the results in this section are taken from the work of Muñoz-Fernández, Révész, and Seoane-Sepúlveda [31]. First, it is convenient to have a formula to calculate the norm $\|\cdot\|_\Delta$.

Theorem 17. *Let $a,b,c \in \mathbb{R}$ and $P(x,y) = ax^2 + by^2 + cxy$. Then,*

$$\|P\|_\Delta = \begin{cases} \max\left\{|a|, |b|, \left|\frac{c^2-4ab}{4(a-c+b)}\right|\right\} & \text{if } a-c+b \neq 0 \text{ and } 0 < \frac{2b-c}{2(a-c+b)} < 1, \\ \max\{|a|,|b|\} & \text{otherwise.} \end{cases} \tag{5}$$

Now we provide a parametrisation of $\mathsf{S}_{\mathcal{P}(^2\Delta)}$ and describe the geometry of $\mathsf{B}_{\mathcal{P}(^2\Delta)}$. We use the notations S_Δ and B_Δ for short.

Theorem 18. *If we define the mappings*

$$f_+(a,b) = 2 + 2\sqrt{(1-a)(1-b)}$$

and

$$f_-(a,b) = -f_+(-a,-b) = -2 - 2\sqrt{(1+a)(1+b)},$$

for every $(a,b) \in [-1,1]^2$, and the set

$$F = \{(a,b,c) \in \mathbb{R}^3 : (a,b) \in \partial[-1,1]^2 \text{ and } f_-(a,b) \leq c \leq f_+(a,b)\},$$

where $\partial[-1,1]^2$ is the boundary of $[-1,1]^2$, then

(a) $\mathsf{S}_\Delta = \operatorname{graph}(f_+|_{[-1,1]^2}) \cup \operatorname{graph}(f_-|_{[-1,1]^2}) \cup F$.

(b) $\operatorname{ext}(\mathsf{B}_\Delta) = \{\pm(1, t, -2-2\sqrt{2(1+t)}), \pm(t, 1, -2-2\sqrt{2(1+t)}) : t \in [-1,1]\}$.

You can find a picture of S_Δ in figure 2.

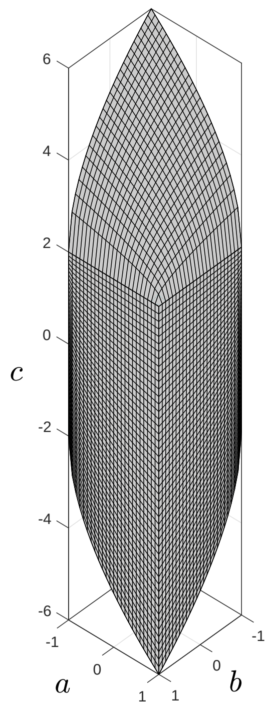

Figure 2: Unit sphere of $\mathcal{P}(^2\Delta)$.

3.3. The geometry of $\mathcal{P}(^2D(\alpha,\beta))$

First, recall that $\mathcal{P}(^2D(\alpha,\beta))$ is the 3-dimensional space of the polynomials ax^2+by^2+cxy endowed with the norm

$$\|P\|_{D(\alpha,\beta)} := \sup\{|P(\mathbf{x})| : \mathbf{x}\in D(\alpha,\beta)\}.$$

It is a simple exercise to show that the spaces $\mathcal{P}(^2D(\alpha,\alpha+\beta))$ and $\mathcal{P}(^2D(0,\beta))$ are isometric. We write $D(\beta)$ instead of $D(0,\beta)$ for simplicity. Actually, the isometry is given by the matrix

$$\begin{pmatrix} \cos^2\alpha & \sin^2\alpha & \frac{\sin 2\alpha}{2} \\ \sin^2\alpha & \cos^2\alpha & -\frac{\sin 2\alpha}{2} \\ -\sin 2\alpha & \sin 2\alpha & \cos 2\alpha \end{pmatrix}.$$

This isometry allows us to restrict our attention to the study of the geometry of $\mathsf{B}_{D(\beta)}$.

A moment's thought reveals that, if $\beta\geq\pi$, then $\mathsf{B}_{D(\beta)}=\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$, where $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ stands for the closed unit ball of the space $\mathcal{P}(^2\ell_2^2)$ of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the sup norm over the unit disk. The extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ were described by Choi and Kim [10]. An alternative approach was provided by Muñoz-Fernández, Pellegrino, Seoane-Sepúlveda, and Weber [30].

Theorem 19. *Let $\beta\geq\pi$ and define $f(a,b)=2\sqrt{1+ab-|a+b|}$ on $[-1,1]^2$. Then,*

(a) $\|P\|_{D(\beta)}=\frac{1}{2}\left(|a+b|+\sqrt{(a-b)^2+c^2}\right)$, *for all $P\in\mathcal{P}(^2D(\beta))$.*

(b) $\mathsf{S}_{\mathcal{P}(^2D(\beta))}=\mathrm{graph}(f)\cup\mathrm{graph}(-f)$.

(c) $\mathrm{ext}(\mathsf{B}_{\mathcal{P}(^2D(\beta))})=\{\pm(t,-t,2\sqrt{1-t^2}) : t\in[-1,1]\}\cup\{\pm(1,1,0)\}$.

The reader can find a graph of $\mathsf{S}_{\mathcal{P}(^2\ell_2^2)}$ in figure 3.

Let us give just another example in this section, taken from the work of Muñoz-Fernández et al. [30].

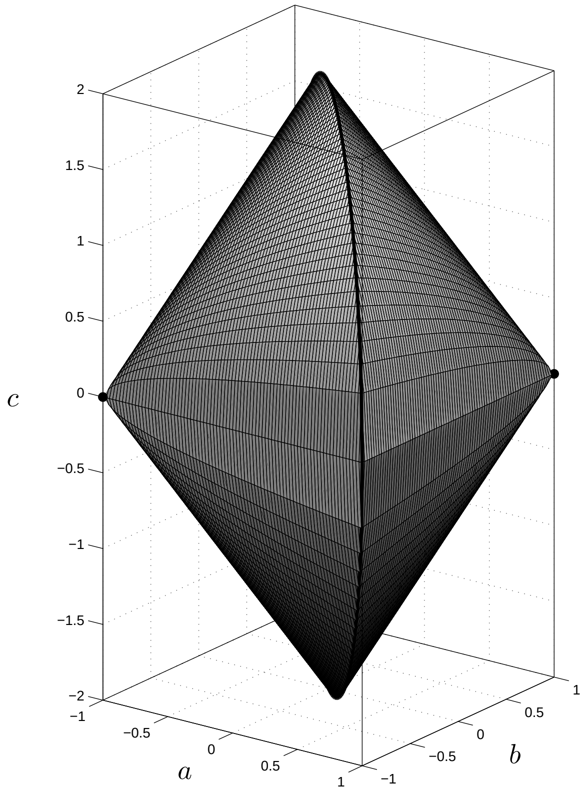

Figure 3: $\mathsf{S}_{\mathcal{P}(^2\ell_2^2)}$. The extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ are drawn with a thick line and dots.

Theorem 20. *If we define the mappings*

$$G_1(a,b) = 2\sqrt{(1-a)(1-b)}$$

and

$$G_2(a,b) = -G_1(-a,-b) = -2\sqrt{(1+a)(1+b)},$$

for every $(a,b) \in [-1,1]^2$*, and the set*

$$F = \{(a,b,c) \in \mathbb{R}^3 \,:\, (a,b) \in \partial[-1,1]^2 \text{ and } G_2(a,b) \le c \le G_1(a,b)\},$$

where $\partial[-1,1]^2$ *is the boundary of* $[-1,1]^2$*, then*

(a) $\|P\|_{D(\frac{\pi}{2})} = \max\left\{|a|, |b|, \frac{1}{2}|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}|\right\}$ *for every* $P \in \mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$.

(b) $\mathsf{S}_{D(\frac{\pi}{2})} = \operatorname{graph}(G_1) \cup \operatorname{graph}(G_2) \cup F$.

(c) $\operatorname{ext}(\mathsf{B}_{D(\frac{\pi}{2})}) = \{\pm\left(1, t, -2\sqrt{2(1+t)}\right), \pm\left(t, 1, -2\sqrt{2(1+t)}\right) \,:\, t \in [-1,1]\} \cup \{\pm(1,1,0)\}$.

The reader can find a sketch of $\mathsf{S}_{\mathcal{P}(^2D(\frac{\pi}{2}))}$ in figure 4.

4. Polynomial inequalities

A number of polynomial inequalities can be tackled using the Krein-Milman approach described right after theorem 15. In this section we will introduce some problems of interest together with a sample of the type of results that can be achieved using the Krein-Milman approach.

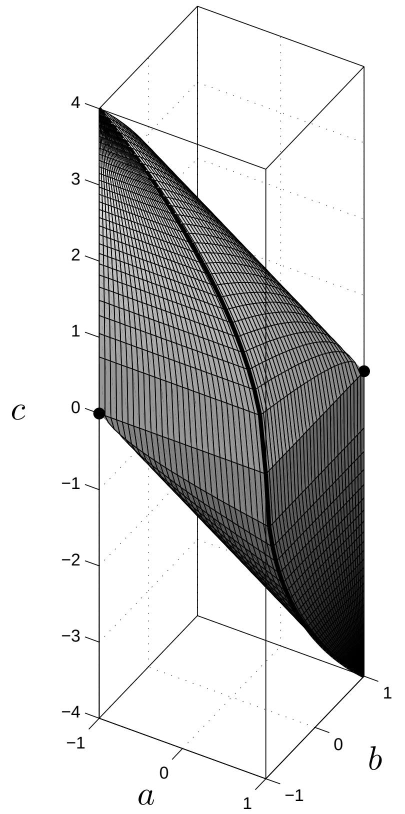

Figure 4: $\mathsf{S}_{D(\frac{\pi}{2})}$. The extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ are drawn with a thick line and dots.

4.1. The Bohnenblust-Hille inequality and related problems

The ℓ_q norm of the coefficients of polynomials in $\mathcal{P}(^m\mathbb{K}^n)$ is given by

$$|P|_q := \begin{cases} \left(\sum_{|\alpha|=m} |a_\alpha|^q\right)^{\frac{1}{q}} & \text{if } 1 \leq q < +\infty, \\ \max\{|a_\alpha| \,:\, |\alpha| = m\} & \text{if } q = +\infty, \end{cases}$$

for every $P \in \mathcal{P}(^m\mathbb{K}^n)$ with coefficients a_α. Observe that

$$|\cdot|_q \leq |\cdot|_s \leq d^{\frac{1}{s}-\frac{1}{q}} |\cdot|_q,$$

for $1 \leq s \leq q$, where d is the dimension of $\mathcal{P}(^m\mathbb{K}^n)$. These norms appear in a number of problems of interest. They can also be used to estimate the difficult-to-calculate polynomial norm of the spaces $\mathcal{P}(^m\ell_p^n)$. Let us denote the norm in $\mathcal{P}(^m\ell_p^n)$ by $\|\cdot\|_p$. Since the norms $|\cdot|_q$ and $\|\cdot\|_p$ are equivalent for all $p, q \geq 1$, there exist k and K depending on p, q, m, n such that

$$k\|P\|_p \leq |P|_q \leq K\|P\|_p,$$

for all $P \in \mathcal{P}(^m\mathbb{R}^n)$. The optimal values of the constants k and K can be calculated in many situations using the Krein-Milman approach. Indeed, as for the constant K, the target function to which the Krein-Milman approach could be applied is

$$B_{\|\cdot\|_p} \ni P \mapsto |P|_q.$$

Here is where the geometry of $B_{\|\cdot\|_p}$ can be used to optimise the target functions in the case when we have a description of its extreme points.

The equivalence constants which we have just introduced are closely related to the famous polynomial Bohnenblust-Hille constants. Let us call $K_{m,n,q,p}$ the best (smallest) value of K in (4.1). The m-th polynomial Bohnenblust-Hille constant is nothing but an upper bound for $K_{m,n,\frac{2m}{m+1},\infty}$. The reason why the specific choice $q = \frac{2m}{m+1}$ and $p = \infty$ is of interest rests on the fact that, if $q \geq \frac{2m}{m+1}$, then there exists a constant $D_{m,q} > 0$ depending only on m and q such that

$$|P|_q \leq D_{m,q} \|P\|_\infty, \tag{6}$$

for all $P \in \mathcal{P}(^m\mathbb{K}^n)$ and every $n \in \mathbb{N}$. Moreover, any constant fitting in (6) for $q < \frac{2m}{m+1}$ depends necessarily on n. This result was proved by Bohnenblust and Hille [7] in 1931. Observe that any plausible choice for $D_{m,q}$ in (6) must satisfy $D_{m,q} \geq \sup\{K_{m,n,q,\infty} \, : \, n \in \mathbb{N}\}$. The best (in the sense of smallest) possible choice for $D_{m,q}$ in (6) when $q = \frac{2m}{m+1}$ is called the polynomial Bohnenblust-Hille constant. It is interesting to notice that there exists a considerable difference between the polynomial Bohnenblust-Hille constants for real and complex polynomials. For this reason, the polynomial Bohnenblust-Hille constants are usually denoted by $D_{\mathbb{K},m}$.

Moreover, if we keep $n \in \mathbb{N}$ fixed, the best (smallest) $D_m(n) > 0$ in

$$|P|_{\frac{2m}{m+1}} \leq D_m(n) \|P\|_\infty,$$

for all $P \in \mathcal{P}(^m\mathbb{K}^n)$, is denoted by $D_{\mathbb{K},m}(n)$. Observe that $D_{\mathbb{K},m}(n) = K_{m,n,\frac{2m}{m+1},\infty}$. The calculation of the Bohnenblust-Hille constants $D_{\mathbb{K},m}$ and $D_{\mathbb{K},m}(n)$ has motivated a large amount of papers, but their exact values are still unknown except for very restricted choices of m and n. The best lower and upper estimates on $D_{\mathbb{K},m}$ and $D_{\mathbb{K},m}(n)$ known nowadays can be found in the literature [3, 8, 9, 12–14, 20, 25].

We present below a simple application of the Krein-Milman approach to calculate the value of $D_{\mathbb{R}}(2)$ based on the following result by Choi, Kim, and Ki [11].

Theorem 21. *The set* $\mathrm{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))})$ *of extreme points of the unit ball of* $\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))$ *is given by*

$$\mathrm{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))}) = \{\pm x^2, \ \pm y^2, \ \pm(tx^2 - ty^2 \pm 2\sqrt{t(1-t)}xy) \, : \, t \in [1/2, 1]\}.$$

Theorem 22. *Let f be the real-valued function given by*

$$f(t) = \left[2t^{\frac{4}{3}} + (2\sqrt{t(1-t)})^{\frac{4}{3}}\right]^{\frac{3}{4}}.$$

We have that $D_{\mathbb{R},2}(2) = f(t_0) \approx 1.837\,373$*, where*

$$t_0 = \frac{1}{36}\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right) \approx 0.867\,835.$$

Moreover, the following normalized polynomials are extreme for this problem:

$$P_2(x,y) = \pm\left(t_0x^2 - t_0y^2 \pm 2\sqrt{t_0(1-t_0)}xy\right).$$

Proof. We just have to notice that, due to the convexity of the ℓ_p-norms and theorem 21, we have

$$D_{\mathbb{R},2}(2) = \sup\{|\mathbf{a}|_{\frac{4}{3}} \, : \, \mathbf{a} \in \mathsf{B}_{\mathcal{P}(^2\ell_\infty^2\mathbb{R})}\} = \sup\{|\mathbf{a}|_{\frac{4}{3}} \, : \, \mathbf{a} \in \mathrm{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2\mathbb{R})})\} = \sup_{t\in[1/2,1]} f(t).$$

The function f is maximized using elementary calculus. The help of computer packages of symbolic calculus such as Matlab may be helpful to prove that f attains its maximum in $[1/2, 2]$ at $t = t_0$, thus concluding the proof. ∎

4.2. Bernstein and Markov type inequalities

Bernstein and Markov inequalities are estimates on the growth of the derivatives of polynomials. The famous Russian chemist D. Mendeleev (the author of the periodic table of elements) was among the pioneers that studied these types of estimates. In particular, he was interested in the following problem:

> If $p(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and we define $\|p\|_{[\alpha,\beta]} := \max\{|p(x)| : x \in [\alpha, \beta]\}$, then what is the smallest possible constant $M_2(\alpha, \beta) > 0$ so that $|p'(x)| \leq M_2(\alpha, \beta)\|p\|_{[\alpha,\beta]}$ for every $x \in [\alpha, \beta]$ and every quadratic polynomial p?

Considering an appropriate change of variable, namely $x \to [\alpha + \beta + (\beta - \alpha)x]/2$, it can be seen that $M_2(\alpha, \beta) = 2/(\beta - \alpha)M_2(-1, 1)$ and, hence, we can restrict ourselves to (quadratic) polynomials on the standard interval $[-1, 1]$. Mendeleev gave his own solution to the problem proving that $M_2(-1, 1) = 4$. Mendeleev's result was generalised by A. A. Markov in 1889 for polynomials of arbitrary degree [26]. What A. A. Markov proved was that

$$\|P'\|_{[-1,1]} \leq n^2\|P\|_{[-1,1]}, \tag{7}$$

with equality for the n-th Chebyshev polynomial of the first kind, defined, for $x \in [-1, 1]$, by $T_n(x) = \cos(n \arccos x)$. V. A. Markov [27] (brother of A. A. Markov) provided in 1892 a sharp estimate on the norm of the k-th derivative of a polynomial of arbitrary degree. A. A. Markov's inequality (7) can be improved in the inner points of $[-1, 1]$. Let $M_n(x)$ be the optimal constant in

$$|P'(x)| \leq M\|P\|_{[-1,1]}, \qquad \text{for all } P \in \mathcal{P}_n(\mathbb{R}).$$

According to a classical result due to Bernstein [6], we have $M_n(x) \leq \frac{n}{\sqrt{1-x^2}}$ in $(-1, 1)$. Both the uniform Markov type estimates on the norm of the derivative and the pointwise estimates due to Bernstein have been generalised in many different ways in the last century. One of the most popular generalisations is due to Harris in 2010 [19]. He proved that the old A. A. Markov constant n^2 is valid for polynomials on any real Banach space, that is, if P is a polynomial of arbitrary degree n on a real Banach space E, then $\|DP(x)\| \leq n^2\|P\|$. Obviously, the constant n^2 is optimal in the general case too. Many Bernstein and Markov type estimates can be obtained by applying the Krein-Milman approach. We present here a worked out example where the Markov and Bernstein optimal estimates are obtained for the space of trinomials $\mathcal{P}_{m,n} = \{ax^m + bx^n + c\}$ endowed with the norm

$$\|ax^m + bx^n + c\|_{m,n} = \sup\{|ax^m + bx^n + c| : x \in [-1, 1]\}.$$

Observe that the polynomial $ax^m + bx^n + c$ in $\mathcal{P}_{m,n}$ can be identified with (a, b, c) in $\mathbb{R}^3$. The geometry of the space $\mathcal{P}_{m,n}$ was studied by Muñoz-Fernández and Seoane-Sepúlveda [33], and the optimal value of the Markov constant $M_{m,n}$ and the Bernstein function $M_{m,n}(x)$ where calculated by Muñoz-Fernández, Sarantopoulos, and Seoane-Sepúlveda [32] when m is odd and n is even. We reproduce here the complete reasoning, based on the Krein-Milman approach and the following characterisation of the extreme points of $\mathsf{B}_{m,n}$ (unit ball of $\mathcal{P}_{m,n}$) when m is odd and n is even.

Theorem 23. *If $m, n \in \mathbb{N}$ are such that m is odd, n is even and $m > n$, the extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{m,n})$ are*

$$\{\pm(0, 2, -1), \pm(1, 1, -1), \pm(1, -1, 1), \pm(0, 0, 1)\}.$$

Theorem 24. *Let $m, n \in \mathbb{N}$ be such that m is odd, n is even and $m > n$. Then,*

$$M_{m,n}(x) = \begin{cases} 2n|x|^{n-1} & \text{if } 0 \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}, \\ mx^{m-1} + n|x|^{n-1} & \text{if } \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq 1. \end{cases} \tag{8}$$

Proof. If $x \in [-1, 1]$, by definition we have that

$$M_{m,n}(x) = \sup_{p \in \mathsf{B}_{m,n}} |p'(x)|.$$

It suffices to work just with the extreme polynomials of $M_{m,n}$, which are given in theorem 23. Notice that the contribution of ± 1 to $M_{m,n}(x)$ is irrelevant. Hence, it suffices to consider the polynomials

$$p_1(x) = \pm(2x^n - 1), \quad p_2(x) = \pm(x^m + x^n - 1) \quad \text{and} \quad p_3(x) = \pm(x^m - x^n + 1).$$

Therefore,

$$\begin{aligned} M_{m,n}(x) &= \max\{|p_1'(x)|, |p_2'(x)|, |p_3'(x)|\} \\ &= \max\{2n|x|^{n-1}, |mx^{m-1} + nx^{n-1}|, |mx^{m-1} - nx^{n-1}|\} \\ &= \max\{2n|x|^{n-1}, mx^{m-1} + n|x|^{n-1}\} \\ &= |x|^{n-1} \max\{2n, m|x|^{m-n} + n\}, \end{aligned}$$

and since $2n \leq m|x|^{m-n} + n$ and $\left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x|$ are equivalent, the result follows immediately. ∎

Corollary 25. *If $m, n \in \mathbb{N}$ are such that m is odd, n is even and $m > n$, then*

$$M_{m,n} = M_{m,n}(\pm 1) = m + n,$$

and equality is attained for the polynomials $p(x) = \pm(x^m + x^n - 1)$ and $p(x) = \pm(x^m - x^n + 1)$.

References

[1] Aron, Richard M. and Klimek, Maciej. “Supremum norms for quadratic polynomials”. In: *Archiv der Mathematik* 76.1 (2001), pp. 73–80. issn: 0003-889X. https://doi.org/10.1007/s000130050544.

[2] Banach, Stefan. “Über homogene Polynome in (L^2)”. In: *Studia Math.* 7 (1938), pp. 36–44. https://doi.org/10.4064/sm-7-1-36-44.

[3] Bayart, Frédéric; Pellegrino, Daniel, and Seoane-Sepúlveda, Juan B. “The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$”. In: *Advances in Mathematics* 264 (2014), pp. 726–746. issn: 0001-8708. https://doi.org/10.1016/j.aim.2014.07.029.

[4] Benítez, Carlos and Sarantopoulos, Yannis. “Characterization of real inner product spaces by means of symmetric bilinear forms”. In: *Journal of Mathematical Analysis and Applications* 180.1 (1993), pp. 207–220. issn: 0022-247X. https://doi.org/10.1006/jmaa.1993.1396.

[5] Bernal-González, Luis; Muñoz-Fernández, Gustavo A.; Rodríguez-Vidanes, Daniel L., and Seoane-Sepúlveda, Juan B. “A complete study of the geometry of 2-homogeneous polynomials on circle sectors”. In: *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas. RACSAM* 114.3 (2020), p. 160. issn: 1578-7303. https://doi.org/10.1007/s13398-020-00892-6.

[6] Bernstein, Serge. “Sur l’ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné”. In: *Memoires de l’Académie Royale de Belgique* 4 (1912), pp. 1–103.

[7] Bohnenblust, Henri Frederic and Hille, Einar. “On the absolute convergence of Dirichlet series”. In: *Annals of Mathematics. Second Series* 32.3 (1931), pp. 600–622. issn: 0003-486X. https://doi.org/10.2307/1968255.

[8] Campos, Jamilson Ramos; Jiménez-Rodríguez, Pablo; Muñoz-Fernández, Gustavo A.; Pellegrino, Daniel, and Seoane-Sepúlveda, Juan B. “On the real polynomial Bohnenblust-Hille inequality”. In: *Linear Algebra and its Applications* 465 (2015), pp. 391–400. issn: 0024-3795. https://doi.org/10.1016/j.laa.2014.09.040.

[9] Carando, Daniel; Defant, Andreas, and Sevilla-Peris, Pablo. “The Bohnenblust-Hille inequality combined with an inequality of Helson”. In: *Proceedings of the American Mathematical Society* 143.12 (2015), pp. 5233–5238. issn: 0002-9939. https://doi.org/10.1090/proc/12551.

[10] Choi, Yun Sung and Kim, Sung Guen. “The unit ball of $\mathcal{P}(^2 l_2^2)$”. In: *Archiv der Mathematik* 71.6 (1998), pp. 472–480. issn: 0003-889X. https://doi.org/10.1007/s000130050292.

[11] Choi, Yun Sung; Kim, Sung Guen, and Ki, Haseo. "Extreme polynomials and multilinear forms on l_1". In: *Journal of Mathematical Analysis and Applications* 228.2 (1998), pp. 467–482. issn: 0022-247X. https://doi.org/10.1006/jmaa.1998.6161.

[12] Defant, Andreas; Frerick, Leonhard; Ortega-Cerdà, Joaquim; Ounaïes, Myriam, and Seip, Kristian. "The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive". In: *Annals of Mathematics. Second Series* 174.1 (2011), pp. 485–497. issn: 0003-486X. https://doi.org/10.4007/annals.2011.174.1.13.

[13] Defant, Andreas; García, Domingo, and Maestre, Manuel. "Bohr's power series theorem and local Banach space theory". In: *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]* 557 (2003), pp. 173–197. issn: 0075-4102. https://doi.org/10.1515/crll.2003.030.

[14] Defant, Andreas and Sevilla-Peris, Pablo. "The Bohnenblust-Hille cycle of ideas from a modern point of view". In: *Uniwersytet im. Adama Mickiewicza w Poznaniu. Wydział Matematyki i Informatyki. Functiones et Approximatio Commentarii Mathematici* 50.1 (2014): *A special issue dedicated to Professor Lech Drewnowski, on the occasion of his 70th birthday, I*, pp. 55–127. issn: 0208-6573. https://doi.org/10.7169/facm/2014.50.1.2.

[15] Dineen, Seán. *Complex analysis on infinite-dimensional spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 1999. https://doi.org/10.1007/978-1-4471-0869-6.

[16] Gámez-Merino, José L.; Muñoz-Fernández, Gustavo A.; Pellegrino, Daniel, and Seoane-Sepúlveda, Juan B. "Bounded and unbounded polynomials and multilinear forms: characterizing continuity". In: *Linear Algebra and its Applications* 436.1 (2012), pp. 237–242. issn: 0024-3795. https://doi.org/10.1016/j.laa.2011.06.050.

[17] Gámez-Merino, José L.; Muñoz-Fernández, Gustavo A.; Sánchez, Viktor M., and Seoane-Sepúlveda, Juan B. "Inequalities for polynomials on the unit square via the Krein-Milman theorem". In: *Journal of Convex Analysis* 20.1 (2013), pp. 125–142. issn: 0944-6532.

[18] Grecu, Bogdan C. "Geometry of 2-homogeneous polynomials on l_p spaces, $1 < p < \infty$". In: *Journal of Mathematical Analysis and Applications* 273.2 (2002), pp. 262–282. issn: 0022-247X. https://doi.org/10.1016/S0022-247X(02)00217-2.

[19] Harris, Lawrence A. "A proof of Markov's theorem for polynomials on Banach spaces". In: *Journal of Mathematical Analysis and Applications* 368.1 (2010), pp. 374–381. issn: 0022-247X. https://doi.org/10.1016/j.jmaa.2010.02.050.

[20] Jiménez-Rodríguez, Pablo; Muñoz-Fernández, Gustavo A.; Murillo-Arcila, Marina, and Seoane-Sepúlveda, Juan B. "Sharp values for the constants in the polynomial Bohnenblust-Hille inequality". In: *Linear and Multilinear Algebra* 64.9 (2016), pp. 1731–1749. issn: 0308-1087. https://doi.org/10.1080/03081087.2015.1115810.

[21] Kellogg, Oliver Dimon. "On bounded polynomials in several variables". In: *Mathematische Zeitschrift* 27.1 (1928), pp. 55–64. issn: 0025-5874. https://doi.org/10.1007/BF01171085.

[22] Kim, Sung Guen. "The unit ball of $\mathcal{P}(^2D_*(1,W)^2)$". In: *Mathematical Proceedings of the Royal Irish Academy* 111A.2 (2011), pp. 79–94. issn: 1393-7197. https://doi.org/10.3318/pria.2011.111.1.9.

[23] Kirwan, Pádraig; Sarantopoulos, Yannis, and Tonge, Andrew M. "Extremal homogeneous polynomials on real normed spaces". In: *Journal of Approximation Theory* 97.2 (1999), pp. 201–213. issn: 0021-9045. https://doi.org/10.1006/jath.1996.3273.

[24] Konheim, Alan G. and Rivlin, Theodore J. "Extreme points of the unit ball in a space of real polynomials". In: *American Mathematical Monthly* 73 (1966), pp. 505–507. issn: 0002-9890. https://doi.org/10.2307/2315472.

[25] Maia, Mariana; Nogueira, Tony, and Pellegrino, Daniel. "The Bohnenblust-Hille inequality for polynomials whose monomials have a uniformly bounded number of variables". In: *Integral Equations and Operator Theory* 88.1 (2017), pp. 143–149. issn: 0378-620X. https://doi.org/10.1007/s00020-017-2372-z.

[26] Markov, Andrey A. "On a problem of D. I. Mendeleev". In: *Zap. Im. Akad. Nauk* 62 (1889), pp. 1–24.

[27] Markov, Vladimir A. “Über Polynome, die in einen gegebenen Intervalle möglichst wenig von Null abweichen”. In: *Mathematische Annalen* 77 (1916), pp. 213–258. issn: 0025-5831. url: http://eudml.org/doc/158723.

[28] Milev, Lozko and Naidenov, Nikola. “Indefinite extreme points of the unit ball in a polynomial space”. In: *Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum* 77.3-4 (2011), pp. 409–424. issn: 0001-6969.

[29] Milev, Lozko and Naidenov, Nikola. “Semidefinite extreme points of the unit ball in a polynomial space”. In: *Journal of Mathematical Analysis and Applications* 405.2 (2013), pp. 631–641. issn: 0022-247X. https://doi.org/10.1016/j.jmaa.2013.04.026.

[30] Muñoz-Fernández, Gustavo A.; Pellegrino, Daniel; Seoane-Sepúlveda, Juan B., and Weber, Andreas. “Supremum norms for 2-homogeneous polynomials on circle sectors”. In: *Journal of Convex Analysis* 21.3 (2014), pp. 745–764. issn: 0944-6532.

[31] Muñoz-Fernández, Gustavo A.; Révész, Szilárd Gy., and Seoane-Sepúlveda, Juan B. “Geometry of homogeneous polynomials on non symmetric convex bodies”. In: *Mathematica Scandinavica* 105.1 (2009), pp. 147–160. issn: 0025-5521. https://doi.org/10.7146/math.scand.a-15111.

[32] Muñoz-Fernández, Gustavo A.; Sarantopoulos, Yannis, and Seoane-Sepúlveda, Juan B. “An application of the Krein-Milman theorem to Bernstein and Markov inequalities”. In: *Journal of Convex Analysis* 15.2 (2008), pp. 299–312. issn: 0944-6532.

[33] Muñoz-Fernández, Gustavo A. and Seoane-Sepúlveda, Juan B. “Geometry of Banach spaces of trinomials”. In: *Journal of Mathematical Analysis and Applications* 340.2 (2008), pp. 1069–1087. issn: 0022-247X. https://doi.org/10.1016/j.jmaa.2007.09.010.

[34] Neuwirth, Stefan. “The maximum modulus of a trigonometric trinomial”. In: *Journal d'Analyse Mathématique* 104 (2008), pp. 371–396. issn: 0021-7670. https://doi.org/10.1007/s11854-008-0028-2.

[35] Rodríguez-Vidanes, Daniel. *Constantes de Polarización*. Master's thesis. Universidad Complutense de Madrid, 2018.

[36] Sarantopoulos, Yannis. “Estimates for polynomial norms on $L^p(\mu)$ spaces”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 99.2 (1986), pp. 263–271. issn: 0305-0041. https://doi.org/10.1017/S0305004100064185.

[37] Sarantopoulos, Yannis. “Extremal multilinear forms on Banach spaces”. In: *Proceedings of the American Mathematical Society* 99.2 (1987), pp. 340–346. issn: 0002-9939. https://doi.org/10.2307/2046637.

[38] Van der Corput, Johannes G. and Schaake, Gerrit. “Berichtigung zu: Ungleichungen für Polynome und trigonometrische Polynome”. In: *Compositio Mathematica* 3 (1936), p. 128. issn: 0010-437X. url: http://www.numdam.org/item?id=CM_1936__3__128_0.

IX *Escuela-Taller de Análisis Funcional* (2019)

Maximal averaging operators: from geometry to boundedness through duality

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Abstract: We study boundedness of maximal averaging operators through covering properties of their corresponding differentiation bases. This link between analysis and geometry is provided by duality arguments, following the original approach of Córdoba and Fefferman. Ultimately, this provides analogues of the Lebesgue differentiation theorem for certain differentiation bases. We pay special attention to the case of the strong maximal operator, where we prove the strong maximal theorem of Jessen, Marcinkiewicz and Zygmund by means of the previous general approach combined with induction on the dimension and interpolation arguments.

Resumen: Se estudia la acotación de operadores maximales en términos de propiedades de recubrimiento de las correspondientes bases de diferenciación. Esta conexión entre análisis y geometría viene dada por argumentos de dualidad, siguiendo el enfoque original de Córdoba y Fefferman. En último término, lo anterior permite obtener análogos al teorema de diferenciación de Lebesgue para ciertas bases de diferenciación. Se presta especial atención al caso del operador maximal fuerte, para el que se prueba el teorema maximal fuerte de Jessen, Marcinkiewicz y Zygmund combinando el enfoque general anterior con un argumento de inducción en la dimensión y argumentos de interpolación.

Keywords: differentiation basis, duality, strong maximal function, Lebesgue differentiation theorem.

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1. Introduction

The Lebesgue differentiation theorem is a classical result in real analysis (see, for instance, Wheeden and Zygmund's book [10, Chapter 7]) which states that, for every $1 \le p \le \infty$, the collection of open euclidean balls in $\mathbb{R}^n$ differentiate every function in $L^p(\mathbb{R}^n)$ almost everywhere. A more precise statement of this result can be found in theorem 1 below; before its formulation, we introduce some notation and several definitions.

In this article, n will denote a positive natural number and p will be a number on the extended real interval $[1, +\infty]$.

A *differentiation basis* is a family $\mathcal{B}$ consisting of bounded open sets in $\mathbb{R}^n$ whose union is the whole space and which is homothecy invariant, that is, for every $x \in \mathbb{R}^n$, every $\lambda \in \mathbb{R}$, and every $B \in \mathcal{B}$ we have that $B + x, \lambda B \in \mathcal{B}$. It is straightforward to check that the collection of all Euclidean balls in $\mathbb{R}^n$ is a differentiation basis. We will denote this basis by $\mathcal{B}_n$. Two other differentiation bases which we will be using in this text are the cubes in $\mathbb{R}^n$, and the rectangular parallelepipeds (rectangles) in $\mathbb{R}^n$, with sides parallel to the coordinate axes; these will be denoted by $\mathcal{Q}_n$ and $\mathcal{R}_n$, respectively.

Now let $\mathcal{B}$ be a differentiation basis and $\phi : \mathcal{B} \to \mathbb{R}_+$ be a set function. We write

$$\lim_{\substack{B \searrow x \\ B \in \mathcal{B}}} \phi(B)$$

to denote the limit of $\phi(B)$ as the diameter of B tends to 0 and $x \in B$, for sets in the differentiation basis $\mathcal{B}$. When we work with only one differentiation basis we will simply write $\lim_{B \searrow x} \phi(B)$.

With this notation we can state the Lebesgue differentiation theorem in the following way.

Theorem 1 (Lebesgue differentiation theorem). *Let $n \in \mathbb{N}$ and $1 \le p \le \infty$. For every $f \in L^p(\mathbb{R}^n)$ we have that, for almost every $x \in \mathbb{R}^n$,*

$$\lim_{\substack{B \searrow x \\ B \in \mathcal{B}_n}} \frac{1}{|B|} \int_B |f| \to f.$$

One of the purposes of this article is to study conditions under which the Lebesgue differentiation theorem holds when we substitute $\mathcal{B}_n$ by other differentiation bases. We carry out this analysis in detail for the basis $\mathcal{R}_n$ consisting of n-dimensional rectangles.

Arguably, the most important tool in the study of differentiation bases is the corresponding *maximal operator*. This is a sublinear operator that can be associated with every differentiation basis and whose properties are closely related to the respective differentiation properties of the bases. We give the definition below.

Definition 2. Let $\mathcal{B}$ be a differentiation basis. The **maximal operator** associated with $\mathcal{B}$ is defined for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ as

$$M_{\mathcal{B}} f(x) := \sup_{\substack{B \in \mathcal{B} \\ x \in B}} \frac{1}{|B|} \int_B f, \qquad x \in \mathbb{R}^n.$$ ◂

It is not hard to see that $M_{\mathcal{B}} f$ is a well-defined measurable function whenever $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. It is also easy to check that $M_{\mathcal{B}}$ is sublinear on $L^1_{\text{loc}}(\mathbb{R}^n)$:

$$M_{\mathcal{B}}(f+g) \le M_{\mathcal{B}}(f) + M_{\mathcal{B}}(g) \qquad \forall f, g \in L^1_{\text{loc}}(\mathbb{R}^n).$$

An easy consequence of the sublinearity that we will use below is that

$$|M_{\mathcal{B}} f - M_{\mathcal{B}} g| \le M_{\mathcal{B}}(f-g) \qquad \forall f, g \in L^1_{\text{loc}}(\mathbb{R}^n). \tag{1}$$

We briefly discuss the importance of maximal operators in the subject of differentiation bases. For an extensive discussion of the properties of differentiation bases and related differentiation theorems we send the interested readers to De Guzmán's monograph [5].

Let $\mathcal{B}$ be a differentiation basis, let $\varepsilon > 0$, and define the sublinear operator T_ε by means of

$$T_\varepsilon f(x) := \sup_{\substack{B\in\mathcal{B},\ \mathrm{diam}(B)<\varepsilon \\ x\in B}} \frac{1}{|B|}\int_B f, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \quad x \in \mathbb{R}^n.$$

It is clear that theorem 1 is true for $\mathcal{B}$ if $(T_\varepsilon f)_\varepsilon$ converges to f almost everywhere as $\varepsilon \to 0^+$, for every function $f \in L^p(\mathbb{R}^n)$. Let us fix f to be such a function and, for simplicity, let us assume that $1 \le p < \infty$. Then, for every $g \in C_c^\infty(\mathbb{R}^n)$ we use sublinearity in the form of (1) to write

$$|T_\varepsilon f(x) - f(x)| \le |T_\varepsilon(f-g)(x)| + |T_\varepsilon g(x) - g(x)| + |g(x) - f(x)|. \tag{2}$$

Now, in order to prove that $(T_\varepsilon f)_\varepsilon$ converges to f almost everywhere, it is enough to show that

$$\limsup_{\varepsilon\to 0^+} |T_\varepsilon f(x) - f(x)| = 0 \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

In turn, this will follow if we manage to show that for every $\lambda > 0$ we have that

$$|\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0^+} |T_\varepsilon f(x) - f(x)| > \lambda\}| = 0. \tag{3}$$

We now fix $\lambda > 0$ and, by estimate (2), we have that

$$\begin{aligned} &|\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0^+} |T_\varepsilon f(x) - f(x)| > \lambda\}| \\ &\quad\le |\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0^+} |T_\varepsilon f(x) - T_\varepsilon g(x)| > \lambda/3\}| + |\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0^+} |T_\varepsilon g(x) - g(x)| > \lambda/3\}| \\ &\qquad + |\{x \in \mathbb{R}^n : |g(x) - f(x)| > \lambda/3\}| \\ &\quad\le |\{x \in \mathbb{R}^n : M_{\mathcal{B}}(f-g) > \lambda/3\}| + \frac{3^p}{\lambda^p}\|f-g\|^p_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{4}$$

Passing to the the last line in the estimate above we have used the easily verifiable fact that

$$\limsup_{\varepsilon\to 0^+} |T_\varepsilon g(x) - g(x)| = 0 \qquad \forall g \in C_c^\infty(\mathbb{R}^n)$$

together with the fact that $\sup_{\varepsilon>0} T_\varepsilon(f-g) \le M_{\mathcal{B}}(f-g)$. The condition that we need on $M_{\mathcal{B}}$ in order to deal with the first term in the last line of (4) is given in the following definition.

Definition 3. Let T be a sublinear operator defined on locally integrable functions, and let $1 \le p < \infty$. We say that T is of **weak-type (p, p)** if there exists $C > 0$, depending only upon T, p, and the dimension n, such that

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \le \frac{C^p}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x, \qquad \forall f \in L^p(\mathbb{R}^n), \quad \forall \lambda > 0.$$ ◂

From the discussion above about estimate (4), it is easy to show that, if $M_{\mathcal{B}}$ is of weak-type (p, p), then theorem 1 holds for the differentiation basis $\mathcal{B}$ and the index p. Indeed, by (4) and the weak-type property of $M_{\mathcal{B}}$ we have that

$$|\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0^+} |T_\varepsilon f(x) - f(x)| > \lambda\}| \le \frac{C^p 3^p}{\lambda^p}\|f-g\|^p_{L^p(\mathbb{R}^n)} + \frac{3^p}{\lambda^p}\|f-g\|^p_{L^p(\mathbb{R}^n)}.$$

Now, for any $\delta > 0$, we can choose $g \in C_c^\infty(\mathbb{R}^n)$ such that $\max(C,1)^p 3^p \lambda^{-p}\|f-g\|^p_{L^p(\mathbb{R}^n)} < \delta/2$, which is possible since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, showing that $|\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0^+} |T_\varepsilon f(x) - f(x)| > \lambda\}| < \delta$.

The discussion above proves (3), and thus theorem 1, under the weak-type (p, p)-assumption for $M_{\mathcal{B}}$. Therefore, our first goal will be to determine under which circumstances the maximal operator associated to a differentiation basis is of weak-type (p, p). To this end, we establish a geometric characterisation of the weak-type (p, p) property, given in terms of the sets of the differential basis, in section 2. This is used to prove that the basis of cubes $\mathcal{Q}_n$ differentiate every function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In the last section, we study the case of the basis of rectangles $\mathcal{R}_n$, where the associated maximal operator is not of weak type $(1, 1)$. Instead, we prove the so called *strong maximal theorem*, which is the appropriate replacement of the weak-type $(1, 1)$ property for the maximal operator associated with the basis $\mathcal{R}_n$.

In the statement of the theorem below we use the standard notation $\log^+ t := \max(0, \log t)$ for $t > 0$.

Theorem 4 (strong maximal theorem [6]). *The following estimate holds for all $\lambda > 0$:*

$$|\{x \in \mathbb{R}^n : M_{\mathcal{R}_n} f(x) > \lambda\}| \le C_n \int_{\mathbb{R}^n} \frac{|f|}{\lambda}\Big(1 + \Big(\log^+ \frac{|f|}{\lambda}\Big)^{n-1}\Big),$$

where C_n depends only upon the dimension. It follows that $\mathcal{R}_n$ differentiates functions f for which

$$\int_K |f(x)|\big(1 + (\log^+ |f(x)|)^{n-1}\big)\,\mathrm{d}x < \infty \qquad \textit{for every compact set } K \subset \mathbb{R}^n.$$

The original proof of this theorem is due to Jessen, Marcinkiewicz, and Zygmund [6]. However, we are not going to reproduce the original analytical proof given there. Instead, we will follow the ideas of Córdoba and Fefferman [2], a geometrical approach using covering properties of a differentiation basis in order to prove boundedness of the corresponding maximal operator. In this context, the precise link between the analytic and geometric statements will be given by duality of suitable function spaces and the adjoint of the (linearised) maximal operator associated with a given differentiation basis.

Note that maximal operators are not linear, but sublinear, so in order to define the adjoint operator we will consider a linear operator T bounded by the maximal operator $M_{\mathcal{B}}$ so that T is a *linearisation* of $M_{\mathcal{B}}$. There are several ways to linearise a maximal operator depending on the differentiation basis $\mathcal{B}$. A useful example of a linearisation technique can be found in the proof of proposition 6.

2. Duality link between analysis and geometry

We start by defining a geometric property for differentiation bases.

Definition 5. Let $1 \le q \le \infty$. We say that $\mathcal{B}$ has the **covering property** V_q if there exist $c_1, c_2 > 0$ depending only on $\mathcal{B}, q$ and the dimension such that, for every finite collection $\{B_j\}_{j=1}^N \subset \mathcal{B}$, there exists a finite subcollection $\{\tilde{B}_k\}_{k=1}^M \subseteq \{B_j\}_{j=1}^N$ satisfying

(i) $\displaystyle\Big|\bigcup_{j=1}^N B_j\Big| \le c_1 \Big|\bigcup_{k=1}^M \tilde{B}_k\Big|$,

(ii) $\displaystyle\Big\|\sum_{k=1}^M \mathbf{1}_{\tilde{B}_k}\Big\|_{L^q(\mathbb{R}^n)} \le c_2 \Big|\bigcup_{j=1}^N B_j\Big|^{\frac{1}{q}}$. ◂

Property (i) of the definition above roughly states that we did not lose too much measure when passing to the subcollection, while property (ii) is an L^q-control of the overlap of the sets in the subcollection.

The next proposition establishes the duality between the latter geometric property on the basis $\mathcal{B}$ and the analytical weak-type (p, p) condition of its maximal operator $M_{\mathcal{B}}$.

Proposition 6. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The maximal operator $M_{\mathcal{B}}$ is of weak-type (p, p) if and only if $\mathcal{B}$ has the covering property $V_{p'}$.*

Proof. We start by showing necessity. Suppose that $\mathcal{B}$ is a differentiation basis with the covering property $V_{p'}$. Let f be a function in $L^p(\mathbb{R}^n)$ and, for $\lambda > 0$, consider the set $E_\lambda \coloneqq \{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \lambda\}$. If a point x is in E_λ, by definition we have that there exists a set $B_x \in \mathcal{B}$ containing x such that

$$|B_x| < \frac{1}{\lambda} \int_{B_x} |f(y)|\,\mathrm{d}y. \tag{5}$$

Therefore, we have that the set E_λ is contained in the union of the family $C_\lambda \coloneqq \{B_x : x \in E_\lambda\} \subset \mathcal{B}$ where the B_x are selected satisfying property (5). It is straightforward to check that this inclusion is actually an equality.

Next, we consider a compact set $K \subset E_\lambda = \bigcup_{x \in E_\lambda} B_x$. By compactness, there exists a finite collection $\{B_j\}_{j=1}^N \subset C_\lambda$ covering K. Now we apply the hypothesis, and there exists a finite subcollection $\{\tilde{B}_k\}_{k=1}^M \subset \{B_j\}_{j=1}^N$ satisfying properties (i) and (ii) from definition 5. By property (i) it follows that

$$|K| \leq c_1 \Big| \bigcup_{k=1}^{M} \tilde{B}_k \Big|. \tag{6}$$

The sets $\tilde{B}_k$ verify inequality (5) for $k = 1, \dots, M$ because they are chosen from the original collection C_λ. Therefore, we have

$$\Big| \bigcup_{k=1}^{M} \tilde{B}_k \Big| \leq \sum_{k=1}^{M} |\tilde{B}_k| \leq \frac{1}{\lambda} \int_{\mathbb{R}^n} \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k}(y)|f(y)|\, \mathrm{d}y. \tag{7}$$

Now, this is the integral of the product of two positive integrable functions, so we can use Hölder's inequality to get

$$\int_{\mathbb{R}^n} \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k}(y)|f(y)|\, \mathrm{d}y \leq \Big\| \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k} \Big\|_{p'} \|f\|_p. \tag{8}$$

Next, combining inequalities (7) and (8) and using property (ii) of the definition of $V_{p'}$, we arrive at

$$\Big| \bigcup_{k=1}^{M} \tilde{B}_k \Big| \leq \frac{c_2}{\lambda} \Big| \bigcup_{k=1}^{M} \tilde{B}_k \Big|^{\frac{1}{p'}} \|f\|_p, \tag{9}$$

which, together with (6), implies

$$|K|^{\frac{1}{p}} \leq c_1 \Big| \bigcup_{k=1}^{M} \tilde{B}_k \Big|^{\frac{1}{p}} \leq \frac{c_1 c_2}{\lambda} \|f\|_p. \tag{10}$$

Finally, since the Lebesgue measure is regular, and we have this inequality for every compact set K contained in E_λ, we conclude that $|E_\lambda|^{\frac{1}{p}} = \sup\{|K| : K \subset E_\lambda,\ K \text{ compact}\}^{\frac{1}{p}} \leq c_1 c_2 \lambda^{-1} \|f\|_p$. Since this holds for every $\lambda > 0$ and for every $f \in L^p(\mathbb{R}^n)$, we get that the maximal operator $M_{\mathcal{B}}$ is of weak-type (p, p).

Now, we are going to see that if $M_{\mathcal{B}}$ is of weak-type (p, p), for $1 < p < \infty$, then $\mathcal{B}$ has the covering property $V_{p'}$. To this end, let us consider a finite collection $\{B_j\}_{j=1}^N \subset \mathcal{B}$. Without loss of generality, we can assume that the sets in this collection are ordered by size in measure, $|B_1| \geq |B_2| \geq \dots \geq |B_N|$; this ordering assumption is just for the sake of specificity. Now, we are going to define a selection algorithm to extract a subcollection $\{\tilde{B}_k\}_{k=1}^M$ satisfying inequalities (i) and (ii) in the definition of $V_{p'}$. Start by taking the biggest set in measure, $\tilde{B}_1 = B_1$. Having chosen $\tilde{B}_1, \dots, \tilde{B}_m$, $m < N$, we choose the next set B to be the largest set in measure from $\{B_{m+1}, \dots, B_N\}$ such that

$$\Big| B \cap \bigcup_{j=1}^{m} \tilde{B}_j \Big| \leq \frac{1}{2}|B|.$$

This condition tells us that the sets we are selecting do not overlap more than 50 % in measure. Since the original collection was finite, the selection algorithm stops in finitely many steps.

We have selected a subcollection $\{\tilde{B}_k\}_{k=1}^M \subseteq \{B_j\}_{j=1}^N$. Now, we have to use that the sets in this subcollection satisfy certain overlapping properties, and that $M_{\mathcal{B}}$ is weak-type (p, p) by hypothesis, to prove that this subcollection verifies the conditions required in the definition of the covering property $V_{p'}$.

To prove condition (i) of $V_{p'}$, it is enough to prove an inequality of the type

$$\Big| \bigcup_{\substack{B \text{ not} \\ \text{selected}}} B \Big| \leq C_p \Big| \bigcup_{k=1}^{M} \tilde{B}_k \Big|,$$

with C_p a constant depending only on p. Recall that, if $B \in \{B_j\}_{j=1}^N$ has not been selected, then we have that $|B \cap \bigcup_{k=1}^M \tilde{B}_k| > |B|/2$. Hence, the following inclusions hold,

$$\bigcup_{\substack{B \text{ not} \\ \text{selected}}} B \subseteq \bigcup \left\{B : \frac{|B \cap \bigcup_{k=1}^M \tilde{B}_k|}{|B|} > \frac{1}{2}\right\} \subseteq \left\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k})(x) > \frac{1}{2}\right\},$$

since, if $x \in B$ with B not selected,

$$M_{\mathcal{B}}(\mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k})(x) = \sup_{B \ni x, B \in \mathcal{B}} \frac{1}{|B|} \int_B \mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k}(y)\,\mathrm{d}y = \sup_{B \ni x, B \in \mathcal{B}} \frac{|B \cap \bigcup_{k=1}^M \tilde{B}_k|}{|B|} > \frac{1}{2}.$$

Therefore, using that $M_{\mathcal{B}}$ is of weak-type (p, p), we have that

$$\left|\bigcup_{\substack{B \text{ not} \\ \text{selected}}} B\right| \le \left|\left\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k})(x) > \frac{1}{2}\right\}\right| \le C_p' 2^p \int_{\mathbb{R}^n} |\mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k}(y)|^p\,\mathrm{d}y = C_p' 2^p \left|\bigcup_{k=1}^M \tilde{B}_k\right|$$

with C_p' a constant depending only on p. Hence,

$$\left|\bigcup_{j=1}^n B_j\right| \le (1 + C_p' 2^p) \left|\bigcup_{k=1}^M \tilde{B}_k\right|$$

and we conclude that the subcollection $\{\tilde{B}_k\}_{k=1}^M$ satisfies condition (i) of the covering property $V_{p'}$.

In order to prove condition (ii) of the covering property $V_{p'}$, let us start by defining the collection of sets $\{\tilde{E}_k\}_{k=1}^M$ by setting $\tilde{E}_k := \tilde{B}_k \setminus \bigcup_{j<k} \tilde{B}_j$. It can be easily seen that the sets $\{\tilde{E}_k\}_{k=1}^M$ are pairwise disjoint; furthermore, we have

$$|\tilde{E}_k| \ge \frac{1}{2}|\tilde{B}_k| \qquad \text{and} \qquad \bigcup_{k=1}^M \tilde{E}_k = \bigcup_{k=1}^M \tilde{B}_k.$$

These properties tell us that this new subcollection covers the same space as the one given by the algorithm and that the sets involved have at least half of the measure of the previous ones.

Let us define the following linear and weak-type (p, p) operator

$$T(f)(x) := \sum_{k=1}^M \left(\frac{1}{|\tilde{B}_k|} \int_{\tilde{B}_k} f(y)\,\mathrm{d}y\right) \mathbf{1}_{\tilde{E}_k}(x), \qquad x \in \mathbb{R}^n.$$

Observe that for fixed $x \in \mathbb{R}^n$ the sum above collapses to a single term because of the fact that the sets $\{\tilde{E}_k\}_{k=1}^M$ are pairwise disjoint. This readily implies that $T(f)(x) \le M_{\mathcal{B}}(f)(x)$ for all $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Using Tonelli's theorem we see that

$$T^*(f)(x) = \sum_{k=1}^M \left(\frac{1}{|\tilde{B}_k|} \int_{\tilde{E}_k} f(y)\,\mathrm{d}y\right) \mathbf{1}_{\tilde{B}_k}(x), \qquad x \in \mathbb{R}^n,$$

is the adjoint operator of T. Evaluating T^* at $\mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k}$ and using the properties of the collection $\{\tilde{E}_k\}_{k=1}^M$ above, we get

$$T^*(\mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k})(x) = \sum_{k=1}^M \left(\frac{|\tilde{E}_k|}{|\tilde{B}_k|}\right) \mathbf{1}_{\tilde{B}_k}(x) \ge \frac{1}{2} \sum_{k=1}^M \mathbf{1}_{\tilde{B}_k}(x)$$

for all $x \in \mathbb{R}^n$.

Now we claim that, if a sublinear operator acting on measurable functions in $\mathbb{R}^n$ is of weak-type (p, p) for some $1 < p < \infty$, then for every measurable set $E \subseteq \mathbb{R}^n$ of finite measure we have

$$\int_E |Tf(x)|\,\mathrm{d}x \le C_{p,n,T} \|f\|_{L^p(\mathbb{R}^n)} |E|^{\frac{1}{p'}}. \tag{11}$$

Assuming this claim for a moment and combining it with the estimate above for T^* we obtain

$$\Big|\int_{\mathbb{R}^n}\sum_{k=1}^{M}\mathbf{1}_{\tilde{B}_k}(x)f(x)\,\mathrm{d}x\Big| \le 2\Big|\int_{\mathbb{R}^n} T^*(\mathbf{1}_{\bigcup_{k=1}^M \tilde{B}_k})(x)f(x)\,\mathrm{d}x\Big| = 2\Big|\int_{\bigcup_{k=1}^M \tilde{B}_k} T(f)(x)\,\mathrm{d}x\Big|$$
$$\le 2C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}\Big|\bigcup_{k=1}^{M}\tilde{B}_k\Big|^{\frac{1}{p'}},$$

with $C_{p,n}$ a constant depending only on p and the dimension n. Finally, taking the supremum over all functions $f \in L^p(\mathbb{R}^n)$ with $\|f\|_{L^p(\mathbb{R}^n)} \le 1$ and using that the dual space of $L^p(\mathbb{R}^n)$ is $L^{p'}(\mathbb{R}^n)$, we conclude

$$\Big\|\sum_{k=1}^{M}\mathbf{1}_{\tilde{B}_k}\Big\|_{L^{p'}(\mathbb{R}^n)} \le 2C_{p,n}\Big|\bigcup_{k=1}^{M}\tilde{B}_k\Big|^{\frac{1}{p'}},$$

so $\{\tilde{B}_k\}_{k=1}^M$ satisfies condition (ii) of the covering property $V_{p'}$ and $\mathcal{B}$ has the covering property $V_{p'}$, as we wanted to see.

It remains to prove the claim, which is however a straightforward calculation using the layer-cake decomposition [3, Proposition 2.3]. We have for any $\beta > 0$

$$\int_E |Tf(x)|\,\mathrm{d}x = \int_0^\infty |\{x \in E : |Tf(x)| > \lambda\}|\,\mathrm{d}\lambda \le \beta|E| + \int_\beta^\infty \frac{C^p}{\lambda^p}\|f\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}\lambda$$
$$\le \beta|E| + \frac{C^p}{(p-1)\beta^{p-1}}\|f\|^p_{L^p(\mathbb{R}^n)} \le \frac{C}{(p-1)^{\frac{1}{p}}}\|f\|_{L^p(\mathbb{R}^n)}|E|^{\frac{1}{p'}}$$

by choosing $\beta^p = \frac{C^p}{p-1}\frac{\|f\|_p^p}{|E|}$. This proves the claim with a constant $C_p \simeq p'$ as $p \to 1^+$. ∎

Before ending this section, let us note a couple of remarks.

Remark 7. The claim in the proof above is only valid for $1 < p < \infty$, as one can see also by inspecting the proof. Furthermore, it can be seen that if $\mathcal{B}$ has the covering property V_∞, then $M_{\mathcal{B}}$ is of weak-type $(1, 1)$, but the converse is not true. Another remark that is of some interest is that the claim is actually a characterisation of the weak-type (p, p) for some operator T and $p \in (1, \infty)$. Indeed, assume that (11) is true for p. Then, for $\lambda > 0$, consider the set

$$E_\lambda := \{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}.$$

One needs here some qualitative assumption that will show that $|E_\lambda| < \infty$. This can be made concrete for specific operators T, such as maximal functions, by proving an *a priori* estimate on a nice function f and then extending by density. If one can guarantee that $|E_\lambda| < \infty$, then applying (11) to E yields

$$|E_\lambda|\lambda \le \int_{E_\lambda} |T(f)| \le C_{p,n,T}\|f\|_p|E_\lambda|^{\frac{1}{p'}},$$

which clearly implies the weak-type (p, p) of T when $|E_\lambda| < \infty$.

The principle behind this claim is slightly more general and can be used to define an actual norm on the space $L^{p,\infty}(\mathbb{R}^n)$ for $1 < p < \infty$, which turns these spaces into Banach spaces. For $p = 1$, the space $L^{1,\infty}$ is not normable and only a restricted weaker analogue holds. We refer the interested reader to Grafakos's book [4, Exercise 1.4.14] for further details. ◂

Remark 8. The differentiation basis given by all cubes $\mathcal{Q}_n$ has the covering property V_∞ (this is the well-known Vitali covering lemma [10, Chapter 7]). Hence, we conclude that $\mathcal{Q}_n$ differentiates $L^1(\mathbb{R}^n)$. ◂

3. The strong maximal theorem

In this section we give the proof of theorem 4. First of all, we remember that $\mathcal{R}_n$ is the basis whose elements are open rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

We begin by describing a negative result.

Proposition 9. *The strong maximal operator $M_{\mathcal{R}_n}$ is not of weak-type* $(1,1)$.

Proof. For simplicity, we provide the details in $\mathbb{R}^2$, but essentially the same construction proves the proposition in any dimension. We are going to see that there is no $c>0$ such that

$$|\{x \in \mathbb{R}^2 : M_{\mathcal{R}_2} f(x) > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{R}^2} |f(y)| \, \mathrm{d}y \tag{12}$$

holds for $f \equiv \mathbf{1}_Q$, where $Q=[0,1]^2$. Consider the set $A := \{x \in \mathbb{R}^2 : x_1, x_2 > 1\}$ and take $x=(x_1,x_2) \in A$; see figure 1. We get that

$$M_{\mathcal{R}_2} f(x) = \sup_{\substack{R \in \mathcal{R}_2 \\ R \ni x}} \frac{1}{|R|} \int_R |f(y)| \, \mathrm{d}y = \sup_{\substack{R \in \mathcal{R}_2 \\ R \ni x}} \frac{1}{|R|} \int_Q \mathrm{d}y = \sup_{\substack{R \in \mathcal{R}_2 \\ R \ni x}} \frac{|Q|}{|R|} \geq \frac{1}{x_1 x_2}.$$

Now, for $0<\lambda<1$, let $E_\lambda := \{x \in \mathbb{R}^2 : M_{\mathcal{R}_2} f(x) > \lambda\}$. We have that

$$|E_\lambda| > \left|\left\{x \in \mathbb{R}^2 : x_1 x_2 < \frac{1}{\lambda}\right\}\right| = \int_{\left\{1 < x_1 < \frac{1}{\lambda x_2}, 1 < x_2 < \frac{1}{\lambda}\right\}} \mathrm{d}x_1 \, \mathrm{d}x_2 = \frac{1}{\lambda} \log \frac{1}{\lambda} + 1 - \frac{1}{\lambda} \simeq \frac{1}{\lambda} \log \frac{1}{\lambda},$$

where in the last step the functions on both sides of $\simeq$ are comparable for $\lambda \in (0,1)$. From here we can conclude that (12) does not hold. Otherwise, we would have that

$$\frac{1}{\lambda} \log \frac{1}{\lambda} \leq \frac{c}{\lambda},$$

which is clearly impossible when $\lambda \to 0$.

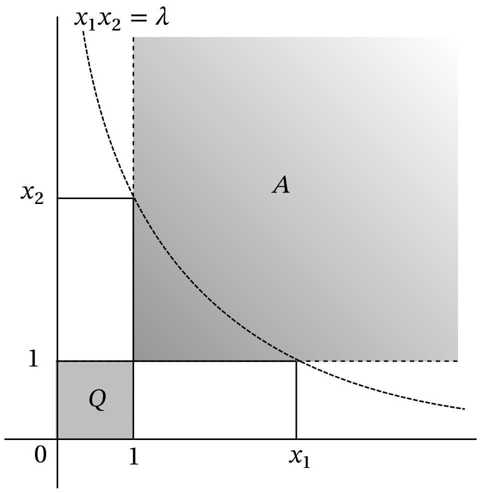

Figure 1: Representation of the function $f \equiv \mathbf{1}_Q$, the set A and the curve $x_1 x_2$.

Thus, the strong maximal operator $M_{\mathcal{R}_2}$ is not of weak-type $(1,1)$. In higher dimensions we just need to work with the higher dimensional unit cube $[0,1]^n$. ∎

In the case of the strong maximal operator, the suitable substitute of the weak $(1,1)$ property is the $L(\log L)^{n-1}$ endpoint estimate of theorem 4. In order to prove this, we will rely on an approach similar to the one outlined in proposition 6, adjusted to the geometry of the basis $\mathcal{R}_n$. The appropriate covering property is given in the following definition.

Definition 10. We say that a differentiation basis $\mathcal{B}$ in $\mathbb{R}^n$ has the **covering property** $V_{\exp,m}$, $m \in \mathbb{N}$, if there exist $c_1, c_2 > 0$ such that for every finite collection $\{R_j\}_{j=1}^N \subset \mathcal{B}$ there is a finite subcollection $\{\tilde{R}_k\}_{k=1}^M$ such that

(i) $\left|\bigcup_{j=1}^N R_j\right| \leq c_1 \left|\bigcup_{k=1}^M \tilde{R}_k\right|$,

(ii) there exists $\theta_o(n) > 0$ such that $\displaystyle\int_{\mathbb{R}^n} \left[\exp\Big(\theta\big(\sum_{k=1}^{M} \mathbf{1}_{\tilde{R}_k}\big)^{\frac{1}{m}}\Big) - 1\right] \leq \theta c_2 \left|\bigcup_{k=1}^{M} \tilde{R}_k\right|$ for every $\theta \in [0, \theta_o(n))$. ◂

The next proposition will be essential in the proof of theorem 4. We give the statement and proof for general bases $\mathcal{B}$ in $\mathbb{R}^n$ although we will only need it for the basis $\mathcal{R}_n$.

Proposition 11. *If the differentiation basis $\mathcal{B}$ has the covering property $V_{\exp,m}$, then there exists $C > 0$ such that*

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \lambda\}| \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^m \mathrm{d}x. \tag{13}$$

Proof. We proceed exactly as in the proof where the covering property $V_{p'}$ implied the weak-type (p, p). Consider the set $E_\lambda \coloneqq \{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \lambda\}$. This set can be written as

$$E_\lambda = \bigcup_{x \in E_\lambda} B_x$$

such that, for all $x \in B_x$, we have

$$\frac{1}{|B_x|} \int_{B_x} |f(y)| \,\mathrm{d}y > \lambda.$$

By taking a compact set $K \subset \bigcup_{x \in E_\lambda} B_x$, we can extract a finite subcover such that $K \subset \bigcup_{j=1}^{N} B_j$ with

$$\frac{1}{|B_j|} \int_{B_j} |f(y)| \,\mathrm{d}y > \lambda.$$

By (i) of definition 10, applying the sub-additive property of the measure and using the last inequality,

$$|K| \leq \left|\bigcup_{j=1}^{N} B_j\right| \leq c_1 \left|\bigcup_{k=1}^{M} \tilde{B}_k\right| \leq c_1 \sum_{k=1}^{M} |\tilde{B}_k| \leq \frac{c_1}{\lambda} \int_{\mathbb{R}^n} \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k}(y) |f(y)| \,\mathrm{d}y.$$

Here we need a generalisation of Hölder's inequality matching the exponential norm in property (ii) of definition 10. In order to prove it, note that $\phi(t) = t(1 + (\log^+ t)^m)$ is a positive and strictly increasing function in $(0, +\infty)$ and $\phi(0) = 0$. Hence, Young's inequality with respect to ϕ guarantees us that

$$st \leq c_{\theta,m} s \left(1 + (\log^+ s)^m\right) + \exp(\theta t^{\frac{1}{m}}) - 1, \tag{14}$$

where $s, t > 0$, θ is a small enough positive value and $c_{\theta,m}$ is a constant value that depends on θ and m; a detailed proof of (14) can be found, for example, in the work of Bagby [1]. Setting

$$s \coloneqq \frac{|f(y)|}{\lambda} \qquad \text{and} \qquad t \coloneqq \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k}(y),$$

we have, by (14) and (ii) of definition 10 that

$$|K| \leq c_1 \left|\bigcup_{k=1}^{M} \tilde{B}_k\right| \leq \frac{c_1}{\lambda} \int_{\mathbb{R}^n} \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k}(y) |f(y)| \,\mathrm{d}y \leq c_1 c_2 \theta \left|\bigcup_{k=1}^{M} \tilde{B}_k\right| + c_1 c_{\theta,m} \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(y)|}{\lambda}\right)^m\right) \mathrm{d}y.$$

Notice that

$$c_1 \left|\bigcup_{k=1}^{M} \tilde{B}_k\right| \leq c_1 c_2 \theta \left|\bigcup_{k=1}^{M} \tilde{B}_k\right| + c_1 c_{\theta,m} \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(y)|}{\lambda}\right)^m\right) \mathrm{d}y$$

and then

$$\frac{1 - c_2 \theta}{c_{\theta,m}} \left|\bigcup_{k=1}^{M} \tilde{B}_k\right| \leq \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(y)|}{\lambda}\right)^m\right) \mathrm{d}y.$$

Choosing θ sufficiently small, letting $K \nearrow E_\lambda$ and using the regularity of the Lebesgue measure, we conclude

$$|E_\lambda| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(y)|}{\lambda}\right)^m\right) \mathrm{d}y,$$

for $C \equiv C(\theta, m)$. ∎

Now, in order to prove our main theorem, it will suffice to show that the differentiation basis $\mathcal{R}_n$ has the covering property $V_{\exp,n-1}$. Note first that we already know the differentiation properties of the strong maximal operator on $\mathbb{R}$, which agrees with the Hardy-Littlewood maximal operator; see remark 8. This sets a first stone on the path for an inductive proof. Indeed, we will prove by induction on the dimension that theorem 4 holds in $\mathbb{R}^n$, with the case $n = 1$, which is the base step of the induction argument, being known to hold true. We will then use the inductive hypothesis, which states that the theorem holds in $\mathbb{R}^{n-1}$, to prove the corresponding covering property $V_{\exp,n-1}$.

With the latter paragraph as a motivation, we introduce two lemmas detailing some precise implications on the boundedness of an operator satisfying (13).

Lemma 12. *Let T be a sublinear operator for which there exists a constant $C > 0$ such that the inequality*

$$|\{x \in \mathbb{R}^n : Tf(x) > \lambda\}| \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^m \mathrm{d}x, \tag{15}$$

holds for every measurable function $f : \mathbb{R}^n \to \mathbb{R}$. In addition, assume that $\|T\|_{L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \leq 1$. Then, T is weak-type (p, p) for every $p > 1$ and the following inequality holds:

$$|\{x \in \mathbb{R}^n : Tf(x) > \lambda\}| \leq C 2^{m+p} \left(\frac{m}{p-1}\right)^m \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^p.$$

Proof. First, fix some $n \in \mathbb{N}$, $p > 1$. For any function f and $\lambda > 0$, define $f_{>\lambda} = f \cdot \mathbf{1}_{>\lambda/2}$, where $\mathbf{1}_{>\lambda/2}$ is the indicator function of the set $\{x \in \mathbb{R}^n : f(x) > \lambda/2\}$. Analogously, define $f_{\leq\lambda/2} = f - f_{>\lambda/2}$. Using the sublinearity of T, it follows that

$$|\{x \in \mathbb{R}^n : Tf(x) > \lambda\}| \leq |\{x \in \mathbb{R}^n : Tf_{>\lambda/2}(x) > \lambda/2\}| + |\{x \in \mathbb{R}^n : Tf_{\leq\lambda/2}(x) > \lambda/2\}|.$$

Note that, since $\|Tf_{\leq\lambda/2}\|_\infty \leq \|T\|_{L^\infty\to L^\infty}\|f_{\leq\lambda/2}\|_\infty \leq \|f_{\leq\lambda/2}\|_\infty \leq \lambda/2$, the latter term in the inequality above equals 0, since an essentially bounded function cannot exceed its essential supremum.

Applying (15) on the surviving term, we get that

$$|\{x \in \mathbb{R}^n : Tf(x) > \lambda\}| \leq 2^m C \int_{\mathbb{R}^n} \frac{|f_{>\lambda/2}(x)|}{\lambda/2} \left(\log \frac{|f_{>\lambda/2}(x)|}{\lambda/2}\right)^m,$$

where we have used the elementary inequality $(1+t)^m/t^m \leq 2^m$ for $t > 1$. In addition, we can use that, for every $\varepsilon > 0$, the estimate $(\log t)^m / t^{\varepsilon m} \leq 1/\varepsilon^m$ holds to obtain

$$|\{x \in \mathbb{R}^n : Tf(x) > \lambda\}| \leq \frac{C 2^m}{\varepsilon^m} \int_{\mathbb{R}^n} \frac{|f_{>\lambda/2}(x)|}{\lambda/2} \left(\frac{|f_{>\lambda/2}(x)|}{\lambda/2}\right)^{\varepsilon m}. \tag{16}$$

Choosing $\varepsilon m = p - 1$ and noting that $f_{\lambda/2} \leq f$ pointwise almost everywhere, we reach at

$$|\{x \in \mathbb{R}^n : Tf(x) > \lambda\}| \leq C 2^{m+p} \left(\frac{m}{p-1}\right)^m \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^p,$$

which is the desired weak-type (p, p) estimate. ∎

In the following lemma we use the weak-type estimates above to obtain strong-type estimates, with appropriate control over the involved constants.

Lemma 13. *Let T be a sublinear operator for which there exists a constant $C > 0$ for which the inequality*

$$|\{x \in \mathbb{R}^n : Tf(x) > \lambda\}| \leq C_1 \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^m, \tag{17}$$

holds for every measurable function $f : \mathbb{R}^n \to \mathbb{R}$. In addition, assume that $\|T\|_{L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \leq 1$. Then, T is strong-type (p, p) for every $p > 1$ and the following estimate holds:

$$\|T\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq C(n, m, C_1) \left(\frac{p}{p-1}\right)^{m+1}. \tag{18}$$

Proof. For any fixed $p > 1$, we estimate the L^p-norm of Tf by using the layer-cake decomposition [3, Proposition 2.3], and then considering again the decomposition $f = f_{>\lambda/2} + f_{\leq\lambda/2}$ together with the sublinearity of T, as follows:

$$\|Tf\|^p_{L^p(\mathbb{R}^n)} = p\int_0^\infty \lambda^{p-1}|\{x\in\mathbb{R}^n : Tf(x) > \lambda\}|\,\mathrm{d}\lambda \leq p\int_0^\infty \lambda^{p-1}|\{x\in\mathbb{R}^n : Tf_{>\lambda/2}(x) > \lambda/2\}|\,\mathrm{d}\lambda.$$

Using the estimate (16) from the proof of lemma 12 and switching the order of integration with Fubini's theorem, we get for any $\varepsilon > 0$ that

$$\begin{aligned}\|Tf\|^p_{L^p(\mathbb{R}^n)} &\leq pC_1\frac{2^{m(1+\varepsilon)}}{\varepsilon^m}\int_{\mathbb{R}^n}|f(x)|^{1+\varepsilon m}\Big(\int_0^{2|f(x)|}\lambda^{p-\varepsilon m-2}\,\mathrm{d}\lambda\Big)\,\mathrm{d}x\\ &= pC_1\frac{2^{m(1+\varepsilon)+p}}{\varepsilon^m}\frac{1}{p-\varepsilon m-1}\int_{\mathbb{R}^n}|f(x)|^p\,\mathrm{d}x\end{aligned}$$

as long as $p-1 > \varepsilon m$. The proof is completed by optimizing in ε under the constraint above, which amounts to the choice $\varepsilon \simeq (p-1)/m$. ∎

We proceed to show the main theorem.

Theorem 14. *Assume that the strong maximal theorem, theorem 4, holds in* $\mathbb{R}^{n-1}$. *Then, the differentiation basis* $\mathcal{R}_n$ *has the covering property* $V_{\exp,n-1}$.

Proof. The statement of this theorem essentially proves the inductive step in the proof of theorem 4, in combination with the results presented previously in this section. Let us denote by $\Pi_n(R)$ the projections of $R\in\mathcal{R}_n$ onto the n-th coordinate axis. With this notation we can order the rectangles by choosing one of their sides, say $\Pi_n(R)$. Hence, we have that $|\Pi_n(R_1)| \geq \ldots \geq |\Pi_n(R_N)|$ for $\{R_j\}_{j=1}^N \subset \mathcal{R}_n$. We construct a subcollection $\{\tilde{R}_k\}_{k=1}^M$ as follows.

First, we choose $\tilde{R}_1 := R_1$. Assuming that the rectangles $\{\tilde{R}_1,\ldots,\tilde{R}_k =: R_{k_0}\}\subset\{R_j\}_{j=1}^N$ for some $k_0 < N$ have been selected, we choose $\tilde{R}_{k+1}$ to be the first rectangle $R\in\{R_{k_0+1},\ldots,R_N\}$ such that either

$$\Big|R\cap\Big(\bigcup_{j\leq k}\tilde{R}_k^*\Big)\Big| \leq \frac{|R|}{2} \tag{19}$$

holds or $R\cap\big(\bigcup_{j\leq k}\tilde{R}_k\big) = \varnothing$. Here, given some $R\in\mathcal{R}_n$, we define R^* to be the rectangle with the same center as R, satisfying $\Pi_n(R^*) = 3\Pi_n(R)$, and having all other sides coinciding with those of R. This selection algorithm terminates in finitely many steps as the original collection was finite.

Let $\{\tilde{R}_j\}_{j=1}^M \subset \{R_j\}_{j=1}^N$, $M\leq N$, denote the subcollection extracted with the previous selection scheme. Our aim is to project our selected rectangles down to $\mathbb{R}^{n-1}$ in such a way that we can exploit the properties of the strong maximal operator in $\mathbb{R}^{n-1}$. For any $R\in\mathcal{R}_n$ and $y\in\mathbb{R}$, let $\Pi_n^{\perp,y}(R)$ denote the slice of the rectangle R by a hyperplane perpendicular to the n-th axis, and crossing the n-th axis at height y; in formulas,

$$\Pi_n^{\perp,y}(R) := \big\{x\in\mathbb{R}^{n-1} : (x,y)\in R\big\}.$$

If a rectangle $R\in\{R_j\}_{j=1}^N$ has not been selected with the previous scheme, then, for some $k\leq N$,

$$\Big|R\cap\Big(\bigcup_{j\leq k}\tilde{R}_j^*\Big)\Big| > \frac{|R|}{2}.$$

Note that R necessarily intersects one of the $\{\tilde{R}_j\}_{j\leq k}$, since otherwise R would have been selected. This implies that $\Pi_n(R)\subseteq\Pi_n(\bigcup_{j\leq k}\tilde{R}_j)$ and so the Π_n projection of the set appearing in the left hand side of the estimate above is $\Pi_n(R)$. Thus, $|\Pi_n(R)|$ can be cancelled from both sides of the estimate, resulting in an analogous sparseness property for the slices

$$\Big|\Pi_n^{\perp,y}(R)\cap\Big(\bigcup_{j\leq k}\Pi_n^{\perp,y}(\tilde{R}_j^*)\Big)\Big| > \frac{|\Pi_n^{\perp,y}(R)|}{2}. \tag{20}$$

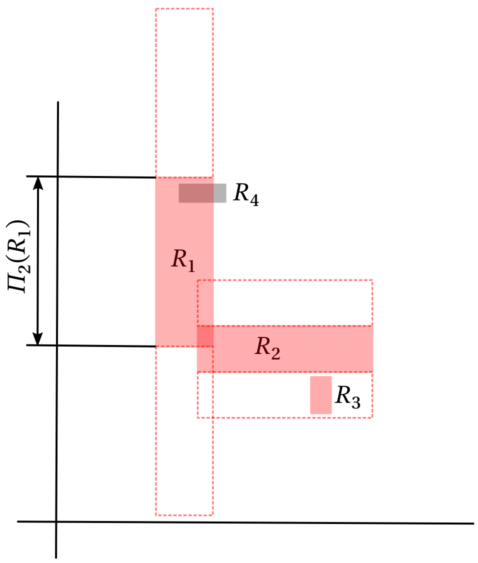

Figure 2: An example of application of the presented selection scheme in $\mathbb{R}^2$. Rectangles are ordered by the size of their vertical projection. Those rectangles shaded in red are the selected ones. The dotted lines show some of the tripled extensions $R \mapsto R^*$.

Estimate (20) must be understood as the statement that the $(n-1)$-dimensional average of the function $\mathbf{1}_{\bigcup_{j\le k}\Pi_n^{\perp,y}(\tilde R_j^*)}$ on the $(n-1)$-*dimensional rectangle* $\Pi_n^{\perp,y}(R)$ is big. Thus, remembering the definition of the strong maximal operator as a supremum, we get that, for every $x \in \Pi_n^{\perp,y}(R)$,

$$M_{\mathcal{R}_{n-1}}[\mathbf{1}_{\bigcup_{j=1}^M \Pi_n^{\perp,y}(\tilde R_j^*)}](x) \ge M_{\mathcal{R}_{n-1}}[\mathbf{1}_{\bigcup_{j\le k}\Pi_n^{\perp,y}(\tilde R_j^*)}](x) \ge \frac{1}{|\Pi_n^{\perp,y}(R)|}\Big|\Pi_n^{\perp,y}(R)\cap\Big(\bigcup_{j\le k}\Pi_1^{y}(\tilde R_j^*)\Big)\Big| > \frac{1}{2}.$$

The estimate above clearly holds also for any $x \in \bigcup_{j=1}^M \tilde R_j$. Thus, we have proved

$$\Pi_n^{\perp,y}\Big(\bigcup_{j=1}^N R_j\Big) = \bigcup_{j=1}^N \Pi_n^{\perp,y}(R_j) \subseteq \Big\{x\in\mathbb{R}^{n-1} : M_{\mathcal{R}_{n-1}}[\mathbf{1}_{\bigcup_{j=1}^M \Pi_n^{\perp,y}(\tilde R_j^*)}](x) > \frac{1}{2}\Big\}. \tag{21}$$

Now we can proceed on showing that our selection scheme actually extracts a subcollection of rectangles satisfying the $V_{\exp,n-1}$ property, namely conditions (i) and (i) of definition 10. Since by assumption $M_{\mathcal{R}_{n-1}}$ satisfies the strong maximal theorem in $\mathbb{R}^{n-1}$, lemma 12 implies that $M_{\mathcal{R}_{n-1}}$ is (say) weak-type $(2,2)$. This and estimate (21) imply that

$$\Big|\Pi_n^{\perp,y}\Big(\bigcup_{j=1}^N R_j\Big)\Big| \le C_1\Big|\Pi_n^{\perp,y}\Big(\bigcup_{j=1}^M \tilde R_j\Big)\Big|$$

for some constant $C_1 > 0$ depending only upon the dimension. Integrating for $y \in \mathbb{R} = \Pi_n(\mathbb{R}^n)$ proves property (i) of $V_{\exp,n-1}$.

In order to prove condition (ii), we may take the following exponential expansion for some $\theta > 0$,

$$\int_{\mathbb{R}^{n-1}}\Big[\exp\Big(\theta\Big(\sum_{j=1}^M \mathbf{1}_{\tilde R_j}\Big)^{\frac{1}{n-1}}\Big) - 1\Big] = \int_{\mathbb{R}^{n-1}}\sum_{\tau=1}^{\infty}\frac{\theta^\tau}{\tau!}\Big(\sum_{j=1}^M \mathbf{1}_{\tilde R_j}\Big)^{\frac{\tau}{n-1}} = \sum_{\tau=1}^{\infty}\frac{\theta^\tau}{\tau!}\Big\|\sum_{j=1}^M \mathbf{1}_{\tilde R_j}\Big\|_{L^{\frac{\tau}{n-1}}(\mathbb{R}^{n-1})}^{\frac{\tau}{n-1}}, \tag{22}$$

by an application of the monotone convergence theorem to the partial sums of the exponential series. Again, we may get a control for the overlap in the right hand side of the estimate above from the $(n-1)$-dimensional properties of the strong maximal operator. For any $k \le M$, it follows after the selection rule of equation (19) that

$$\Big|\Pi_n^{\perp,y}(\tilde R_k)\cap\Big(\bigcup_{j<k}\Pi_n^{\perp,y}(\tilde R_j)\Big)\Big| \le \Big|\Pi_n^{\perp,y}(\tilde R_k)\cap\Big(\bigcup_{j<k}\Pi_n^{\perp,y}(\tilde R_j^*)\Big)\Big| \le \frac{1}{2}|\Pi_n^{\perp,y}(\tilde R_k)|,$$

for every $y \in \mathbb{R}$. In order to simplify the notation, fix any $y \in \mathbb{R}$ and write $I_k := \Pi_n^{y,\perp}(\tilde{R}_k)$, for $k \leq M$. In this way, the previous inequality turns into the $(n-1)$-dimensional *sparseness* property

$$\Big|I_k \cap \Big(\bigcup_{j<k} I_j\Big)\Big| \leq \frac{1}{2}|I_k|, \qquad k \leq M. \tag{23}$$

Consider the disjoint increment sets $E_k := I_k \setminus \bigcup_{j<k} I_j$ for $k \leq M$ and define the auxiliary linear operator

$$Tf := \sum_{j=1}^{M} \left(\frac{1}{|I_j|}\int_{I_j} f\right)\mathbf{1}_{E_j} \leq M_{\mathcal{R}_{n-1}} f.$$

It follows that T has the same boundedness properties as $M_{\mathcal{R}_{n-1}}$ and so, combining the assumption with lemma 13 gives the estimate

$$\|T\|_{L^q(\mathbb{R}^{n-1})\to L^q(\mathbb{R}^{n-1})} \leq C_n (q')^{n-1}, \qquad \frac{1}{q}+\frac{1}{q'} = 1, \quad q \geq 2. \tag{24}$$

Following the path of the proof of proposition 6, we get that its adjoint is given by the formula

$$T^* f = \sum_{j=1}^{M} \left(\frac{1}{|I_j|}\int_{E_j} f\right)\mathbf{1}_{I_j}, \qquad T^*(\mathbf{1}_{\bigcup_{k=1}^M I_k}) = \sum_{j=1}^{M} \frac{|E_j|}{|I_j|}\mathbf{1}_{I_j}.$$

Note that (23) implies that $|E_k| \geq \frac{1}{2}|I_k|$, so that $T^*(\mathbf{1}_{\bigcup_{k=1}^M I_k}) > \frac{1}{2}\sum_{j=1}^M \mathbf{1}_{I_j}$. This fact, together with (24), allows us to estimate, for every integer $1 \leq p < \infty$,

$$\begin{aligned}\left(\int_{\mathbb{R}^{n-1}} \Big|\sum_{k=1}^{M}\mathbf{1}_{I_k}\Big|^p\right)^{\frac{1}{p}} &\leq 2\left(\int_{\mathbb{R}} \Big|T^*\Big(\mathbf{1}_{\bigcup_{k=1}^M I_k}\Big)\Big|^p\right)^{\frac{1}{p}} \leq \|T^*\|_{L^{p'}(\mathbb{R}^{n-1})\to L^{p'}(\mathbb{R}^{n-1})} \Big\|\mathbf{1}_{\bigcup_{k=1}^M I_k}\Big\|_{L^p(\mathbb{R}^{n-1})} \\ &\leq C_n p^{n-1}\Big|\bigcup_{k=1}^M I_k\Big|^{\frac{1}{p}},\end{aligned} \tag{25}$$

for some constant $C_n > 0$ depending only upon the dimension. Note that the estimate for $p = 1$ is a straightforward application of (23), without appealing to (24).

We can now complete the proof of condition (ii) of $V_{\exp,n-1}$. Remember that from (22) we have

$$\int_{\mathbb{R}^2}\left[\exp\left(\theta\Big(\sum_{k=1}^M \mathbf{1}_{\tilde{R}_k}\Big)^{\frac{1}{n-1}} - 1\right)\right] \leq \left(\sum_{\tau=1}^{2(n-1)} + \sum_{\tau=2n-1}^{\infty}\right)\frac{\theta^\tau}{\tau!}\int_{\mathbb{R}^{n-1}}\Big|\sum_{k=1}^{M}\mathbf{1}_{I_k}\Big|^{\frac{\tau}{n-1}} =: I + II.$$

Now for I, since $\tau/(n-1) \leq 2$, we can use Hölder's inequality together with (25) for $p = 2$ to get

$$I \leq C_n' \theta e \Big|\bigcup_{k=1}^M I_k\Big|,$$

with C_n' only depending on the dimension. For II we will use the asymptotic estimates for $\tau!$ provided by Stirling's formula [8, Exercise 12-22] in the form

$$\lim_{j\to\infty}\frac{j!\mathrm{e}^j}{j^j\sqrt{2\pi j}} = 1.$$

This, together with (25) for $p = \tau/(n-1)$, yields the estimate

$$II \leq C_n \sum_{\tau=2n-1}^{\infty}\frac{(\theta \mathrm{e})^\tau}{\tau^\tau\sqrt{\tau}}\left(\frac{\tau}{n-1}\right)^\tau \Big|\bigcup_{k=1}^M I_k\Big| \leq C_n''\theta\Big|\bigcup_{k=1}^M I_k\Big|$$

provided that $\theta < \theta_o(n)$ is sufficiently small; the constant $C_n'' > 0$ above depends only upon the dimension. Summing the estimates for I and II completes the proof of property (ii) of $V_{\exp,n-1}$, and thus the proof of the theorem. ∎

Proof of Theorem 4. We can now put together the full proof of theorem 4, which is by way of induction on the dimension n. For $n = 1$ the theorem holds because of remark 8, namely because $\mathcal{R}_1$ is the basis of intervals of $\mathbb{R}$ and $M_{\mathcal{R}_1}$ is just the Hardy-Littlewood maximal operator. So assume that the theorem holds for $M_{\mathcal{R}_{n-1}}$ in $\mathbb{R}^{n-1}$. Then, theorem 14 tells us that the basis of n-dimensional rectangles has the property $V_{\exp,n-1}$ on $\mathbb{R}^n$, and proposition 11, applied for $m = n-1$, yields the conclusion of theorem 4 in $\mathbb{R}^n$. The inductive step and thus the proof of the main estimate of the theorem is complete. In order to show that $\mathcal{R}_n$ differentiates functions which are locally in the space $L(\log L)^{n-1}$, namely the second conclusion of the theorem, one follows the argument on p. 3 of section 1, replacing the weak (p, p) type of $M_{\mathcal{B}}$ with the main estimate just proved for $M_{\mathcal{R}_n}$. We omit the details. ∎

4. Concluding remarks

In lemma 13 we presented a particular case of a more general interpolation theorem called the *Marcinkiewicz interpolation theorem*; see Stein's book [9, § I.4], for example. Whenever we have certain boundedness properties of an operator at two particular endpoints $1 \le p < q \le \infty$, we can use interpolation arguments to recover boundedness for every other r in between p and q. In our particular case, we had strong-type (∞, ∞) properties plus the strong maximal theorem, which implies we can use any $p > 1$ as a weak-type (p, p) endpoint. For a more detailed exposition of interpolation theorems for operators acting on different Banach spaces and applications to several problems in harmonic analysis see for example Grafakos's book [4].

This text is intended to be an introduction to the study of maximal operators given by differentiation bases beyond the one consisting of Euclidean balls or cubes in $\mathbb{R}^n$. Several other bases are of interest and give rise to intriguing problems in harmonic analysis. For example one can consider the bases of rectangles in $\mathbb{R}^2$ with short side of length $\delta \ll 1$ and long side of length 1, whose longest side points lie in a given finite set of directions $V \subset \mathbb{S}^1$. If these directions are uniformly distributed on $\mathbb{S}^1$, then this basis gives rise to the so-called *Kakeya maximal function*, an object which is central in one of the main conjectures in modern harmonic analysis. The study of the basis of rectangles with sides parallel to the coordinate sides is a toy model, allowing the development of geometric and combinatorial arguments which are suitable for these more general bases. As mentioned in the introduction, the interested reader could consult De Guzmán's work [5] for a thorough discussion on the theory of differentiation bases and the analysis of several different approaches for their study, together with corresponding conjectures.

References

[1] Bagby, Richard J. "Maximal functions and rearrangements: some new proofs". In: *Indiana University Mathematics Journal* 32.6 (1983), pp. 879–891. issn: 0022-2518. https://doi.org/10.1512/iumj.1983.32.32060.

[2] Córdoba, Antonio J. and Fefferman, Robert A. "A geometric proof of the strong maximal theorem". In: *Annals of Mathematics. Second Series* 102.1 (1975), pp. 95–100. issn: 0003-486X. https://doi.org/10.2307/1970976.

[3] Duoandikoetxea, Javier. *Fourier analysis.* Vol. 29. Graduate Studies in Mathematics. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. Providence: American Mathematical Society, 2001. isbn: 978-0-8218-2172-5.

[4] Grafakos, Loukas. *Classical Fourier analysis.* 3rd ed. Graduate Texts in Mathematics 249. New York: Springer, 2014. https://doi.org/10.1007/978-1-4939-1194-3.

[5] Guzmán, Miguel de. "Differentiation of integrals in $\mathbf{R}^n$". In: *Measure theory.* Conference on Measure Theory (Oberwolfach, 1975). Lecture Notes in Mathematics 541. 1976, pp. 181–185. https://doi.org/10.1007/BFb0081051.

[6] Jessen, Børge; Marcinkiewicz, Józef, and Zygmund, Antoni. "Note on the differentiability of multiple integrals". In: *Fundamenta Mathematicae* 25 (1935), pp. 217–234. issn: 0016-2736. https://doi.org/10.4064/fm-25-1-217-234.

[7] Moriyón, Roberto. *The degenerate complex Monge-Ampère equation.* PhD thesis. Princeton University, 1979.

[8] Schröder, Bernd S. W. *Mathematical analysis. A concise introduction.* Hoboken: John Wiley & Sons, 2008. isbn: 978-0-470-10796-6.

[9] Stein, Elias M. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series 30. Princeton: Princeton University Press, 1971. isbn: 978-0-691-08079-6.

[10] Wheeden, Richard L. and Zygmund, Antoni. *Measure and integral. An introduction to real analysis.* Pure and Applied Mathematics 43. New York: Marcel Dekker, 1977. isbn: 978-0-8247-6499-9.

IX *Escuela-Taller de Análisis Funcional* (2019)

Hardy spaces and holomorphic functions of infinitely many variables

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Abstract: We take a classical result on the interplay between complex and Fourier analysis in one variable (that the space of bounded holomorphic functions on the unit disc and the Hardy space on the torus are isomorphic), and extend it to functions in infinitely many variables. This will present several difficulties that we sort out.

Resumen: Tomamos un resultado clásico sobre la interacción del análisis complejo y de Fourier de una variable (que el espacio de funciones holomorfas y acotadas en el disco y el espacio de Hardy en el toro son isomorfos) y lo extendemos a funciones en infinitas variables. Esto presenta varias dificultades que deberemos sortear.

Keywords: holomorphic function, Fourier analysis, Banach space, isometry.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ its boundary (that we call the *torus*). We denote by $H_\infty(\mathbb{D})$ the space of bounded holomorphic functions on the disc and by $H_\infty(\mathbb{T})$ the Hardy space on the torus (precise definitions are given in section 2). A classical result in analysis states that

$$H_\infty(\mathbb{D}) = H_\infty(\mathbb{T});$$

that is: both spaces are isometrically isomorphic as Banach spaces. Our aim in this note is to present an analogous result for functions of infinitely many variables (or, to be more precise, defined on subsets of infinite dimensional spaces, see theorem 43). This was done for the first time by Cole and Gamelin [4]. We follow here a different approach, based in the one given by Defant et al. [5] using results from Rudin [6]. We do it in several steps. First we are going to analyse the proof of the 1-dimensional case, so that we can transfer it to functions of several complex variables and, finally to infinite dimensional spaces. In order to achieve this goal we have to face several issues: to find proper analogues to $\mathbb{D}$ and $\mathbb{T}$ for several and infinitely many variables, find a good definition of holomorphy in this setting, and to find a device that allows to extend the results from the finite to the infinite dimensional case. We assume some knowledge of the basic concepts of complex, harmonic and functional analysis.

2. The 1-dimensional case

As we explained before, we are going to look at the interplay between complex and harmonic analysis, and all the time we will keep one foot in each side. We begin by defining the spaces we will be dealing with. First of all, the space of holomorphic functions on $\mathbb{D}$ is denoted by $H(\mathbb{D})$. We consider the following subspace.

Definition 1. We define the space $H_\infty(\mathbb{D}) = \{f : \mathbb{D} \longrightarrow \mathbb{C} : f \text{ is bounded and holomorphic}\}$. ◂

Theorem 2. *$H_\infty(\mathbb{D})$ with the norm*

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$$

is a Banach space.

Proof. Let $\{f_n\}_{n=0}^\infty$ be a Cauchy sequence in $H_\infty(\mathbb{D})$. The space $C_\infty(\mathbb{D})$ of bounded continuous functions on the disc (that obviously contains $H_\infty(\mathbb{D})$) is Banach (see, e.g., Cerdà's book [2, Chapter 2]). Then, the sequence converges uniformly on $\mathbb{D}$ to some bounded continuous $f : \mathbb{D} \to \mathbb{C}$. But then $\{f_n\}_{n=0}^\infty$ converges uniformly on the compact subsets of $\mathbb{D}$ to the function f, and a straightforward application of Morera's theorem (see Stein and Shakarchi's book [8, Theorem 5.2]) shows that f is holomorphic. ∎

Remark 3. If U is an open subset of $\mathbb{C}$, then a function $f : U \to \mathbb{C}$ is analytic on U if, for every point $z_0 \in U$, there exist $r > 0$ and a sequence $\{c_n\}_{n=0}^\infty \subset \mathbb{C}$ which depend on z_0 such that

$$f(z) = \sum_{n=0}^\infty c_n (z - z_0)^n, \text{ for every } z \in (z_0 + r\mathbb{D}) \subset U,$$

where $z_0 + r\mathbb{D} = \{z \in \mathbb{C} : |z - z_0| < r\}$. This is, f admits a power series expansion in a neighbourhood of each point $z_0 \in U$. One of the key results (probably one of the most important ones in complex analysis) is that every holomorphic function is analytic [8, Theorem 4.4]. In our particular case, that is, on the disc $\mathbb{D}$, it is known that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic if and only if there exist coefficients $\{c_n\}_{n=1}^\infty \subset \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^\infty c_n z^n,$$

for every $z \in \mathbb{D}$. This convergence is, moreover, absolute on $\mathbb{D}$ and uniform on $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$ for every $0 < r < 1$. ◂

This is our main object in the side of complex analysis. Let us explore now the side of harmonic analysis. On $\mathbb{T}$ we consider the normalised Lebesgue measure, and the corresponding space $L_1(\mathbb{T})$. This means that, for each f, the integral has to be understood in the following sense:

$$\int_{\mathbb{T}} f(w)\,\mathrm{d}w = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t.$$

If $f \in L_1(\mathbb{T})$, then $w \in \mathbb{T} \mapsto f(w)w^{-n}$ is again in $L_1(\mathbb{T})$ for every $n \in \mathbb{Z}$ (because $|w^{-n}| = 1$). Then, we can define the Fourier coefficients of f in the following way.

Definition 4. Given $f \in L_1(\mathbb{T})$ and $n \in \mathbb{Z}$, the n-th Fourier coefficient is defined as

$$\hat{f}(n) = \int_{\mathbb{T}} f(w)w^{-n}\,\mathrm{d}w. \quad ◂$$

Let us note that

$$|\hat{f}(n)| \leq \int_{\mathbb{T}} |f(w)w^{-n}|\,\mathrm{d}w = \|f\|_1, \tag{1}$$

and the operator $L_1(\mathbb{T}) \to \mathbb{C}$ defined by $f \mapsto \hat{f}(n)$ is continuous. We can now define the second space we are going to be dealing with.

Definition 5. The Hardy space on the circumference is defined as

$$H_\infty(\mathbb{T}) = \{f \in L_\infty(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0\}. \quad ◂$$

Theorem 6. *$H_\infty(\mathbb{T})$ is a closed subspace of $L_\infty(\mathbb{T})$, hence Banach.*

Proof. The result follows as a straightforward consequence of the fact that the operator $f \mapsto \hat{f}(n)$ is continuous. ■

Thus, the goal of this section is to prove that

$$H_\infty(\mathbb{T}) = H_\infty(\mathbb{D}) \tag{2}$$

as Banach spaces. That is: there is an isometric isomorphism between these two spaces.

Remark 7. Let us give the first step towards the proof. Each $f \in H_\infty(\mathbb{T})$ defines a family of Fourier coefficients $\{\hat{f}(n)\}_{n=0}^{\infty}$, and we may consider the (in principle only formal) power series given by $\sum_{n=0}^{\infty} \hat{f}(n)z^n$. Note that (recall (1) and the fact that $\|f\|_1 \leq \|f\|_\infty$)

$$\sum_{n=0}^{\infty} |\hat{f}(n)|\,|z^n| \leq \|f\|_\infty \sum_{n=0}^{\infty} |z|^n < \infty \iff |z| < 1.$$

Then, the function $g : \mathbb{D} \to \mathbb{C}$ given by $g(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ is well defined and, by remark 3, is holomorphic. In other words, the operator $H_\infty(\mathbb{T}) \to H(\mathbb{D})$ given by $f \mapsto g(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ is well defined. It is an easy exercise to check that it is linear and injective. The main problem now is to show that in fact it takes values in $H_\infty(\mathbb{D})$ (that is, the function g defined in this way is bounded) and is surjective. ◂

Our first concern is to show that the function defined by the power series is indeed bounded on $\mathbb{D}$. To do this, we will reformulate the function in more convenient terms. We bring now our tool for this purpose.

Definition 8. The Poisson kernel $p : \mathbb{D} \times \mathbb{T} \to \mathbb{C}$ is defined as

$$p(z, w) = \sum_{n\in\mathbb{Z}} w^{-n} r^{|n|} u^n, \tag{3}$$

for $z \in \mathbb{D}$ and $w \in \mathbb{T}$, where $z = ru$ with $u = z/|z| \in \mathbb{T}$ and $r = |z|$. ◂

Remark 9. Let us note that, since $|w| = |u| = 1$ and $0 \leq r < 1$, we have

$$\sum_{n\in\mathbb{Z}} |w^{-n} r^{|n|} u^n| \leq \sum_{n\in\mathbb{Z}} r^{|n|} = 1 + 2\sum_{n=1}^{\infty} r^n < \infty\,.$$

Hence, the series in (3) converges (even absolutely) for each fixed $w \in \mathbb{T}$ and $z \in \mathbb{D}$ and p is well defined. Moreover, by the Weierstrass M-test (see Rudin's book [6, Theorem 7.10]), the series converges uniformly on $r\mathbb{D} \times \mathbb{T}$ for every $0 < r < 1$. ◂

Proposition 10. *The following statements hold:*

1. $p(z,w) = \dfrac{|w|^2 - |z|^2}{|w-z|^2} > 0$ *for all* $w \in \mathbb{T}$ *and* $z \in \mathbb{D}$*;*
2. $\displaystyle\int_{\mathbb{T}} p(z,w)\,\mathrm{d}w = 1$ *for every fixed* $z \in \mathbb{D}$.

Proof.

1. Let $w \in \mathbb{T}$ and $z \in \mathbb{D}$. Observe that

$$\begin{aligned}
p(z,w) &= \sum_{n\in\mathbb{Z}} w^{-n} r^{|n|} u^n = \sum_{n=1}^{\infty} \left(\frac{wr}{u}\right)^n + \sum_{n=0}^{\infty} \left(\frac{ru}{w}\right)^n \\
&= \frac{wr}{u - wr} + \frac{w}{w - ru} = \frac{wu(1-r^2)}{uw - ru^2 - rw^2 + r^2 wu} \\
&= \frac{1-r^2}{1 - r\frac{u}{w} - r\frac{w}{u} + r^2} = \frac{1-r^2}{1 - ru\overline{w} - r\overline{u}w + r^2} \\
&= \frac{|w|^2 - |z|^2}{|w|^2 - z\overline{w} - \overline{z}w + |z|^2} = \frac{|w|^2 - |z|^2}{|w-z|^2} > 0.
\end{aligned}$$

2. If we fix $z \in \mathbb{D}$, the series in (3) converges uniformly on $\mathbb{T}$. Then, we may change the sum and the integral as follows:

$$\int_{\mathbb{T}} p(z,w) = \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} w^{-n} r^{|n|} u^n \,\mathrm{d}w = \sum_{n\in\mathbb{Z}} r^{|n|} u^n \int_{\mathbb{T}} w^{-n}\,\mathrm{d}w.$$

A straightforward computation shows that

$$\int_{\mathbb{T}} w^{-n}\,\mathrm{d}w = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\pi} \frac{\mathrm{d}w}{2\pi} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases} \tag{4}$$

■

A direct consequence of proposition 10.1 is that $p(z,w)$ is bounded for each fixed $z \in \mathbb{D}$. Then, for every $f \in L_1(\mathbb{T})$, the function given by $w \mapsto p(z,w)f(w)$ again belongs to $L_1(\mathbb{T})$ and the function $P[f]\colon \mathbb{D} \to \mathbb{C}$ given by

$$P[f](z) = \int_{\mathbb{T}} p(z,w) f(w)\,\mathrm{d}w$$

is well defined. In fact, since the series defining p is uniformly convergent on $\mathbb{T}$ (for fixed $z \in \mathbb{D}$), then

$$\begin{aligned}
\int_{\mathbb{T}} p(z,w) f(w)\,\mathrm{d}w &= \int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} w^{-n} r^{|n|} u^n\right) f(w)\,\mathrm{d}w \\
&= \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{T}} f(w) w^{-n}\,\mathrm{d}w\right) r^{|n|} u^n \\
&= \sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} u^n
\end{aligned}$$

(note that this series converges because $|\hat{f}(n)| \leq \|f\|$, $0 \leq r \leq 1$ and $|u| = 1$). In this way, we may define an operator P (that we call *Poisson operator*) acting on $L_1(\mathbb{T})$ by doing $f \mapsto P[f]$.

Remark 11. If $f \in H_\infty(\mathbb{T})$, then $\hat{f}(n) = 0$ for $n < 0$ and, for $z = ru \in \mathbb{D}$, we have

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} u^n = \sum_{n=0}^{\infty} \hat{f}(n) r^{|n|} u^n = \sum_{n=0}^{\infty} \hat{f}(n) r^n u^n = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

Then, P (restricted to $H_\infty(\mathbb{T})$) is exactly the operator that we already considered in remark 7. We have

$$|P[f](z)| \le \int_{\mathbb{T}} |p(z,w)||f(w)| \,\mathrm{d}w \le \|f\|_\infty \int_{\mathbb{T}} |p(z,w)| \,\mathrm{d}w = \|f\|_\infty$$

and, hence,

$$\sup_{z\in\mathbb{D}} |P[f](z)| \le \|f\|_\infty. \tag{5}$$

This shows that $P[f]$ is bounded or, in other words, $P\colon H_\infty(\mathbb{T}) \to H_\infty(\mathbb{D})$ is well defined and continuous. ◂

Roughly speaking, what the operator P does is to "extend" functions on $\mathbb{T}$ to $\mathbb{D}$. Let us see how this operator acts on some particularly nice functions.

Remark 12. We begin by considering *trigonometric polynomials*. There are functions $Q\colon \mathbb{T} \to \mathbb{C}$ that can be written as

$$Q(w) = \sum_{n=N}^{M} c_n w^n,$$

where $a_n \in \mathbb{C}$, $N < M \in \mathbb{Z}$. First of all, for each $n \in \mathbb{Z}$, taking (4) we have

$$\hat{Q}(n) = \int_{\mathbb{T}} \sum_{k=N}^{M} c_k w^k w^{-n} \,\mathrm{d}w = \sum_{k=N}^{M} c_k \int_{\mathbb{T}} w^{k-n} \,\mathrm{d}w = \begin{cases} c_n & \text{if } N \le n \le M, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$P[Q](z) = \sum_{n\in\mathbb{Z}} \hat{Q}(n) r^{|n|} u^n = \sum_{n=N}^{M} c_n r^{|n|} u^n.$$

This function is continuous on all $\overline{\mathbb{D}}$ and coincides with Q on $\mathbb{T}$. ◂

This, in fact, also happens to every continuous function.

Proposition 13. *Let $f \in C(\mathbb{T})$. Then, $P[f]$ extends to a continuous function on $\overline{\mathbb{D}}$ which is equal to f on $\mathbb{T}$.*

Proof. By the Stone-Weierstrass theorem [6, Chapter 7], there is a sequence of trigonometric polynomials $\{Q_n\}_{n=0}^\infty$ converging to f with the supremum norm $\|\cdot\|_\infty$. As we already observed in remark 12, each $P[Q_n]$ is continuous on $\overline{\mathbb{D}}$. On the other hand, (5) gives

$$\|P[Q_n] - P[Q_m]\|_\infty = \|P[Q_n - Q_m]\|_\infty \le \|Q_n - Q_m\|_\infty$$

for every n and m. This implies that $\{P[Q_n]\}_{n=0}^\infty$ is a Cauchy sequence in $C(\overline{\mathbb{D}})$ and, hence, converges uniformly to some continuous function F on $\overline{\mathbb{D}}$. For each $w \in \mathbb{T}$ we have

$$F(w) = \lim_{n\to\infty} P[Q_n](w) = \lim_{n\to\infty} Q_n(w) = f(w).$$ ∎

One would expect that, whenever a function is defined on $\overline{\mathbb{D}}$ and is restricted to $\mathbb{T}$, then "extending" it to $\mathbb{D}$ with P would give us the original function. We have that, at least if the function is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, then this is the case. This shows that, when restricted to this space, the operator P is surjective.

Proposition 14. *Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ be holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Let $f_{|\mathbb{T}}$ denote the restriction of f to $\mathbb{T}$. Then, $P[f_{|\mathbb{T}}] = f$ on $\mathbb{D}$.*

The last tools we need to prove (2) are some basic concepts of functional analysis related with the weak topologies. We just recall here what we are going to need later. For a deeper study, the reader is referred to Brezis's book [1]. Given a Banach space E, we denote its topological dual by E^*, and given $x \in E$ and $x \in E^*$, we write $\langle x, x^* \rangle := x^*(x)$. For each $x^* \in E^*$, we consider the function $\varphi_{x^*} : E \to \mathbb{C}$, defined by $\varphi_{x^*}(x) = \langle x, x^* \rangle$. Then, the weak topology $\sigma(E, E^*)$ on E is the finest topology that makes all the maps $(\varphi_{x^*})_{x^* \in E^*}$ continuous. A sequence $\{x_n\}_{n=0}^{\infty}$ in E converges to x in the weak topology if and only if $\{\langle x_n, x^* \rangle\}_{n=0}^{\infty}$ converges to $\langle x, x^* \rangle$ for all $x^* \in E^*$. Similarly, for each $x \in E$ we may consider the function $\psi_x : E^* \to \mathbb{C}$ defined by $\varphi_x(x^*) = \langle x, x^* \rangle$, and the weak-star (or weak*) topology $\sigma(E^*, E)$ is defined as the finest topology on E^* making all the maps $(\psi_x)_{x \in E}$ continuous. A sequence $\{x_n^*\}_{n=0}^{\infty}$ in E^* converges to x^* in the weak* topology if and only if $\{\langle x, x_n^* \rangle\}_{n=0}^{\infty}$ converges to $\langle x, x^* \rangle$ for all $x \in E$.

Remark 15. A key fact when dealing with weak topologies is the Banach-Alaoglu theorem [1, Theorem 3.16], by which the closed ball $B_{E^*} = \{f \in E^* : \|f\| \leq 1\}$ is compact in the weak* topology.

Let us also recall that $(L_1(\mathbb{T}))^* = L_\infty(\mathbb{T})$, and the duality is given by

$$\langle f, g \rangle = \int_{\mathbb{T}} f(w) g(w) \, \mathrm{d}w$$

for $f \in L_1(\mathbb{T})$ and $g \in L_\infty(\mathbb{T})$. An important fact for us is that, since $L_1(\mathbb{T})$ is separable, the closed unit ball of $L_\infty(\mathbb{T})$ is metrizable in the $\sigma(L_\infty, L_1)$-topology [1, Theorem 3.28]. These two facts imply that every bounded sequence in $L_\infty(\mathbb{T})$ has a subsequence that converges in the $\sigma(L_\infty, L_1)$-topology. ◂

We are finally ready to state and prove the main result of this section. Given a function $g \in H_\infty(\mathbb{D})$, we will denote by $c_n(g)$ the n-th coefficient of its power series expansion centered on 0.

Theorem 16. *The Poisson operator $P : H_\infty(\mathbb{T}) \to H_\infty(\mathbb{D})$ defined as $f \mapsto P[f]$ is an isometric isomorphism so that $c_n(P[f]) = \hat{f}(n)$ for all $n \in \mathbb{N}_0$.*

Proof. From remarks 7 and 11 we already know that it is well defined, continuous and injective. It is only left, then, to see that it is onto. Let $g \in H_\infty(\mathbb{D})$, and consider its power series expansion (recall remark 3)

$$g(z) = \sum_{n=0}^{\infty} c_n(g) z^n, \tag{6}$$

which converges absolutely and uniformly on $r\mathbb{D}$ for every $0 < r < 1$. Now, for each $n \in \mathbb{N}$ we consider the function $f_n : \mathbb{T} \to \mathbb{C}$ given by $f_n(w) = g((1 - 1/n)w)$. Note that

$$\|f_n\|_\infty = \sup_{w \in \mathbb{T}} |f_n(w)| = \sup_{w \in \mathbb{T}} |g((1 - 1/n)w)| \leq \sup_{z \in \mathbb{D}} |g(z)| = \|g\|_\infty.$$

Then, $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $L_\infty(\mathbb{T})$ that, in view of remark 15, has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges to some $f \in L_\infty(\mathbb{T})$. Notice that $\|f\|_\infty \leq \|g\|_\infty$. Our aim now is to see that, in fact, $P[f] = g$. First, if $n \in \mathbb{Z}$, the weak* convergence implies

$$\hat{f}(n) = \int_{\mathbb{T}} f(w) w^{-n} \, \mathrm{d}w = \langle f, w^{-n} \rangle = \lim_{k \to \infty} \langle f_{n_k}, w^n \rangle = \lim_{k \to \infty} \widehat{f_{n_k}}(n).$$

But, since the series in (6) converges uniformly on $\mathbb{T}$, we have (recall again (4))

$$\begin{aligned}
\widehat{f_{n_k}}(n) &= \int_{\mathbb{T}} f_{n_k}(w) w^{-n} \, \mathrm{d}w = \int_{\mathbb{T}} \sum_{m=0}^{\infty} c_m(g) \left(1 - \tfrac{1}{n_k}\right)^m w^m w^{-n} \, \mathrm{d}w \\
&= \sum_{m=0}^{\infty} c_m(g) \left(1 - \tfrac{1}{n_k}\right)^m \int_{\mathbb{T}} w^m w^{-n} \, \mathrm{d}w = \begin{cases} \left(1 - \tfrac{1}{n_k}\right)^n c_n(g) & \text{if } n \geq 0, \\ 0 & \text{if } 0 > n. \end{cases}
\end{aligned}$$

Hence,

$$\hat{f}(n) = \begin{cases} c_n(g) & \text{if } n \geq 0, \\ 0 & \text{if } 0 > n, \end{cases}$$

thus $f \in H_\infty(\mathbb{T})$. Moreover, for $z \in \mathbb{D}$,

$$P[f](z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n = \sum_{n=0}^{\infty} c_n(g) z^n = g(z),$$

and by remark 11 we have $\|g\|_\infty \leq \|f\|_\infty$. ∎

We have the result for functions of one variable. Our aim is to extend this to an analogous result in infinitely many variables. As an intermediate step we have to look at it for functions of several (but finitely many) variables. We do this in the following section.

3. The N-dimensional case

We want to reproduce in this section the program that we presented in section 2. As there, we have to keep a foot in the world of holomorphic functions and another foot in the world of Fourier analysis. But before we proceed we have to define the concepts and spaces that we are going to work with.

Definition 17. Let U be open in $\mathbb{C}^N$. A function $f\colon U \to \mathbb{C}$ is holomorphic if for every $z \in U$ there exists a unique $\nabla f(z) \in \mathbb{C}^N$ so that

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - \langle \nabla f(z), h\rangle}{\|h\|} = 0,$$

where $\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$ for $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in \mathbb{C}^N$, and $h \to 0$ with the usual topology on $\mathbb{C}^N$. ◂

Remark 18. Given a holomorphic function $f\colon U \subset \mathbb{C}^N \to \mathbb{C}$ and fixed $N-1$ coordinates, that is, $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_N \in \mathbb{C}$, then the restricted function $g(z) = f(a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_N)$ is holomorphic in its domain of definition and $g'(z) = \big(\nabla f(a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_N)\big)_j$. Because then

$$\lim_{h\to 0} \frac{g(z+h) - g(z) - g'(z)}{|h|} = \lim_{h\to 0} \frac{f(\overline{z} + \overline{h}) - f(\overline{z}) - \langle \nabla f(\overline{z}), \overline{h}\rangle}{\|\overline{h}\|},$$

where $\overline{z} = (a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_N)$ and $\overline{h} = (0, \dots, 0, h, 0, \dots, 0)$ with h in the j-th coordinate, and $\langle \nabla f(\overline{z}), \overline{h}\rangle$ as in the previous definition (not a scalar product). This limit equals 0 since f is holomorphic. ◂

A key fact of the bounded holomorphic functions space is the following version of the Weierstrass theorem [5, Theorem 2.4].

Theorem 19. *Let $(f_n)_n$ be a sequence of holomorphic functions on $r\mathbb{D}^N$ that converges uniformly on all compact subsets of $r\mathbb{D}^N$ to some $f\colon r\mathbb{D}^N \to \mathbb{C}$. Then, f is holomorphic.*

Following exactly the same proof as in the one-variable case (theorem 2) we have that it is a Banach space.

Theorem 20. *$H_\infty(\mathbb{D}^N) = \{f\colon \mathbb{D}^N \to \mathbb{C} \,:\, f$ is bounded and holomorphic$\}$ is a Banach space with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}^N} |f(z)|$.*

In the one variable case we had that every holormorphic function is also analytic. This is also true for finitely many variables, as we shall see in theorem 23. But before we get into that we have to make clear what it means that a series converges in this context.

Remark 21. If $(c_i)_{i\in I}$ is a family of scalars with I being a countable family of indexes, we say that $(c_i)_{i\in I}$ is summable if there exists $s \in \mathbb{C}$ (which we call the sum of $(c_i)_{i\in I}$ and write $s = \sum_{i\in I} c_i$) such that for all $\varepsilon > 0$ there exists a finite set $F_0 \subseteq I$ such that $|s - \sum_{i\in F} c_i| < \varepsilon$ for all finite sets $F_0 \subseteq F \subseteq I$. This is equivalent to the following three statements:

1. (Cauchy's criterion) for all $\varepsilon > 0$ there exists a finite set $F_0 \subseteq I$ such that $|\sum_{i\in F} c_i| < \varepsilon$ for all finite sets $F \subseteq I \backslash F_0$;

2. (absolute summability) $(|c_i|)_{i\in I}$ is summable, which equivalently means that $\sup_{F\subseteq I \text{ finite}} \sum_{i\in F} |c_i| < \infty$; in this case $\sum_{i\in I} |c_i| = \sup_{F\subseteq I \text{ finite}} \sum_{i\in F} |c_i|$;
3. (unconditional summability) for every bijection $\sigma\colon \mathbb{N} \to I$, $\sum_{n=1}^{\infty} c_{\sigma(n)}$ converges; in this case $\sum_{i\in I} c_i = \sum_{n=1}^{\infty} c_{\sigma(n)}$. ◂

Let us fix some notation. A *multi-index* is an N-tuple $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$. Given such a multi-index and some $z = (z_1, \dots, z_N) \in \mathbb{C}^N$, we denote

$$z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N} .$$

A *monomial* is any mapping of the form $z \mapsto z^\alpha$.

Example 22. The first example of power series in N variables is

$$\sum_{\alpha\in\mathbb{N}_0^N} z^\alpha ,$$

that converges if and only if $|z_j| < 1$ for every $j = 1, \dots, N$ and, in this case,

$$\sum_{\alpha\in\mathbb{N}_0^N} z^\alpha = \prod_{j=1}^{N} \frac{1}{1-z_j}.$$

This follows immediately from the fact that every finite subset of $\mathbb{N}_0^N$ is contained in $\{0,1,\dots,M\}^N$ for some M, that

$$\sum_{\alpha\in\{0,1,\dots,M\}^N} |z|^\alpha = \Big(\sum_{k=0}^{M} |z_1|^k\Big) \cdots \Big(\sum_{k=0}^{M} |z_N|^k\Big)$$

and the formula of the geometric series. ◂

Theorem 23. *Let $f\colon \mathbb{D}^N \to \mathbb{C}$. The following two statements are equivalent:*

1. *f is holomorphic;*
2. *there exist coefficients $\big(c_\alpha(f)\big)_{\alpha\in\mathbb{N}_0^N} \subset \mathbb{C}$ so that*

$$f(z) = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f) z^\alpha,$$

for every $z \in \mathbb{D}^N$.

Moreover, in this case, the convergence is absolute and uniform on each compact set of $\mathbb{D}^N$, and the coefficients are unique and can be computed as

$$c_\alpha(f) = \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \dots \int_{|\zeta_N|=\rho_N} \frac{f(\zeta_1, \dots, \zeta_N)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1$$

for any $0 < \rho_i < 1$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$.

Proof. Suppose first that f is holomorphic. Let $z = (z_1, \dots, z_N) \in \mathbb{D}^N$ and choose $= (\rho_1, \dots, \rho_N)$ such that $|z_j| < \rho_j < 1$ for all $1 \leq j \leq N$. Then, applying N times the Cauchy integral formula for one one variable, we obtain

$$f(z) = \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \dots \int_{|\zeta_N|=\rho_N} \frac{f(\zeta_1, \dots, \zeta_N)}{(\zeta_1 - z_1) \dots (\zeta_N - z_N)} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1.$$

But $1/(\zeta_j - z_j) = \sum_{k_j=0}^{\infty} z_j^{k_j} / \zeta_j^{k_j+1}$, so this can be rewritten as

$$f(z) = \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \dots \int_{|\zeta_N|=\rho_N} \Big(\sum_{k_1=0}^{\infty} \frac{z_1^{k_1}}{\zeta_1^{k_1+1}}\Big) \dots \Big(\sum_{k_N=0}^{\infty} \frac{z_N^{k_N}}{\zeta_N^{k_N+1}}\Big) f(\zeta_1, \dots, \zeta_N) \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1.$$

Since these series converge uniformly on compact subsets, they commute with the integration and, therefore,

$$f(z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_N=0}^{\infty} \left(\frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \cdots \int_{|\zeta_N|=\rho_N} \frac{f(\zeta_1, \dots, \zeta_N)}{\zeta_1^{k_1+1} \dots \zeta_N^{k_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 \right) z_1^{k_1} \dots z_N^{k_N}.$$

So for each $\alpha = (k_1, \dots, k_N)$ we can define

$$c_\alpha(f) = \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \cdots \int_{|\zeta_N|=\rho_N} \frac{f(\zeta_1, \dots, \zeta_N)}{\zeta_1^{k_1+1} \dots \zeta_N^{k_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 \tag{7}$$

and notice that it does not depend on the choice of . With this, we get that f is analytic as

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) z^\alpha \quad \text{for all } z \in \mathbb{D}^N$$

and the convergence is absolute (recall remark 21). Let us see that the series converges uniformly in every K, compact subset of $\mathbb{D}^N$. Since $\mathbb{D}^N$ is open, there exists $\epsilon > 0$ such that $(1+\epsilon)K \subset \mathbb{D}$. Now, given $z = (z_1, \dots, z_N) \in K$, define $s(z) = \big((1+\epsilon)|z_1|, \dots, (1+\epsilon)|z_N|\big) \in [0,1)^N$, so $z \in s(z)\mathbb{D}^N$ and, therefore, $\{s(z)\mathbb{D}^N : z \in K\}$ is an open cover of K. Hence, there exist $z_1, \dots, z_n \in K$ such that $K \subset \cup_{j=1}^n s(z_j)\mathbb{D}^N$. Then it is enough to check the uniform convergence on subsets of the form $s\mathbb{D}^N$ with $s \in (0,1)^N$. To see this we choose $= (\rho_1, \dots, \rho_N)$ such that $s_j < \rho_j < 1$ for all j and use (7) to get

$$|c_\alpha(f)| s^\alpha \leq \frac{s^\alpha}{\rho^\alpha} \|f\|_\infty.$$

Since $\sum_{\alpha \in \mathbb{N}_0^N} s^\alpha / \rho^\alpha$ converges (recall example 22), the Weierstrass M-test [6, Theorem 7.10] implies that $\sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) z^\alpha$ converges uniformly in $\overline{s\mathbb{D}^N}$. Let us assume now that $f(z) = \sum_{\alpha \in \mathbb{N}_0^N} b_\alpha z^\alpha$ (pointwise) for every $z \in \mathbb{D}^N$. Pick some $0 < r < 1$ and note that $|b_\alpha z^\alpha| < |b_\alpha r^{\alpha_1} \cdots r^{\alpha_N}|$ for every $z \in r\mathbb{D}^N$. But the series $\sum_\alpha |b_\alpha r^{\alpha_1} \cdots r^{\alpha_N}|$ converges (by assumption) and, by the Weierstrass M-test, the series $\sum_\alpha b_\alpha z^\alpha$ converges uniformly on $s\mathbb{D}^N$ for every $0 < s < r$. In particular, the polynomials given by

$$\sum_{\alpha \in \{0,1,\dots,k\}^N} b_\alpha z^\alpha$$

converge (as $k \to \infty$) uniformly to f on $s\mathbb{D}^N$ for every $0 < s < r$. By theorem 19 we have that f is holomorphic and bounded on $s\mathbb{D}^N$ and, since s is arbitrary, f is holomorphic in $\mathbb{D}^N$. In particular,

$$\begin{aligned} c_\alpha(f) =& \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \cdots \int_{|\zeta_N|=\rho_N} \frac{f(\zeta_1, \dots, \zeta_N)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 = \\ =& \sum_{\beta \in \mathbb{N}_0^N} b_\beta \frac{1}{(2\pi \mathrm{i})^N} \int_{|\zeta_1|=\rho_1} \cdots \int_{|\zeta_N|=\rho_N} \frac{\zeta^\beta}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 = b_\alpha. \end{aligned}$$

■

Now we move to the Fourier side. Analogously to the one dimensional case, we have to define the Fourier coefficients and then the Hardy space which we are going to work with.

Definition 24. Given $f \in L_1(\mathbb{T}^N)$, for each $\alpha \in \mathbb{Z}^N$ the α-th Fourier coefficient is defined as

$$\hat{f}(\alpha) = \int_{\mathbb{T}^N} f(w) w^{-\alpha} \, \mathrm{d}w.$$

◂

Just as in (1), we have $|\hat{f}(\alpha)| \leq \|f\|_1$ and the operator $L_1(\mathbb{T}^N) \to \mathbb{C}$ given by $f \mapsto \hat{f}(\alpha)$ is continuous. This immediately gives the following.

Theorem 25. *The Hardy space*

$$H_\infty(\mathbb{T}^N) = \{ f \in L_\infty(\mathbb{T}^N) : \hat{f}(\alpha) = 0 \text{ for } \alpha \notin \mathbb{N}_0^N \}$$

is a closed subspace of $L_\infty(\mathbb{T}^N)$ and, therefore, it is a Banach space.

The goal now is to show that there exists an isometric isomorphism between these two spaces, the space of holomorphic functions and the Hardy space. For that purpose we define the analogous tool to the one we used in the one dimensional case:

Definition 26. The N-dimensional Poisson kernel $p_N\colon \mathbb{D}^N \times \mathbb{T}^N \to \mathbb{C}$ is defined as

$$p_N(z,w) = \prod_{j=1}^{N} p(z_j, w_j)$$

for $z \in \mathbb{D}^N$ and $w \in \mathbb{T}^N$, where $u \in \mathbb{T}^N$ is given by $u_j = z_j/|z_j|$ and $r = (r_1, \dots, r_N)$ by $r_j = |z_j|$, and we write $r^{|\alpha|} = r_1^{|\alpha_1|} \cdots r_N^{|\alpha_N|}$ and $z = ru$. ◂

The absolute convergence of the series defining $p(z,w)$ in (4) gives

$$p_N(z,w) = \sum_{\alpha \in \mathbb{Z}^N} w^{-\alpha} r^{|\alpha|} u^{\alpha},$$

for every $z \in \mathbb{D}^N$ and $w \in \mathbb{T}^N$. Also, the series converges uniformly on $r\mathbb{D}^N$ for every $0 < r < 1$.

Proposition 27. *The following statements hold:*

1. $p_N(z,w) > 0$ *for every* $w \in \mathbb{T}^N$ *and* $z \in \mathbb{D}^N$*;*
2. $\displaystyle\int_{\mathbb{T}^N} p_N(z,w)dw = 1$ *for every fixed* $z \in \mathbb{D}^N$.

Proof. Both statements follow from the one-dimensional case (proposition 10) and Fubini's theorem. ■

Definition 28. Given $f \in L_1(\mathbb{T}^N)$, we define the N-dimensional Poisson operator for f by the function $P_N[f]\colon \mathbb{D}^N \to \mathbb{C}$ given by

$$P_N[f](z) = \int_{\mathbb{T}^N} p_N(z,w) f(w)\,\mathrm{d}w = \sum_{\alpha \in \mathbb{Z}^N} \hat{f}(\alpha) r^{|\alpha|} u^{\alpha}.$$ ◂

$P_N[f]$ is well defined for every $f \in L_1(\mathbb{T}^N)$ since $|\hat{f}(\alpha)| \le \|f\|_1$, $r \in [0,1]^N$ and $u \in \mathbb{T}^N$.

Theorem 29. *For each $N \in \mathbb{N}$, the N-dimensional Poisson operator*

$$P_N\colon H_\infty(\mathbb{T}^N) \to H_\infty(\mathbb{D}^N)$$

is an isometric isomorphism such that $c_\alpha(P_N[f]) = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{N}_0^N$.

Proof. On the one hand, observe that, for $f \in H_\infty(\mathbb{T}^N)$,

$$P_N[f](z) = \sum_{\alpha \in \mathbb{N}_0^N} \hat{f}(\alpha) z^{\alpha} \tag{8}$$

for every $z \in \mathbb{D}^N$ and (recall theorem 23) $P_N[f]$ is holomorphic on $\mathbb{D}^N$. Moreover, using the properties of the N-dimensional Poisson kernel (proposition 27), we have that

$$\|P_N[f]\|_\infty \le \sup_{z \in \mathbb{D}^N} \int_{\mathbb{T}^N} |p_N(z,w) f(w)|\,\mathrm{d}w \le \|f\|_\infty \sup_{z \in \mathbb{D}^N} \int_{\mathbb{T}^N} p_N(z,w)\,\mathrm{d}w = \|f\|_\infty.$$

Therefore, $P_N[f] \in H_\infty(\mathbb{D}^N)$ and the operator P_N is well defined and continuous such that $\hat{f}(\alpha) = c_\alpha(P_N[f])$ for every $f \in H_\infty(\mathbb{T}^N)$ and $\alpha \in \mathbb{N}_0^N$. The uniqueness of the coefficients shows that the operator defined is injective. It only remains to see that P_N is surjective. We take some $g \in H_\infty(\mathbb{D}^N)$ and, for each $n \in \mathbb{N}$, consider the function defined by $f_n(u) = g((1-1/n)u)$ for every $u \in \mathbb{T}^N$, which is in $H_\infty(\mathbb{T}^N)$ and has Fourier coefficients $\hat{f}_n(\alpha) = c_\alpha(g)(1-1/n)^{|\alpha|}$. Indeed, since the monomial series expansion of g converges uniformly on $r\mathbb{T}^N$ for every $0 < r < 1$, we have that

$$\hat{f}_n(\alpha) = \int_{\mathbb{T}^N} f_n(w) w^{-\alpha}\,\mathrm{d}w = \begin{cases} c_\alpha(g)(1-1/n)^{|\alpha|} & \text{for } \alpha \in \mathbb{N}_0^N, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\|f_n\|_\infty \leq \|g\|_\infty$ for every $n \in \mathbb{N}$. With exactly the same argument as in remark 15 we can find a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which $\sigma(L_\infty, L_1)$-converges to some $f \in B_{L_\infty(\mathbb{T}^N)}(0, \|g\|_\infty)$. As a consequence of the weak convergence,

$$\hat{f}(\alpha) = \int_{\mathbb{T}^N} f(w)w^{-\alpha}\,\mathrm{d}w = \lim_{k\to\infty} \int \mathbb{T}^N f_{n_k}(w)w^{-\alpha}\,\mathrm{d}w = \begin{cases} c_\alpha(g) & \text{for } \alpha \in \mathbb{N}_0^N, \\ 0 & \text{otherwise.} \end{cases}$$

This implies $f \in H_\infty(\mathbb{T}^N)$. Moreover, by (8) we get $P_N[f](z) = g$, and then $\|f\|_\infty = \|g\|_\infty$. ■

4. The infinite-dimensional case

We jump now from finitely to infinitely many variables. To do so, we will restrict the problem to finite variables, we will apply the finite-dimensional theorem and, using some powerful tools, we will go back to infinitely many variables. We have, then, to face two problems:

- to define a proper setting for our problem in the infinite dimensional setting,
- to find tools that allow us to jump from the finite to the infinite dimensional case.

We begin by tackling the first issue: to translate our problem to the setting of infinite dimensions. We start by defining the main components of our result. Firstly, we need to find a proper substitute for $\mathbb{D}^N$. Then, we need to understand the concept of holomophic function in infinite dimensions. Finally, we define the Fourier coefficients for infinitely many variables.

As substitute for $\mathbb{D}^N$ we could think of the unit ball of the Banach space ℓ_∞ (the space of bounded sequences with the supremum norm $\|\cdot\|$). However, this candidate presents some problems, since the space c_{00} (of sequences with only finitely many non-zero elements) is not dense in $(\ell_\infty, \|\cdot\|)$. So, we may choose a "smaller" space: this is going to be the Banach space $c_0 = \{\{z_n\}_{n=1}^\infty \subset \mathbb{C} : \lim_{n\to\infty} z_n = 0\}$, and we will consider its unit ball B_{c_0} as analogous to $\mathbb{D}^N$. Recall that the dual space of c_0 is the space $(c_0)^* = \ell_1 = \{\{z_n\}_{n=1}^\infty \subset \mathbb{C} : \sum_{n=1}^\infty |z_n| < \infty\}$.

The analogue to $\mathbb{T}^N$ will be the infinite dimensional torus $\mathbb{T}^\infty = \{\{w_n\}_{n=1}^\infty : w_n \in \mathbb{T} \text{ for each } n \in \mathbb{N}\}$, which is a compact space by Tychonoff's theorem. Also, since $\mathbb{T}^\infty$ is a grup with the product coordinate to coordinate, we are able to work with the Haar measure (see Cohn's book [3, Chapter 9]).

The definition of holomorphic functions on B_{c_0} is a particular case of the Fréchet differentiability, which is valid for any normed space and any open subset.

Definition 30. Let U be an open subset of a normed space X. A function $f : U \to \mathbb{C}$ is said to be holomorphic if it is Fréchet differentiable at every $x \in U$, that is, if there exists a continuous linear functional $x^* \in X^*$ such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - x^*(h)}{\|h\|} = 0.$$

In that case we denote the unique x^* by $\mathrm{d}f(x)$, and call it the differential of f at x. ◂

Remark 31. The restriction of every holomorphic function to finite dimensional subspaces is again holomorphic. More precisely, given a holomorphic function $f : U \to \mathbb{C}$ and M an N dimensional subspace of X with basis $e_1, \dots, e_N$, then we just take the inclusion $i_M : M \to X$, $e_i \mapsto i_M(e_i) = b_i$ and consider, for each $z_0 \in U \cap M$, the vector $\nabla(f \circ i_M)(z_0) = \big([\mathrm{d}f(z_0)](i_M(e_k))\big)_{k=1}^N = \big([\mathrm{d}f(z_0)](b_k)\big)_{k=1}^N$, which is the differential of $f_{|U\cap M}$. Indeed,

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - \langle \nabla(f \circ i_M)(z_0), h\rangle}{\|h\|} = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - \sum_k [\mathrm{d}f(z_0)](b_k)h_k}{\|h\|} = 0.$$

In particular, if $f : B_{c_0} \to \mathbb{C}$ is a holomorphic function and $N \in \mathbb{N}$, the restriction $f_N : \mathbb{D}^N \to \mathbb{C}$ defined by $f_N(z_1, \dots, z_N) = f(z_1, \dots, z_N, 0, 0, \dots)$ is holomorphic. ◂

Remark 32. Given a holomorphic function $f : \mathbb{D}^N \to \mathbb{C}$, we may see it as a function on B_{c_0} (let us denote it by $\tilde{f}$) just by adding zeros $\tilde{f}(z) = f(z_1, \dots, z_N, 0, \dots)$, which is holomorphic. ◂

Theorem 33. *The space $H_\infty(B_{c_0}) = \{f : B_{c_0} \to \mathbb{C} : f$ is holomorphic and bounded$\}$ with the norm $\|f\|_\infty = \sup_{z \in B_{c_0}} |f(z)|$ is a Banach space.*

This fundamental fact is a consequence of the following simplified Weierstrass type theorem, a proof of which can be found in the book of Defant et al. [5, Theorem 2.13].

Theorem 34. *Let X be a normed space, and (f_n) a bounded sequence in $H_\infty(B_X)$ that converges to $f : B_X \to \mathbb{C}$ uniformly on each compact subset of B_X (i.e., with respect to the compact-open topology). Then, $f \in H_\infty(B_X)$ and $\|f\|_\infty \leq \sup_n \|f_n\|_\infty$.*

Before introducing Taylor coefficients, let us fix some notation that will be used frequently throughout this section. We will write

$$\mathbb{N}_0^{(\mathbb{N})} = \bigcup_{N \in \mathbb{N}} \mathbb{N}_0^N \quad \text{and} \quad \mathbb{Z}^{(\mathbb{N})} = \bigcup_{N \in \mathbb{Z}} \mathbb{Z}^N.$$

When convenient, we will also identify $(\alpha_1, \alpha_2, \dots, \alpha_N)$ with $(\alpha_1, \alpha_2, \dots, \alpha_N, 0, 0, \dots)$. Given $f \in H_\infty(B_{c_0})$ and N, let $f_N : \mathbb{D}^N \to \mathbb{C}$ be its restriction, that is, $f_N(z_1, \dots, z_N) = f(z_1, \dots, z_N, 0, 0, \dots)$. This is a holomorphic function on $\mathbb{D}^N$ (see remark 31) and, by theorem 23, can be expanded as a power series

$$f_N(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f_N) z^\alpha,$$

for every $z \in \mathbb{D}^N$. This, in principle, provides us a set of coefficients $\{c_\alpha(f_N)\}_{\alpha \in \mathbb{N}_0^N}$ for each N. But, as a matter of fact, when we increase the dimension we only add "new" coefficients for the "new" dimensions. Let us be more precise. If $M \geq N$ and $\alpha \in \mathbb{N}_0^N$ (that we identify with $(\alpha_1, \dots, \alpha_N, 0, \dots, 0) \in \mathbb{N}_0^M$ and call again α), then

$$c_\alpha(f_N) = c_\alpha(f_M).$$

Indeed, by theorem 23 we have

$$c_{(\alpha,0)}(f_{N+1}) = \frac{1}{(2\pi \mathrm{i})^{N+1}} \int_{\rho_1 \mathbb{T}} \cdots \int_{\rho_{N+1} \mathbb{T}} \frac{f(\zeta_1, \dots, \zeta_{N+1}, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1} \zeta_{N+1}} \, \mathrm{d}\zeta_{N+1} \cdots \mathrm{d}\zeta_1,$$

and using Cauchy's integral formula,

$$\begin{aligned} c_{(\alpha,0)}(f_{N+1}) &= \frac{2\pi \mathrm{i}}{(2\pi \mathrm{i})^{N+1}} \int_{\rho_1 \mathbb{T}} \cdots \int_{\rho_{N+1} \mathbb{T}} \frac{f(\zeta_1, \dots, \zeta_N, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 \\ &= \frac{1}{(2\pi \mathrm{i})^N} \int_{\rho_1 \mathbb{T}} \cdots \int_{\rho_N \mathbb{T}} \frac{f(\zeta_1, \dots, \zeta_N, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_N^{\alpha_N+1}} \, \mathrm{d}\zeta_N \cdots \mathrm{d}\zeta_1 = c_\alpha(f_N). \end{aligned}$$

Then, each function $f \in H_\infty(B_{c_0})$ defines a unique family of coefficients $\{c_\alpha(f)\}_{\alpha \mathbb{N}_0^{(\mathbb{N})}}$, that we call *Taylor coefficients*. In other words, each function f defines a **formal** power series

$$f \sim \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(f) z^\alpha.$$

The problem now is that $f(z)$ may not coincide with $\sum c_\alpha(f) z^\alpha$. Toeplitz [9] gave an example of a function $f \in H_\infty(B_{c_0})$ and a point $z \in B_{c_0}$ for which $\sum c_\alpha(f) z^\alpha$ does not converge. In other words, when we deal with functions of infinitely many variables, *holomorphic* and *analytic* are no longer equivalent. This is a problem for us, since the proof of the isometry between the spaces, both in the case of one and several variables (recall theorems 16 and 29) depends heavily on the fact that a holomorphic function has a representation as a power series.

We now move on to the Fourier part, as we did in the previous sections. Given $f \in L_1(\mathbb{T}^\infty)$ and $\alpha \in \mathbb{Z}^{(\mathbb{N})}$, we define the α-th Fourier coefficient as

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha} \, \mathrm{d}w$$

and the Hardy space

$$H_\infty(\mathbb{T}^\infty) = \big\{ f \in L_\infty(\mathbb{T}^\infty) : \hat{f}(\alpha) = 0 \text{ if } \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \big\}.$$

Once again, we have that $|\hat{f}(\alpha)| \leq \|f\|_1$ for every α. We are also going to use the following two facts, the proof of which can be found, for example, in the book of Defant et al. [5, Chapter 5].

Proposition 35. *If $f_1, f_2 \in L_1(\mathbb{T}^\infty)$ are such that $\hat{f_1}(\alpha) = \hat{f_2}(\alpha)$ for every α, then $f_1 = f_2$.*

Definition 36. An **analytic trigonometric polynomial** is a function $Q \in L_1(\mathbb{T}^\infty)$ of the form

$$Q = \sum_{\substack{\alpha \in F \subseteq \mathbb{N}_0^{(\mathbb{N})} \\ F \text{ finite}}} a_\alpha w^\alpha. \qquad ◂$$

Proposition 37. *The set of analytic trigonometric polynomials is dense in $L_1(\mathbb{T}^\infty)$.*

With this we have accomplished the first goal that we stated at the very beginning of this section: to define a proper setting on which to formulate our problem. So, our goal now is to show that

$$H_\infty(\mathbb{T}^\infty) = H_\infty(B_{c_0})$$

isometrically as Banach spaces. Since we have lost the equivalence between holomorphy and analiticity, we cannot adapt the proof of the finite-dimensional case, and we have to go a different way. What we are going to do is to go "down" in each side (holomorphic and harmonic) to N variables, apply the result that we already know (theorem 29) and then "climb up" again to the infinite dimensional setting.

$$\begin{tikzcd}
H_\infty(\mathbb{T}^\infty) \arrow[r] \arrow[d, dashed, leftrightarrow] & H_\infty(B_{c_0}) \arrow[d, dashed, leftrightarrow] \\
H_\infty(\mathbb{T}^N) \arrow[r] & H_\infty(\mathbb{D}^N)
\end{tikzcd}$$

So, we now need to find tools that allow us to go "down" and "up" in each side (this was the second goal that we stated at the beginning of the section). We start with the holomorphic part. Given $f \in H_\infty(B_{c_0})$ and $N \in \mathbb{N}$, by remark 31, $f_N : \mathbb{D}^N \to \mathbb{C}$, where $f_N(z_1, \ldots, z_N) = f(z_1, \ldots, z_N, 0, 0, \ldots)$, is a holomorphic function on $\mathbb{D}^N$. This is the way to go from infinite to finite dimensions. We will use the next theorem (taken from the book of Defant et al. [5, Theorem 2.21], sometimes known as "Hilbert's criterion") to go the opposite way (that is, to "jump" from finitely to infinitely many variables). But before we need a tiny observation.

Remark **38.** If $g : \mathbb{D}^N \to \mathbb{C}$ is a holomorphic function with $g(0) = 0$ and $|g(u)| < C$ for every $u \in \mathbb{D}^N$, then

$$|g(u)| \leq C \max_{1 \leq n \leq N} |u_n|, \tag{9}$$

for each such u. Indeed, for $0 \neq u \in \mathbb{D}^N$, define $h : \mathbb{D} \to \mathbb{D}$ by $h(\zeta) = 1/Cg(\zeta \cdot \frac{u}{\max_n |u_n|})$. Then, the classical Schwarz' lemma (see Rudin's book [7, Theorem 12.2]) yields $|h(\zeta)| \leq |\zeta|$ for all $\zeta \in \mathbb{D}$, which for $\zeta = \max_n |u_n|$ gives our claim. ◂

Theorem 39. *Let $(c_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \subset \mathbb{C}$ be so that*

$$\sum_{\alpha \in \mathbb{N}_0^N} |c_\alpha z^\alpha| < \infty \qquad \textit{for every } z \in \mathbb{D}^N \textit{ and every } N \in \mathbb{N}, \textit{ and} \tag{10}$$

$$\sup_{N \in \mathbb{N}} \sup_{z \in \mathbb{D}^N} \Big| \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha \Big| < \infty. \tag{11}$$

Then, there exists a unique $f \in H_\infty(B_{c_0})$ such that $c_\alpha(f) = c_\alpha$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$. Moreover, $\|f\|_\infty$ equals the supremum in (11).

Proof. For each $N \in \mathbb{N}$ we define the function $f_N : \mathbb{D}^N \to \mathbb{C}$ by

$$f_N(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha.$$

By (10) and (11), this function is in $H_\infty(\mathbb{D}^N)$ (every analytic function on $\mathbb{D}^N$ is holomorphic by theorem 23). Moreover, $|f_N(z)| \leq \eta$ for all $z \in \mathbb{D}^N$ and all N (where η is the supremum in (11)); in other words, $\|f_N\|_\infty \leq \eta$ for every N. We look now at these functions as defined on B_{c_0} (recall remark 32), and our aim is to show that $(f_N)_N$ converges uniformly on every compact subset $K \subset B_{c_0}$. We choose then some compact $K \subset B_{c_0}$ and we want to see that $(f_N)_N$ is uniformly Cauchy on K. We fix $z \in K$ and define for $1 \leq N < M$ the holomorphic function (remember that we look to the functions f_N as defined on B_{c_0})

$$f_{N,M} : \prod_{n=N+1}^{M} \mathbb{D} \to \mathbb{C} \quad \text{by} \quad f_{N,M}(u) = f_N(z_1, \dots, z_N, 0, 0, \dots) - f_M(z_1, \dots, z_N, u, 0, 0, \dots).$$

Then, $f_{N,M}(0) = 0$ and by (11) we have $|f_{N,M}(u)| < 3\eta$ for $u \in \prod_{n=N+1}^{M}$, and hence by (9), for these u,

$$|f_{N,M}(u)| \leq 3\eta \max_{N+1 \leq n \leq M} |u_n|.$$

Now we pick $r \in c_0$ such that $K \subset \{x \in c_0 : |z_j| \leq |r_j| \text{ for all } j \in \mathbb{N}\}$ (take $r_j := \sup_{z \in K} \sup_{k \geq j} |z_k|$) and then, taking $u = (z_{N+1}, \dots, z_M)$,

$$|f_N(z) - f_M(z)| = |f_{N,M}(z_{N+1}, \dots, z_M)| \leq 3\eta \max_{N+1 \leq n \leq M} |r_n|.$$

Using the fact that $r \in c_0$, we obtain that $\{f_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $H_\infty(B_{c_0})$ with respect to the uniform convergence on compact subsets, and then converges to a certain function f that, by theorem 34, belongs to $H_\infty(B_{c_0})$ and satisfies $\|f\|_\infty \leq \eta$. Let us see that $c_\alpha(f) = c_\alpha$ for all α. Take $\alpha \in \mathbb{N}_0^M$ and $0 < r < 1$. Then (note that $c_\alpha(f) = c_\alpha$ for all $N \geq M$), if we take $N \geq M$,

$$\begin{aligned} c_\alpha = c_\alpha(f) = \lim_N c_\alpha(f_N) &= \lim_N \frac{1}{(2\pi \mathrm{i})^M} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_M|=r} \frac{f_N(\zeta_1, \dots, \zeta_M, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_M^{\alpha_M+1}} \, \mathrm{d}\zeta_M \cdots \mathrm{d}\zeta_1 \\ &= \frac{1}{(2\pi \mathrm{i})^M} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_M|=r} \frac{f(\zeta_1, \dots, \zeta_M, 0, \dots)}{\zeta_1^{\alpha_1+1} \dots \zeta_M^{\alpha_M+1}} \, \mathrm{d}\zeta_M \cdots \mathrm{d}\zeta_1 = c_\alpha(f). \end{aligned}$$

Finally, we have $\eta \leq \|f\|_\infty$ since

$$\eta = \sup_N \sup_{z \in \mathbb{D}^N} \Big| \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha \Big| = \sup_N \sup_{z \in \mathbb{D}^N} |f_N(z)| = \|f_N\|_\infty \leq \|f\|_\infty. \qquad \blacksquare$$

We move now to the side of Fourier analysis. To begin with, we need a way to go from $\mathbb{T}^\infty$ to $\mathbb{T}^N$ in a reasonable way. Given $f \in L_1(\mathbb{T}^\infty)$ and $N \in \mathbb{N}$, we define

$$f_{[N]}(w) = \int_{\mathbb{T}^\infty} f(w, z) \, \mathrm{d}z.$$

Recall that $H_\infty(\mathbb{T}^\infty) \subset L_\infty(\mathbb{T}^\infty) \subset L_1(\mathbb{T}^\infty)$. Let us see that with this definition everything works fine.

Lemma 40. *Given $f \in L_p(\mathbb{T}^\infty)$ with $p = 1, \infty$, and $N \in \mathbb{N}$, we have the following:*

(i) *$f_{[N]} \in L_p(\mathbb{T}^N)$, and $\|f_{[N]}\|_p \leq \|f\|_p$;*

(ii) *$\hat{f}_{[N]}(\alpha) = \hat{f}(\alpha)$ for every $\alpha \in \mathbb{Z}_0^N$;*

(iii) *if $p = 1$, then $f_{[N]} \to f$ in $L_1(\mathbb{T}^\infty)$; if $p = \infty$, then $f_{[N]} \to f$ in the $w(L_\infty, L_1)$-topology.*

Proof. Let us first look at (i). If $p = 1$, using the monotonicity of the integral together with Fubini's theorem we have

$$\begin{aligned} \|f_{[N]}\|_1 = \int_{\mathbb{T}^N} |f_{[N]}(w)| \, \mathrm{d}w &= \int_{\mathbb{T}^N} \left| \int_{\mathbb{T}^\infty} f(w, u) \, \mathrm{d}u \right| \mathrm{d}w \leq \int_{\mathbb{T}^N} \left(\int_{\mathbb{T}^\infty} |f(w, u)| \, \mathrm{d}u \right) \mathrm{d}w \\ &= \int_{\mathbb{T}^N \times \mathbb{T}^\infty} |f(w, u)| \, \mathrm{d}(w, u) = \int_{\mathbb{T}^\infty} |f(z)| \, \mathrm{d}z = \|f\|_1. \end{aligned}$$

If $p = \infty$, recall that

$$\begin{aligned} |f_{[N]}(w)| &= \left| \int_{\mathbb{T}^\infty} f(w,u)\,\mathrm{d}u \right| \leq \int_{\mathbb{T}^\infty} |f(w,u)|\,\mathrm{d}u \\ &\leq \int_{\mathbb{T}^\infty} \|f\|_\infty \,\mathrm{d}u \leq \|f\|_\infty \int_{\mathbb{T}^\infty} \mathrm{d}u = \|f\|_\infty, \text{ almost everywhere,} \end{aligned}$$

so we have $\|f_{[N]}\|_\infty \leq \|f\|_\infty$.

For the proof of (ii) take $\alpha \in \mathbb{Z}^N$ and $f \in L_1(\mathbb{T}^\infty)$. Then, again by Fubini's theorem,

$$\hat{f}_{[N]}(\alpha) = \int_{\mathbb{T}^N} \left(\int_{\mathbb{T}^\infty} f(w,u)\,\mathrm{d}u \right) w^{-\alpha}\,\mathrm{d}w = \int_{\mathbb{T}^N \times \mathbb{T}^\infty} f(w,u)(w,u)^{-\alpha}\,\mathrm{d}(w,u) = \hat{f}(\alpha).$$

We begin the proof of (iii) by considering $L_1(\mathbb{T}^\infty)$. Let us suppose first that $f \in L_1(\mathbb{T}^k)$ for some k. Then, a straightforward calculation shows that $f_{[N]} = f$ for every $N \geq k$. In particular, $f_{[N]} \to f$ for every $f \in \bigcup_{k\in\mathbb{N}} L_1(\mathbb{T}^k)$, and as an immediate consequence of the density of trigonometric polynomials on $L_1(\mathbb{T}^\infty)$ (proposition 37), these functions are dense in $L_1(\mathbb{T}^\infty)$. Now, by (i) for $p = 1$, the projection $L_1(T^\infty) \to L_1(T^N)$ given by $f \mapsto f_{[N]}$ is a contraction. Given $f \in L_1(\mathbb{T}^\infty)$ and $\varepsilon > 0$, we can take $g \in \bigcup_k L_1(\mathbb{T}^k)$ such that $\|g - f\|_1 < \varepsilon/3$ and, by the previous coment, $\|f_{[N]} - g_{[N]}\|_1 < \varepsilon/3$. Since $g_N \to g$ in $L_1(\mathbb{T}^\infty)$, there exists $N_0 \in \mathbb{N}$ such that $\|g_{[N]} - g\|_1 < \varepsilon/3$ for every $N \geq N_0$. Then, for every $N \geq N_0$ we have

$$\|f_{[N]} - f\|_1 \leq \|f_{[N]} - g_{[N]}\|_1 + \|g_{[N]} - g\|_1 + \|g - f\|_1 < \varepsilon.$$

It is only left to show (iii) for $p = \infty$. Given $f \in L_\infty(\mathbb{T}^\infty)$, we have to show that $\langle f_{[N]}, \cdot\rangle \to \langle f, \cdot\rangle$ pointwise on $L_1(\mathbb{T}^\infty)$. Using (iii) for $L_1(\mathbb{T}^\infty)$, this holds true on the dense subspace $L_\infty(\mathbb{T}^\infty)$ of $L_1(\mathbb{T}^\infty)$, and using (i) for $L_\infty(\mathbb{T}^\infty)$, all functionals $\langle f_{[N]}, \cdot\rangle$ are uniformly bounded on $L_1(\mathbb{T}^\infty)$; that is, for every $h \in L_1(\mathbb{T}^\infty)$, $|\langle f_{[N]}, h\rangle| \leq \|f\|_\infty \|h\|_1$.

Then, given $h \in L_1(\mathbb{T}^\infty)$ and $\varepsilon > 0$, we can take $g \in L_\infty(\mathbb{T}^\infty)$ such that $\|h - g\|_1 < \frac{\varepsilon}{4\|f\|_\infty}$. Since $\langle f_{[N]}, g\rangle \to \langle f, g\rangle$, there exists $N_0 \in \mathbb{N}$ such that $|\langle f_{[N]} - f, g\rangle| < \varepsilon/2$ for every $N \geq N_0$. Then, for every $N \geq N_0$ we have

$$\begin{aligned} |\langle f_{[N]}, h\rangle - \langle f, h\rangle| &\leq |\langle f_{[N]}, h - g\rangle| + |\langle f, h - g\rangle| + |\langle f_{[N]}, g\rangle - \langle f, g\rangle| \\ &\leq 2\|f\|_\infty \|h - g\|_1 + |\langle f_{[N]} - f, g\rangle| < \varepsilon. \end{aligned}$$

∎

Proposition 41. *Given $f \in H_\infty(\mathbb{T}^\infty)$ and $N \in \mathbb{N}$, we have the following:*

(i) *$f_{[N]} \in H_\infty(\mathbb{T}^N)$, and $\|f_{[N]}\|_\infty \leq \|f\|_\infty$;*

(ii) *$\hat{f}_{[N]}(\alpha) = \hat{f}(\alpha)$ for every $\alpha \in \mathbb{N}_0^N$;*

(iii) *$f_{[N]} \to f$ in $H_\infty(\mathbb{T}^\infty)$ in the $w(L_\infty, L_1)$-topology.*

Proof. It is a consequence of lemma 40 and the fact that $H_\infty(\mathbb{T}^\infty)$ is a closed subspace of $L_\infty(\mathbb{T}^\infty)$. ∎

Theorem 42. *Let $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} \subset \mathbb{C}$. The following are equivalent:*

(i) *there exists $f \in H_\infty(\mathbb{T}^\infty)$ so that $\hat{f}(\alpha) = c_\alpha$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$;*

(ii) *for each $N \in \mathbb{N}$, there exists $f_N \in H_\infty(\mathbb{T}^N)$ so that $\hat{f}_N(\alpha) = c_\alpha$ for every $\alpha \in \mathbb{N}_0^N$ satisfying $\sup\limits_{N\in\mathbb{N}} \|f_N\|_\infty < \infty$.*

Moreover, in this case $\|f\|_\infty = \sup_N \|f_N\|_\infty$.

Proof. Taking $f_N = f_{[N]}$ in proposition 41 immediately gives that (i) implies (ii) and $\sup_{N\in\mathbb{N}} \|f_N\|_\infty = \sup_{N\in\mathbb{N}} \|f_{[N]}\|_\infty \leq \|f\|_\infty$.

Assume that (ii) holds, consider the sequence $\{f_N\}_{N\in\mathbb{N}}$ as a bounded sequence in $L_\infty(\mathbb{T}^\infty)$, and let $K = \sup_{N\in\mathbb{N}} \|f_N\|_\infty$. Using remark 15 we can find a subsequence $\{f_{N_k}\}_{k\in\mathbb{N}}$ that $\sigma(L_\infty, L_1)$-converges to some $f \in L_\infty(\mathbb{T}^\infty)$ with $\|f\| \leq \sup_{N\in\mathbb{N}} \|f_N\|$. Take now $\alpha \in \mathbb{Z}^{(\mathbb{N})}$ and find $L \geq 1$ such that $\alpha = (\alpha_1, \dots, \alpha_L, 0, 0 \dots)$

with $\alpha_L \neq 0$, and some k_0 such that for all $k \geq k_0$ we have $N_k \geq L$. As a consequence, we have $\hat{f_{N_k}}(\alpha) = c_\alpha$ for all $k \geq k_0$, and therefore

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha}\,\mathrm{d}w = \lim_{k\to\infty} \int_{\mathbb{T}^\infty} f_{N_k}(w_1, \dots, w_{N_k}) w^{-\alpha}\,\mathrm{d}w = \lim_{k\to\infty} \hat{f}_{N_k}(\alpha) = \begin{cases} c_\alpha & \text{if } \alpha \in \mathbb{N}_0^{(\mathbb{N})}, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the uniqueness of the Fourier coeffients (proposition 35) shows that $f_N = f_{[N]}$ and completes the proof of the equivalence. ∎

We finally have at hand everything we need to prove the result we are aiming for.

Theorem 43. *There exists a (unique) isometric isomorphism*

$$P_\infty : H_\infty(\mathbb{T}^\infty) \to H_\infty(B_{c_0})$$

so that $c_\alpha(P_\infty[f]) = \hat{f}(\alpha)$ *for every* $f \in H_\infty(\mathbb{T}^\infty)$ *and all* $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$.

Proof. Our aim now is to define

$$P_\infty : H_\infty(\mathbb{T}^\infty) \to H_\infty(B_{c_0}),$$

satisfying our requests. First of all, given $f \in H_\infty(\mathbb{T}^\infty)$ we consider $f_{[N]} \in H_\infty(\mathbb{T}^N)$ defined in (41), that satisfies $\hat{f}(\alpha) = \hat{f}_{[N]}(\alpha)$ for every $\alpha \in \mathbb{N}_0^N$. Theorem 29 provides us with $g_N = P[f_{[N]}] \in H_\infty(\mathbb{D}^N)$ satisfying $\hat{f}_{[N]}(\alpha) = c_\alpha(g_N)$ for all $\alpha \in \mathbb{N}_0^N$ and $\|g_N\|_\infty = \|f_{[N]}\|_\infty \leq \|f\|_\infty$. Now, for each $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ we consider $c_\alpha = \hat{f}(\alpha)$ and, by theorem 29, the family $\{c_\alpha\}_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ satisfies (10) and (11). Then, by theorem 39, we can find $g \in H_\infty(B_{c_0})$ with $\hat{f}(\alpha) = c_\alpha(g)$ for all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and $\|g\|_\infty \leq \|f\|_\infty$. In this way, P_∞ is well defined and a contraction.

Conversely, given $g \in H_\infty(B_{c_0})$, define for each N the function $g_N \in H_\infty(\mathbb{D}_N)$ as the restriction of g to the first N variables. Then, again using theorem 29, look at $f_N = P_N^{-1}[g_N] \in H_\infty(\mathbb{T}^N)$. Considering this time $c_\alpha = c_\alpha(g)$ for each $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and using theorem 42 we obtain $f \in H_\infty(\mathbb{T}^\infty)$ with $\|f\|_\infty \leq \|g\|_\infty$. Finally, the uniqueness of the Fourier coeffients shows that $f_{[N]} = f_N$ for every N and, therefore, $P_\infty[f] = g$. ∎

References

[1] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. New York: Springer, 2011. https://doi.org/10.1007/978-0-387-70914-7_1.

[2] Cerdà, Joan. *Linear functional analysis.* Graduate Studies in Mathematics 116. Providence and Madrid: American Mathematical Society and Real Sociedad Matemática Española, 2010. https://doi.org/10.1090/gsm/116.

[3] Cohn, Donald L. *Measure Theory.* 2nd ed. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser, 2013. isbn: 978-1-4614-6955-1.

[4] Cole, Brian J. and Gamelin, Theodore W. “Representing measures and Hardy spaces for the infinite polydisk algebra”. In: *Proceedings of the London Mathematical Society* s3-53.1 (1986), pp. 112–142. https://doi.org/10.1112/plms/s3-53.1.112.

[5] Defant, Andreas; García, Domingo; Maestre, Manuel, and Sevilla-Peris, Pablo. *Dirichlet Series and Holomorphic Funcions in High Dimensions.* New Mathematical Monographs 37. Cambridge: Cambridge University Press, 2019. https://doi.org/10.1017/9781108691611.

[6] Rudin, Walter. *Principles of mathematical analysis.* 3rd ed. International Series in Pure and Applied Mathematics. New York: McGraw-Hill, 1976. isbn: 978-0-07-085613-4.

[7] Rudin, Walter. *Real and complex analysis.* 3rd ed. McGraw-Hill series in higher mathematics. New York: McGraw-Hill, 1987. isbn: 978-0-07-054234-1.

[8] Stein, Elias M. and Shakarchi, Rami. *Complex analysis.* Princeton Lectures in Analysis 2. Princeton: Princeton University Press, 2003. isbn: 978-0-691-11385-2.

[9] Toeplitz, Otto. "Über eine bei den Dirichletschen Reihen auftretende Aufgabe aus der Theorie der Potenzreihen von unendlichvielen Veränderlichen". In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (1913), pp. 417–432. url: https://eudml.org/doc/58883.

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