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Bringing Young Mathematicians Together

BYMAT

Bringing Young Mathematicians Together

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En este volumen de *TEMat monográficos* se recogen los resúmenes extendidos de las contribuciones presentadas en el tercer *Congreso BYMAT - Bringing Young Mathematicians Together*, celebrado telemáticamente del 1 al 3 de diciembre de 2020. Organizado conjuntamente por la Universitat de València y la Universitat Politècnica de València, el congreso BYMAT tiene entre sus objetivos proporcionar un espacio cálido y abierto para que jóvenes investigadores den a conocer su trabajo, desde el inicio de su carrera científica.

This volume of *TEMat monográficos* contains the extended abstracts of the contributions presented at the third *BYMAT - Bringing Young Mathematicians Together Conference*, held online from 1 to 3 December 2020. Organised jointly by the Universitat de València and the Universitat Politècnica de València, the BYMAT Conference aims to provide a warm and open space for young researchers to present their work, since the beginning of their scientific career.

3rd *BYMAT Conference* (2020)
Invited article

Characters of finite groups

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Abstract: Groups are the mathematical objects formally describing our idea of symmetry. They appear naturally acting on vector spaces as groups of invertible matrices. Group representation theory is the branch of mathematics that studies such actions. More specifically, character theory studies the trace maps associated to those actions. A fundamental question in the field is to understand how much information about a finite group G and its local subgroups can be extracted from the knowledge of the character theory of G. In this note, I will report on recent advances in this topic.

Resumen: La teoría de grupos es la rama de las matemáticas que describe y estudia las simetrías. Los grupos aparecen de forma natural actuando sobre espacios vectoriales como matrices invertibles. La teoría de representaciones se encarga de estudiar dichas acciones como homomorfismos entre grupos y grupos de matrices. En particular, la teoría de caracteres estudia las trazas asociadas a tales homorfismos. Un problema central en el área es descubrir qué información acerca de la estructura de un grupo G y sus subgrupos locales puede leerse en su tabla de caracteres. El propósito de esta monografía es exponer recientes contribuciones a este problema en el marco de las conjeturas globales-locales.

Keywords: finite groups, character tables, Sylow subgroups, global-local conjectures.

MSC2010: 20C15, 20C20.

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1. Character theory: a brief historical remark

The theory of groups is the abstract mathematical framework to describe the intuitive human concept of symmetry. It has been said by the mathematician and sci-fi writer E. T. Bell that “wherever groups disclose themselves or can be introduced, simplicity crystallizes out of comparative chaos”, a sentence that highlights the usefulness of group theory in other branches of mathematics. In his Erlangen program, F. Klein proposed to study the geometry of a space through its group of symmetries. Outside mathematics, groups play a crucial role in several other disciplines like quantum physics, chemistry and cryptography.

The birth of group theory goes back to the work on polynomial equations of J. L. Lagrange and E. Galois, where groups appeared as permutations of the roots of a polynomial. In his investigations, Galois already introduced key concepts in group theory like those of normal subgroup, simple group and solvable group. However, we owe the first abstract definition of group to Cayley by the end of the 19th century.

Character theory and, more generally, representation theory study how groups act on a vector space. Properties of representations and characters of a group are intimately connected to the algebraic structure of the group itself. This mutual influence is successfully used to study one in terms of the other. Indeed, the study of finite groups bloomed in the early 20th century thanks to the springtime of representation and character theory. The pioneering work of W. Burnside, F. G. Frobenius and I. Schur laid the foundations of this area of mathematics. Burnside proved in 1904 that finite groups whose order is divisible by at most two primes are solvable [3]. This was the first main application of character theory to group theory, and we care to remark that a proof of Burnside’s $p^a q^b$ theorem not involving character theoretical arguments was not found until 1972 by D. Goldschmidt [6] and H. Bender [1].

In 1963, W. Feit and J. Thompson proved that groups of odd order are solvable [5]. Their (225 pages long) proof requires a mixture of deep group and character theoretical arguments. For his contributions to this success, Thompson was awarded a Fields Medal in 1970 and an Abel Prize in 2008. Moreover, the solvability of groups of odd order lies at the heart of one of the greatest achievements of mathematics in the last two centuries: the classification of finite simple groups [4].

2. Character tables and Sylow subgroups

In 1963, R. Brauer published an inspiring survey article on the representations of finite groups [2]. In its introduction, Brauer writes that “A tremendous effort has been made by mathematicians for more than a century to clear up the chaos in group theory. Still, we cannot answer some of the simplest questions.” Far from being critical of the theory of finite groups, Brauer claims to be fascinated by its mysteries. In that landmark survey, he set up a list of 42 problems that still guides the research on representation theory. Among the most significant problems contained in his article we find the so-called Brauer’s $k(B)$-conjecture (Problem 20) and Brauer’s height zero conjecture (Problem 23). These conjectures remain open today, although a great deal of work has been devoted to them (see [9, 11, 17], for instance).

In this note, we focus our attention on Brauer’s Problem 12. For a finite group G, we denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G (those characters afforded by G-actions on vector spaces without G-invariant subspaces). If $\chi \in \mathrm{Irr}(G)$, then $\chi : G \to \mathbb{C}$ is a function constant on G-conjugacy classes. It is well known, and follows from the Wedderburn decomposition of the algebra $\mathbb{C}G$, that $|\mathrm{Irr}(G)| = k$ is the number of G-conjugacy classes of G. Hence, we can arrange the values of the irreducible characters of G in a $(k \times k)$ matrix $X(G)$ known as the character table of G. The value $\chi(1)$ is the degree of χ, and coincides with the dimension of a G-vector space affording χ. It is customary to arrange $X(G)$ so that its first column is the column corresponding to irreducible character degrees. Also, the first row of $X(G)$ usually contains the values of the principal character $1_G : G \to \mathbb{C}$, coming from the trivial action of G on $\mathbb{C}$. For example, the character table of S_3, the symmetric group on 3 symbols, is

$$X(\mathsf{S}_3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

Character tables are invertible matrices whose rows and columns satisfy amazing numerical relations (see [7, Chapter 2] for details on the Schur orthogonality relations).

Problem (Brauer's Problem 12). Given the character table $X(G)$ of a group G and a prime p dividing the order $|G|$ of G, how much information about the structure of the Sylow p-subgroup P of G can be obtained? In particular, can it be decided whether or not P is abelian? ◂

We recall that a Sylow p-subgroup P of G is a p-subgroup of G of order $|G|_p$, the largest p-power dividing $|G|$. The set $\mathrm{Syl}_p(G)$ of Sylow p-subgroups of G is non-empty, its elements form a G-conjugacy class and they *dominate* the p-subgroups of G. Sylow theory is a cornerstone of group theory. We care to mention that the character table of a group does not determine the isomorphism class of its Sylow subgroups. For instance, the dihedral group D_8 of order 8 and the quaternion group Q_8 have the same character table (after possibly rearranging rows and columns).

The Sylow subgroups of S_3 are abelian for every prime p. How can this information be extracted from $X(\mathsf{S}_3)$? For $p = 3$, we notice that every irreducible character of S_3 has degree coprime to p. The Itô-Michler theorem guarantees that such a condition is equivalent to S_3 having a normal and abelian Sylow 3-subgroup. For $p = 2$, we observe that not every irreducible character degree of S_3 is odd. Nevertheless, the degree of every irreducible character belonging to the *principal* 2*-block* of S_3 is odd (the characters in the principal 2-block of S_3 are the ones corresponding to the first and second row of $X(\mathsf{S}_3)$, but we do not wish to get into technical details at this point of the exposition). This condition is equivalent to S_3 having an abelian Sylow 2-subgroup, by the principal block case of Brauer's height zero conjecture (a key case that has been recently proven [12]).

The character table $X(G)$ of a group G determines the order $|G|$ of the group by the well-known formula

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

In particular, we do not need to appeal to the Itô-Michler theorem nor the principal block case of Brauer's height zero conjecture to deduce that the Sylow subgroups of S_3 are abelian (but we thought it could be a good way of informally introducing their statements). Once we know that $X(G)$ easily determines $|P|$, the order of a Sylow p-subgroup P of G, it makes sense to wonder whether $X(G)$ determines the order $|\mathbf{N}_G(P)|$ of the normalizer of P (see [15, Question 7]). At the time of this writing, that is an open question. A positive answer is known in the special case where $\mathbf{N}_G(P) = P$. In other words, we can tell whether $|\mathbf{N}_G(P)| = |G|_p$ after an easy inspection of $X(G)$. This follows from the main results of [19] and [24] for p odd and $p = 2$, respectively.

What more can be said about the structure of $\mathbf{N}_G(P)$ in $X(G)$? Apart from determining if $\mathbf{N}_G(P)$ is a p-group, we can also determine if $\mathbf{N}_G(P)$ is p-*nilpotent*, that is, if $\mathbf{N}_G(P) = P \times X$. This fact follows from [20] and [21] for p odd and from [22] and [26] for $p = 2$.

Let us go back to the structure of P. We have mentioned above that, by the main result of [12], one can determine if P is abelian by looking at the character degrees of irreducible characters in the principal p-block of $X(G)$. The set of irreducible characters of G belonging to the principal p-block is

$$\{\chi \in \mathrm{Irr}(G) \mid \sum_{x \in G^0} \chi(x) \neq 0\},$$

where $G^0 \subseteq G$ consists of those elements of order not divisible by p. Without getting into further technical details, we care to remark that, by Higman's theorem [7, Theorem 8.21], the set of irreducible characters of G belonging to the principal p-block can be determined after an easy inspection of $X(G)$.

It is also possible to determine whether P is cyclic by looking only at $X(G)$ (an elementary proof can be found in [15, Theorem 8]). A huge step further is to consider whether $X(G)$ determines if P is 2-generated. For example, if a group G has a cyclic (1-generated) Sylow 2-subgroup, then G has a normal 2-complement. In particular, such a group is solvable by Feit-Thompson's odd order theorem [5]. In opposition, there are many nonsolvable groups possessing a 2-generated Sylow 2-subgroup. Actually, the number of isomorphism classes of 2-groups of order 2^n that are 2-generated grows exponentially with n. Despite the greater degree of difficulty, we have recently shown in [16] that $X(G)$ determines if a Sylow 2-subgroup is 2-generated. What happens for odd primes, we do not know. We expect that $X(G)$ determines if P is 2-generated if $p = 3$, but for larger primes the situation might be very different.

3. Global-local conjectures

The purpose of this last section is to briefly describe a deep system of interconnected conjectures underlying most of the results mentioned in the context of Brauer's Problem 12 above. This system consists of the so-called global-local conjectures. The common philosophy behind all these conjectures is that certain essential information on the character theory of a finite group G is encoded in its local subgroups (we refer the reader to [10] for a detailed account on the global-local principle in representation theory). By local subgroups of G we mean its nontrivial p-subgroups and their normalizers, where p is any fixed prime. The most important local subgroups are the Sylow subgroups and their normalizers.

One of the most paradigmatic global-local conjectures is the McKay conjecture.

Conjecture (McKay, 1971). *Let G be a finite group, let p be a prime and $P \in \mathrm{Syl}_p(G)$. The number of $\mathrm{Irr}(G)$ of degree not divisible by p equals the number of $\mathrm{Irr}(\mathbf{N}_G(P))$ of degree not divisible by p.*

We write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) \text{ is not divisible by } p\}$. As a global-local statement, the McKay conjecture is telling us that global invariant $|\mathrm{Irr}_{p'}(G)|$ is a local invariant, in the sense that it can be computed as $|\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$ in the local subgroup $\mathbf{N}_G(P)$.

In Section 2, we said that wether a group G has a self-normalizing Sylow p-subgroup P, that is, $\mathbf{N}_G(P) = P$, can be read off from $X(G)$. What does the McKay conjecture predict in such a situation? Assume that $\mathbf{N}_G(P) = P$; then, the McKay conjecture asserts that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(P)|$. As the degrees of irreducible characters divide the order of the group, we have that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Lin}(P)|$, where $\mathrm{Lin}(P) = \mathrm{Hom}(P, \mathbb{C}^\times) \cong P/P'$. Hence, if $\mathbf{N}_G(P) = P$, then the McKay conjecture predicts that $|\mathrm{Irr}_{p'}(G)| = |P : P'|$. Unfortunately, this property does not characterize groups with a self-normalizing Sylow p-subgroup (as shown by S_3 for $p = 3$). More than that, we do not know if $X(G)$ determines $|P : P'|$ in the case where $P' > 1$ (not even if we restrict ourselves to the realm of p-solvable groups, as explained in [15]).

Nevertheless, we have mentioned that global-local conjectures underlie most of the results contained in Section 2. The key is held by the so-called Galois version of the McKay conjecture proposed by Navarro [14], also known as the McKay-Navarro conjecture.

The values of $\chi \in \mathrm{Irr}(G)$ are sums of roots of unity and lie in $\mathbb{Q}(e^{2\pi i/|G|})$. Given $\sigma \in \mathcal{G} = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$, the function $\chi^\sigma = \sigma(\chi)$ is an irreducible character of G. Hence, $\mathcal{G}$ acts on $\mathrm{Irr}(G)$ (so on the rows of $X(G)$). As the McKay conjecture predicts the existence of a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$, and the Galois group $\mathcal{G}$ acts on both sets, it is natural to wonder if such a bijection can be expected to commute with the action of $\mathcal{G}$ (that is, if such bijection can be expected to be $\mathcal{G}$-equivariant). The general linear group $\mathrm{GL}(2,3)$ provides a negative answer to that question for $p = 3$, as $\mathrm{GL}(2,3)$ has less rational-valued irreducible characters of degree not divisible by 3 than the dihedral group D_{12}, the normalizer of a Sylow 3-subgroup.

Let $\mathcal{H}_p \leq \mathcal{G}$ be the subgroup consisting of those Galois automorphisms $\sigma \in \mathcal{G}$ for which there exists a fixed integer f such that $\sigma(\xi) = \xi^{p^f}$ for every root of unity $\xi \in \mathbb{Q}(e^{2\pi i/|G|})$ of order not divisible by p.

Conjecture (McKay-Navarro, 2004). *Let G be a finite group, let p be a prime and $P \in \mathrm{Syl}_p(G)$. There exists an $\mathcal{H}_p$-equivariant bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$.*

In [14, Theorem 5.2 and Theorem 5.3], Navarro proves that the McKay-Navarro conjecture implies the main results of [19] and [24]. We say those results are *proven consequences* of the conjecture. The interest of proving consequences of global-local conjectures is twofold: they help us understand new connections between global and local invariants of a group and, at the same time, they provide new evidence for the validity of these elusive conjectures. In a similar way, the McKay-Navarro conjecture is behind the main results of [26], [21] and [16].

In general, the method used to prove such consequences of the McKay-Navarro conjecture is based on a reduction to simple groups of the statement and then on an exhaustive study of the character theory of the finite nonabelian simple groups (and related groups, as the decorated groups). The first part of the method is what we call proving a *reduction theorem.* The origin of the term goes back to the McKay conjecture.

The McKay conjecture was formulated in 1971 (originally just for simple groups and the prime $p = 2$), after evidence found on symmetric groups and the known sporadic groups. It immediately attracted the interest of the community for the simplicity of its formulation, and celebrated group-theorists started verifying it for different families of groups such as solvable groups and general linear groups. The different verifications used *ad hoc* methods, specific to each family. Despite overwhelming evidence, for several decades no general strategy for proving this easy-to-state conjecture was envisaged. It was not until 2007 that a method was proposed. I. M. Isaacs, G. Malle and G. Navarro [8] showed that in order to prove the McKay conjecture for all finite groups, it was enough to verify the so-called inductive McKay condition only for all finite simple groups. Such a result is what we call a reduction theorem. We care to remark that it is not enough that the conjecture is satisfied for all finite simple groups. The inductive McKay condition is much stronger than the conjecture itself and its verification constitutes a true challenge for simple-group theorists. (We omit a description of the inductive condition here due to its highly technical nature, instead we refer the interested reader to [27].) However, the above strategy has proven to be successful. In 2016, Malle and B. Späth [13] verified the inductive McKay condition for all simple groups at the prime $p = 2$. This has lead to one of the highlights in representation theory of the 21st century: the McKay conjecture holds for the prime 2.

Inspired by the success of the *reduction approach* to the McKay conjecture, we have recently obtained a reduction theorem for the McKay-Navarro conjecture [18]. Namely, we have shown that, in order to prove the McKay-Navarro conjecture in full generality, it is enough to verify the *inductive McKay-Navarro condition* for all finite simple groups. We have already mentioned that verifying the inductive McKay condition is a challenge for simple-groups theorists. It is no surprise that verifying the inductive McKay-Navarro condition constitutes a bigger challenge, as it requires a vast knowledge of the character values of decorated simple groups and the interplay between Galois action and the action of group automorphisms on characters. Some examples of finite simple groups satisfying the inductive McKay-Navarro condition have appeared so far [23, 25], and we are aware that exciting new results will appear soon.

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A probabilistic proof of Meir-Moon theorem

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Abstract: In this work we introduce a probabilistic proof of the Meir-Moon theorem. This theorem gives an asymptotic formula for the coefficients of the solution to Lagrange's equation. Let ψ be an analytic function, with non-negative coefficients, on a disk around $z = 0$ and $f(z) = z\psi(f(z)) = \sum_{n\geq 0} a_n z^n$ Lagrange's equation with data ψ. Under certain conditions over ψ, the coefficients of f satisfy the asymptotic formula

$$a_n \sim \frac{1}{\sqrt{2\pi}} \frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)} \frac{1}{n^{3/2}\tau^n}, \qquad \text{as } n \to \infty,$$

for certain $\tau > 0$. We make no use of saddle point approximation methods: we cast the question in the probabilistic setting of Khinchin families and the local central limit theorem for lattice random variables.

This is based on a joint work with José L. Fernández (Universidad Autónoma de Madrid).

Resumen: En este trabajo presentamos una prueba probabilística del teorema de Meir-Moon. Este teorema da una fórmula asintótica para los coeficientes de la solución de la ecuación de Lagrange. Sea ψ una función analítica en un disco alrededor de $z = 0$, con coeficientes no negativos, y $f(z) = z\psi(f(z)) = \sum_{n\geq 0} a_n z^n$ la ecuación de Lagrange con dato ψ. Bajo ciertas condiciones sobre ψ, los coeficientes de f satisfacen la fórmula asintótica

$$a_n \sim \frac{1}{\sqrt{2\pi}} \frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)} \frac{1}{n^{3/2}\tau^n}, \qquad \text{cuando } n \to \infty,$$

para cierto $\tau > 0$. En esta demostración no utilizamos métodos de punto de silla, los ingredientes son las familias de Khinchin combinadas con cierto teorema local central del límite para variables aleatorias reticulares.

Esto está basado en un trabajo conjunto con José L. Fernández (Universidad Autónoma de Madrid).

Keywords: Khinchin families, probability, Meir-Moon, Lagrange's equation, local central limit theorem, asymptotic analysis, analytic combinatorics.

MSC2010: 05A16, 60C05, 05C05, 05A15.

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1. Introduction and preliminaries

Our objective is to give an asymptotic formula for the coefficients of certain analytic functions. See [5] and [6]. In particular, we want to study generating functions of combinatorial sequences, that is, analytic functions with non-negative coefficients, which are solutions to Lagrange's equation.

In this section, we introduce some definitions and auxiliary results that will be useful later. First, we introduce Khinchin families. Later on, we enumerate some of their properties. We will continue here with Lagrange's equation and its solution: Lagrange inversion formula. Finally, we amalgamate all these tools into a sketch of a proof of Meir-Moon Theorem. For complete details see [1] and [3].

1.1. Khinchin families

We start by defining the class of analytic functions $\mathcal{K}$. This class is convenient for the study of generating functions of combinatorial sequences.

Definition 1. We say that $\psi(z) = \sum_{n\geq 0} b_n z^n$ is in the class $\mathcal{K}$ if ψ is an analytic function with radius of convergence $R > 0$, has non-negative coefficients, $b_0 > 0$ and there exists certain integer $n_0 > 1$ such that $b_{n_0} > 0$. ◂

Now we define the Khinchin family associated to a power series $\psi \in \mathcal{K}$.

Definition 2. Let $\psi \in \mathcal{K}$, an analytic function with power series representation $\psi(z) = \sum_{n\geq 0} b_n z^n$. We define the Khinchin family associated to ψ as the indexed family of discrete random variables $(Y_t)_{[0,R)}$. For each $t \in [0, R)$ we have

$$\mathbb{P}(Y_t = n) = \frac{b_n t^n}{\psi(t)}, \quad \text{for all } n \in \{0, 1, 2, \dots\}.$$ ◂

We will write (Y_t) instead of $(Y_t)_{[0,R)}$ when the radius of convergence is clear from the context. After this line, unless explicitly stated, $R > 0$ will denote the radius of convergence of ψ and (Y_t) the Khinchin family associated to ψ.

The mean and variance functions are

$$\mathbb{E}(Y_t) = m(t) = \frac{t\psi'(t)}{\psi(t)},$$
$$\mathbb{V}(Y_t) = \sigma^2(t) = tm'(t),$$

for all $t \in [0, R)$.

The characteristic function of (Y_t) is

$$\mathbb{E}\left(e^{i\theta Y_t}\right) = \frac{\psi(te^{i\theta})}{\psi(t)}, \quad \text{for all } t \in [0, R) \text{ and } \theta \in \mathbb{R}.$$

In particular, for the normalized Khinchin family $\breve{Y}_t = (Y_t - m(t))/\sigma(t)$ we have

$$\mathbb{E}\left(e^{i\theta \breve{Y}_t}\right) = \mathbb{E}\left(e^{i\theta Y_t/\sigma(t)}\right) e^{-i\theta m(t)/\sigma(t)} = \frac{\psi(te^{i\theta/\sigma(t)})}{\psi(t)} e^{-i\theta m(t)/\sigma(t)}, \quad \text{for all } t \in [0, R) \text{ and } \theta \in \mathbb{R}.$$

The following property will be crucial

Lemma 3. *Suppose $f, g \in \mathcal{K}$, both with radius of convergence at least $R > 0$, and let (X_t) and (W_t) be its Khinchin families, respectively. Denote (Z_t) the Khinchin family associated to $f \cdot g$. Then, for each $t \in [0, R)$, we have*

$$Z_t \stackrel{d}{=} X_t \oplus W_t.$$

Here $\oplus$ denotes sum of independent random variables.

1.2. Lagrange's equation

Let $\psi \in \mathcal{K}$. We will refer to the equation

$$f(z) = z\psi(f(z))$$

as Lagrange equation with data ψ. The coefficients of f, the unique solution to Lagrange's equation with data ψ, are given by the following theorem.

Theorem 4 (Lagrange inversion formula). *Let $f(z)$ and $\psi(z)$ be two analytic functions at certain neighborhood of $z = 0$ such that $\psi(0) \neq 0$ and*

$$f(z) = z\psi(f(z))$$

for $z \in D(0,\delta)$. Then,

$$a_n = COEFF_n\left[f(z)\right] = \frac{1}{n}COEFF_{n-1}\left[\psi(z)^n\right].$$

See, for instance, [4].

1.3. A Hayman type formula

Denote $S_t^{(n)} = Y_t^{(1)} \oplus \cdots \oplus Y_t^{(n)}$, where the random variables $Y_t^{(i)}$ are (i.i.d.) copies of Y_t.

Lemma 5. *With the hypotheses above,*

$$a_n = \frac{1}{2\pi}\frac{1}{\sigma_\psi(t)}\frac{\psi(t)^n}{t^{n-1}n^{3/2}}\int_{-\pi\sigma_\psi(t)\sqrt{n}}^{\pi\sigma_\psi(t)\sqrt{n}} \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta\breve{S}_t^{(n)}/\sqrt{n}}\right)\mathrm{e}^{\mathrm{i}\theta(n-nm_\psi(t))/(\sigma_\psi(t)\sqrt{n})}\,\mathrm{d}\theta.$$

Corollary 6. *Suppose there exists $\tau \in (0,R)$ such that $m_\psi(\tau) = 1$, then*

$$a_n = \frac{1}{2\pi}\frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)}\frac{1}{n^{3/2}\tau^n}\int_{-\pi\sigma_\psi(\tau)\sqrt{n}}^{\pi\sigma_\psi(\tau)\sqrt{n}} \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta\breve{S}_\tau^{(n)}/\sqrt{n}}\right)\mathrm{d}\theta. \tag{1}$$

See [3].

2. Meir-Moon theorem

Theorem 7 (Meir-Moon). *Let $\psi(z) \in \mathcal{K}$ be a holomorphic function in a disc of radius $R > 0$ around $z = 0$. Let $\psi(z) = \sum_{n=0}^\infty b_n z^n$. Suppose that*

- $\gcd\{n \geq 1 : b_n > 0\} = \gcd(\psi) = 1$,
- *there exists $\tau \in (0,R)$ such that*

$$m_\psi(\tau) = \frac{\tau\psi'(\tau)}{\psi(\tau)} = 1.$$

Then, the coefficients of $f(z) = z\psi(f(z)) = \sum_{n=1}^\infty a_n z^n$ satisfy the asymptotic formula

$$a_n \sim \frac{1}{\sqrt{2\pi}}\frac{\tau\psi(\tau)^n}{\sigma_\psi(\tau)}\frac{1}{n^{3/2}\tau^n}, \qquad \text{as } n \to \infty.$$

Sketch of the proof. Apply the local central limit theorem for lattice random variables to the integral I_n on the right-hand side of formula (1), see, for instance, [2] and [3]. Then we have that $\lim_{n\to\infty} I_n = \sqrt{2\pi}$. ∎

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An introduction to knot homology theories

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Abstract: We give a brief exposition of how the Alexander polynomial $\Delta_K(t)$ and the Jones polynomial $J_K(t)$, two classical and powerful knot invariants, can be upgraded to (bi)graded groups $\widehat{HFK}(K)$ and $Kh(K)$ such that the polynomials can be recovered by taking some (graded) Euler characteristic. These groups are called *knot Floer homology* and *Khovanov homology*, respectively, as they share common features with the homology groups of spaces. These groups retain more information about the knots than the polynomials in the sense that they not only encode but also strengthen their properties.

Resumen: Damos una breve exposición de cómo el polinomio de Alexander $\Delta_K(t)$ y el polinomio de Jones $J_K(t)$, dos invariantes de nudos clásicos pero robustos, generalizan a grupos (bi)graduados $\widehat{HFK}(K)$ y $Kh(K)$, de modo que los polinomios se pueden recuperar tomando cierta característica de Euler (graduada). Estos grupos se llaman *homología de nudos de Floer* y *homología de Khovanov*, respectivamente, ya que comparten características similares a las de la homología de espacios. Estos grupos retienen más información sobre los nudos que los polinomios en el sentido de que no solo heredan sus propiedades sino que las hacen más fuertes.

Keywords: knots, Khovanov homology, knot Floer homology.

MSC2010: 57K10, 57K18.

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1. Categorification

We start off by briefly discussing the idea of categorification. Let us borrow from Lurie [3] the term *category number* as a loose measure of the amount of abstraction involved in a mathematical idea, construction, theorem, etc. The most concrete kind of mathematics, such as numbers, or polynomials (arrays of numbers) belong to category number zero. One level up, in category number one, we find mathematical structures such as sets, groups, topological spaces, etc. These objects have certain structure and maps between them which preserve the structure. Category number two refers to classes of mathematical structures, that is, categories, where not only do we have arrows between the structures but also arrows between the categories. And the ladder continues all the way up.

The problem of *categorification* consists of taking an object, statement, construction, etc. which happens in some category number, and lifting it to another such taking place at a higher level, being able to recover the original object, statement, construction, etc. in a rather simple way (*decategorification*).

Example 1. The category of finite sets is a categorification of the natural numbers: every $n \in \mathbb{N}$ is lifted to the finite set S_n of n elements. Decategorification consists of taking cardinality, $\#S_n = n$. ◂

Example 2. The category of finite dimensional chain complexes over a field k is a categorification of the integers. Decategorification sends a chain complex C_* to its Euler characteristic $\chi(C_*) \coloneqq \sum_i (-1)^i \dim_k C_i$. ◂

Example 3. Singular homology categorifies the Euler characteristic of finite-dimensional CW-complexes (and hence the (non-)orientable genus of closed surfaces):

$$\chi(X) = \chi(H_*(X;k)) = \sum_i (-1)^i \dim_k H_i(X,k).$$

The homology groups of a CW-complex carry much more information about it than its Euler characteristic. The success of homology relies on the following properties:

- $H\colon \mathsf{Top} \to \mathsf{grVect}_k$ is a functor.
- $H(X)$ only depends on the homotopy type of X, and the homology of the one-point space is one copy of k concentrated in degree 0.
- There is an isomorphism $H(X \times Y; k) \cong H(X;k) \otimes_k (H(Y;k)$ for spaces X, Y (Künneth formula).
- There are computational tools: Mayer-Vietoris, long exact sequences, etc. ◂

We will try to mimic the previous example 3 for knot polynomial invariants in the next section.

2. Knots and Khovanov homology

Knot theory has proved to be an important subject with many connections with category theory, physics, quantum algebra, manifold theory, biology, etc. We recall that a knot K is a smooth[1] embedding $S^1 \to S^3$. A classical problem in knot theory consists of distinguishing knots up to isotopy. Roughly speaking, two knots are isotopic when one can be deformed in the three-dimensional space into the other without cutting the rope and pasting the endpoints later. A similar discussion follows for links (multiple component knots).

There are two classical link polynomial invariants that we present here:

- The *Alexander polynomial* $\varDelta_L(t) \in \mathbb{Z}[t, t^{-1}]$ of a link L captures topological information about the link embedding, more precisely about the complement of the link in S^3. It is completely determined by the condition $\varDelta_{\text{unknot}} = 1$ and the *skein relation*

$$\varDelta_{L_+} - \varDelta_{L_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\varDelta_{L_0},$$

[1] If you do not like derivatives you can safely replace “smooth” by “piecewise linear”.

where L_+, L_- and L_0 are links that identical except in a small ball where they look like , and , respectively.

- The *Jones polynomial* $J_L(q) \in \mathbb{Z}[q, q^{-1}]$ of a link L has a combinatorial nature, but it can be interpreted as a certain path integral in terms of Chern-Simons theory [7]. It is determined by the condition[2] $J_{\text{unknot}} = q + q^{-1}$ and the skein relation

$$q^2 J_{L_+} - q^2 J_{L_-} = (q - q^{-1}) J_{L_0},$$

where L_+, L_- and L_0 are as before.

Just like with the Euler characteristic of CW-complexes, the Alexander and Jones polynomials have category number zero, and we would like to lift them to some "homology-like" theories, with similar features to the ones singular homology has.

Let Link be the category whose objects are isotopy classes of oriented links in S^3 and whose arrows $L \to L'$ are orientation-preserving homeomorphism classes of bordisms from L to L', that is, compact oriented surfaces $\Sigma \subseteq S^3 \times I$ such that $\partial\Sigma = -L \amalg L'$. We also let $\mathsf{bigrVect}_{\mathbb{Z}/2}$ be the category of bigraded $\mathbb{Z}/2$-vector spaces.

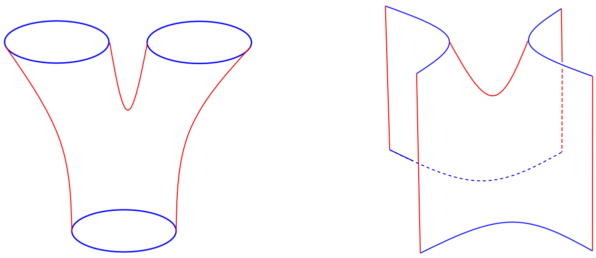

Figure 1: Some (local) pictures of link bordisms.

Theorem 4 (Khovanov [2]). *There exists a functor*

$$Kh\colon \mathsf{Link} \to \mathsf{bigrVect}_{\mathbb{Z}/2}$$

satisfying

(i) *If $\Sigma\colon L \to L'$ is an isotopy, then $Kh(\Sigma)\colon Kh(L) \xrightarrow{\cong} Kh(L')$ is an isomorphism.*

(ii) $Kh(\text{unknot}) = \mathbb{Z}/2_{(0,1)} \oplus \mathbb{Z}/2_{(0,-1)}$

(iii) $Kh(L_1 \amalg L_2) \cong Kh(L_1) \otimes_{\mathbb{Z}/2} Kh(L_2)$.

(iv) *If L is a link, denote by L_0 and L_∞ links identical to L except around one crossing of the form where they have been modified as and , respectively. Then there is an exact triangle*

$$\begin{tikzcd} Kh(L_0) \arrow[rr] & & Kh(L) \arrow[dl] \\ & Kh(L_\infty) \arrow[ul] & \end{tikzcd}$$

(v) *The Jones polynomial is the graded Euler characteristic of Kh:*

$$J_L(q) = \chi_{gr}(Kh(L)) = \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Z}/2} Kh^{i,j}(L).$$

[2]Sometimes it is normalised so that $J_{\text{unknot}} = 1$, but for the purpose of the exposition we do not do that.

Bar-Natan's work [1] has been very influential. Another excellent exposition is [6].

Example 5. The Khovanov homology of the right-handed trefoil $\overline{3}_1$ is $Kh^{i,j}(\overline{3}_1) = \mathbb{Z}/2$ for $(i,j) = (0,1)$, $(0,3)$, $(2,5)$, $(3,9)$ and trivial otherwise. Therefore,

$$\chi_{gr}(Kh(\overline{3}_1)) = \sum_{i,j}(-1)^i q^j \dim_{\mathbb{Z}/2} Kh^{i,j}(K) = q + q^3 + q^5 - q^9 = J_{\overline{3}_1}(q)$$

as expected. ◂

Remark 6. Khovanov homology is strictly stronger than the Jones polynomial: there is a pair of knots called 5_1 and 10_{132} with the same Jones polynomial but with non-isomorphic Khovanov homology. ◂

3. Knot Floer homology

Discovered independently by Ozsváth and Szabó [4] and Rasmussen [5], the knot Floer homology of a knot $K \subset S^3$ is a bigraded $\mathbb{Z}/2$-vector space

$$\widehat{HFK}(K) = \bigoplus_{m,s\in\mathbb{Z}} \widehat{HFK}_m(K,s)$$

which only depends on the isotopy type of K.

One of the major achievements of knot Floer homology is that it categorifies the Alexander polynomial:

Theorem 7 (Oszváth-Szabó, Rasmussen). *For any knot K we have*

$$\Delta_K(t) = \chi_{gr}(\widehat{HFK}(K)) = \sum_{m,s}(-1)^m t^s \dim_{\mathbb{Z}/2} \widehat{HFK}_m(K,s).$$

Example 8. Knot Floer homology is strictly stronger than the Alexander polynomial: there is a pair of celebrated knots, called the *Conway knot* $11n_{34}$ and the *Kinoshita-Terasaka knot* $11n_{42}$, with the same Alexander polynomial (equal to one) but with non-isomorphic knot Floer homology. ◂

Remark 9. Knot Floer homology strengthens some well-known properties of the Alexander polynomial. For instance, Alexander gives a lower bound for the knot genus[3], $g(K) \geq \frac{1}{2}\deg \Delta_K(t)$; whereas knot Floer detects the knot genus [4]: $g(K) = \max\{s \in \mathbb{Z} : \widehat{HFK}(K,s) \neq 0\}$. ◂

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[3] This is the smallest genus among all one-boundary component closed surfaces in S^3 whose boundary is the given knot.

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Free Banach lattices

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Abstract: In this paper we introduce the free Banach lattices generated by certain structures, such as Banach spaces and lattices, and show some of their properties.

Resumen: En este artículo introducimos los retículos de Banach libres generados por determinadas estructuras, tales como los espacios de Banach y los retículos, y mostramos algunas de sus propiedades.

Keywords: free Banach lattices.

MSC2010: 46B40, 46B42.

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1. Introduction

We all know that the starting point of functional analysis was the investigation of the classical function spaces, which provide its most important applications. However, the natural order in these spaces was neglected almost completely. A first attempt to include a compatible order structure in the study of linear and normed spaces was due to F. Riesz, H. Freudenthal and L. V. Kantorovič in the mid-thirties. In the following years, schools of research on vector lattices were subsequently founded and these investigations were continued by various mathematicians in the Soviet Union (B. Z. Vulikh, A. G. Pinsker, A. I. Judin), in Japan (H. Nakano, T. Ogasawara, K. Yosida), and in United States (G. Birkhoff, S. Kakutani, H. F. Bohnenblust, M. H. Stone).

L. V. Kantorovič and his school first recognized the importance of studying vector lattices in connection with Banach's theory of normed spaces; they investigated normed vector lattices as well as order-related linear operators between such vector lattices.

This paper is about the basic theory of free Banach lattices. We define the free Banach lattices generated by a set, by a Banach space, and by a lattice, and show some of their properties. We refer the reader to [1–9] for more background on free Banach lattices.

2. Free Banach lattices

Recall that a *Banach lattice* is a Banach space $(X, \|\cdot\|)$ together with a partial order $\leq$ with the following properties:

(i) For every pair of elements $x, y \in X$ there exist $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$.

(ii) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in X$,

(iii) $0 \leq x$ implies $0 \leq tx$ for every $x \in X$ and $t \in \mathbb{R}^+$,

(iv) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for every $x, y \in X$, where $|x| := x \vee (-x)$.

Properties (i), (ii) and (iii) together can be read as $(X, \leq)$ is a *vector lattice*, while property (iv) means that $\|\cdot\|$ is a *lattice norm*.

The natural morphisms in this category are those maps that preserve the structure of Banach space and vector lattice. A map $T\colon X \to Y$ between two Banach lattices, X and Y, is said to be a *Banach lattice homomorphism* if it is a bounded linear operator and preserves the lattice operations. If T is also bijective and T^{-1} is a Banach lattice homomorphism, we say that T is a *Banach lattice isomorphism*. If moreover, T preserves the norm (that is, $\|T(x)\| = \|x\|$ for every $x \in X$), we say that T is a *Banach lattice isometry*.

The first authors who introduced the concept of free object within the category of Banach lattices were B. de Pagter and A. W. Wickstead in 2015, who defined and studied properties about the free Banach lattice generated by a set [9].

Definition 1. Let A be a non-empty set. A *free Banach lattice over* or *generated by* A is a Banach lattice F together with a bounded map $\phi\colon A \to F$ with the property that for every Banach lattice X and every bounded map $T\colon A \to X$ there is a unique Banach lattice homomorphism $\hat{T}\colon F \to X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "\phi"'] & X \\ F \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

Here, the norm of T is $\|T\| := \sup\{\|T(a)\| : a \in A\}$, while the norm of $\hat{T}$ is the usual for Banach spaces.

This property uniquely determines F up to Banach lattices isometries, and so we can speak of *the* free Banach lattice generated by A, denoted by $FBL(A)$.

Now, the question is whether such an object exists. The answer is affirmative. B. de Pagter and A. W. Wickstead prove it in [9], but A. Avilés, J. Rodríguez and P. Tradacete give an alternative and more tangible way of constructing it in [4]. They describe it as a space of functions:

For $a \in A$, let $\delta_a \colon [-1,1]^A \to \mathbb{R}$ be the evaluation function given by $\delta_a(x^*) = x^*(a)$ for every $x^* \in [-1,1]^A$, and for $f \colon [-1,1]^A \to \mathbb{R}$ define

$$\|f\| = \sup\left\{\sum_{i=1}^n |f(x_i^*)| \,:\, n \in \mathbb{N},\ x_1^*,\dots,x_n^* \in [-1,1]^A,\ \sup_{a\in A}\sum_{i=1}^n |x_i^*(a)| \le 1\right\}.$$

Theorem 2 ([4, Corollary 2.9]). *The free Banach lattice generated by a set A is the closure of the vector lattice generated by $\{\delta_a : a \in A\}$ under the above norm inside the Banach lattice of all functions $f \in \mathbb{R}^{[-1,1]^A}$ with $\|f\| < \infty$, endowed with the norm $\|\cdot\|$, the pointwise order and the pointwise operations.*

The natural identification of A inside $FBL(A)$ is given by the map $\phi \colon A \to FBL(A)$ where $\phi(a) = \delta_a$ for every $a \in A$. Since every function in $FBL(A)$ is a uniform limit of such functions, they are all continuous (in the product topology) and positively homogeneous (that is, $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in [-1,1]^A$ and for every $\lambda \ge 0$ such that $\lambda x^* \in [-1,1]^A$, or equivalently, $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in [-1,1]^A$ and for every $0 \le \lambda \le 1$).

This definition was soon generalized by A. Avilés, J. Rodríguez and P. Tradacete in [4] to the free Banach lattice generated by a Banach space E in the following sense:

Definition 3. Let E be a Banach space. A *free Banach lattice over* or *generated by E* is a Banach lattice F together with a bounded operator $\phi \colon E \to F$ with the property that for every Banach lattice X and every bounded operator $T \colon E \to X$ there is a unique Banach lattice homomorphism $\hat{T} \colon F \to X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi"'] & X \\ F \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

This property uniquely determines F up to Banach lattices isometries, and so we can speak of *the* free Banach lattice generated by E, denoted by $FBL[E]$. This definition generalizes the notion of the free Banach lattice generated by a set A in the sense that the free Banach lattice generated by a set A is the free Banach lattice generated by the Banach space $\ell_1(A)$ (see [4, Corollary 2.9]).

This definition is not very friendly to work with it. However, similar to the previous case, it is possible to give an explicit description of it as a space of functions:

Let us denote by $H[E]$ the vector subspace of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions $f \colon E^* \to \mathbb{R}$ (that is, all functions that satisfy $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in E^*$ and for every $\lambda \ge 0$). For any $f \in H[E]$ let us define

$$\|f\| = \sup\left\{\sum_{i=1}^n |f(x_i^*)| \,:\, n \in \mathbb{N},\ x_1^*,\dots,x_n^* \in E^*,\ \sup_{x\in B_E}\sum_{i=1}^n |x_i^*(x)| \le 1\right\}.$$

Let us take $H_0[E] = \{f \in H[E] : \|f\| < \infty\}$. It is easy to check that $H_0[E]$ is a Banach lattice when equipped with the norm $\|\cdot\|$ and the pointwise order.

Now, given $x \in E$, let $\delta_x \colon E^* \to \mathbb{R}$ be the evaluation function given by $\delta_x(x^*) = x^*(x)$ for every $x^* \in E^*$.

Theorem 4 ([4, Theorem 2.5]). *The free Banach lattice generated by a Banach space E is the closure of the vector lattice generated by $\{\delta_x : x \in E\}$ under the above norm inside $H_0[E]$.*

The natural identification of E inside $FBL[E]$ is given by the map $\phi \colon E \to FBL[E]$ where $\phi(x) = \delta_x$ for every $x \in E$ (it is a linear isometry between E and its image in $FBL[E]$). Moreover, all the functions in $FBL[E]$ are weak*-continuous when restricted to the closed unit ball B_{E^*} (see [4, Lemma 4.10]).

On the other hand, the notion of the free Banach lattice generated by a lattice $\mathbb{L}$ is due to A. Avilés and J. D. Rodríguez Abellán [5].

Definition 5. Given a lattice $\mathbb{L}$, a *free Banach lattice over* or *generated by* $\mathbb{L}$ is a Banach lattice F together with a bounded lattice homomorphism $\phi\colon \mathbb{L} \to F$ with the property that for every Banach lattice X and every bounded lattice homomorphism $T\colon \mathbb{L} \to X$ there is a unique Banach lattice homomorphism $\hat{T}\colon F \to X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} \mathbb{L} \arrow[r, "T"] \arrow[d, "\phi"'] & X \\ F \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$ ◂

Here, the norm of T is $\|T\| \coloneqq \sup\{\|T(x)\| : x \in \mathbb{L}\}$, while the norm of $\hat{T}$ is the usual for Banach spaces.

This definition determines a Banach lattice that we denote by $FBL\langle\mathbb{L}\rangle$ in an essentially unique way. When $\mathbb{L}$ is a distributive lattice (which is a natural assumption in this context, see [5, Section 3]) the function ϕ is injective and, loosely speaking, we can view $FBL\langle\mathbb{L}\rangle$ as a Banach lattice which contains a subset lattice-isomorphic to $\mathbb{L}$ in a way that its elements work as free generators modulo the lattice relations on $\mathbb{L}$.

In order to give an explicit description of it similar to the previous cases, define

$$\mathbb{L}^* = \{x^*\colon \mathbb{L} \to [-1,1] : x^* \text{ is a lattice homomorphism}\}.$$

For every $x \in \mathbb{L}$ consider the evaluation function $\delta_x\colon \mathbb{L}^* \to \mathbb{R}$ given by $\delta_x(x^*) = x^*(x)$, and for $f \in \mathbb{R}^{\mathbb{L}^*}$, define

$$\|f\| = \sup\left\{\sum_{i=1}^{n} |f(x_i^*)| : n \in \mathbb{N},\ x_1^*, \dots, x_n^* \in \mathbb{L}^*,\ \sup_{x\in\mathbb{L}} \sum_{i=1}^{n} |x_i^*(x)| \le 1\right\}.$$

Theorem 6 ([5, Theorem 1.2]). *Consider F to be the closure of the vector lattice generated by $\{\delta_x : x \in \mathbb{L}\}$ under the norm $\|\cdot\|$ inside the Banach lattice of all functions $f \in \mathbb{R}^{\mathbb{L}^*}$ with $\|f\| < \infty$, endowed with the norm $\|\cdot\|$, the pointwise order and the pointwise operations. Then F, together with the assignment $\phi(x) = \delta_x$, is the free Banach lattice generated by $\mathbb{L}$.*

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Crossing limit cycles for piecewise linear differential centers separated by a reducible cubic curve

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Abstract: As for the general planar differential systems, one of the main problems for the piecewise linear differential systems is to determine the existence and the maximum number of crossing limits cycles that these systems can exhibit. But in general to provide a sharp upper bound on the number of crossing limit cycles is a very difficult problem. In this work we study the existence of crossing limit cycles and their distribution for piecewise linear differential systems formed by linear differential centers and separated by a reducible cubic curve, formed either by a circle and a straight line, or by a parabola and a straight line.

Resumen: Así como para los sistemas diferenciales planares, uno de los principales problemas para los sistemas diferenciales planares lineares por partes es determinar la existencia y el número máximo de ciclos límite de cruce que estos sistemas pueden tener. Pero, en general, proporcionar una cota superior ajustada para ese número máximo de ciclos límite es un problema difícil. En este trabajo estudiamos la existencia de ciclos límite y su distribución para sistemas diferenciales lineales por partes formados por centros diferenciales lineales y separados por una curva cúbica reducible, formada o por un círculo y una línea recta o por una parábola y una línea recta.

Keywords: discontinuous piecewise linear differential centers, limit cycles, conics.

MSC2010: 34C05, 34C07, 37G15.

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1. Introduction and main statements

One of the main problems for the piecewise linear differential systems (pwls) is to determine the existence and the maximum number of limits cycles that these systems can exhibit. That is the version of Hilbert's 16th problem for pwls. In the plane the class of pwls separated by a straight line is apparently the simplest class to study, and has been studied in several papers, see [1, 2] and references quoted therein, but it is still an open problem to know if three is the maximum number of crossing limit cycles that this class can have. In particular when the class of pwls separated by a straight line is formed by linear differential centers we know that these systems have no crossing limit cycles (clc), see [4]. However, there are more recent works which study planar discontinuous piecewise linear differential centers (pwlc) where the curve of discontinuity is not a straight line, see [5], there it was proved that there are clc in those systems. Moreover in the paper [3] it was provided the maximum number of clc for pwlc separated by any conic. In this work we study the existence of crossing limit cycles for piecewise linear differential systems formed by linear differential centers and separated by a reducible cubic curve, formed by a parabola and a straight line. Namely

$$\Sigma_k = \{(x, y) \in \mathbb{R}^2 : (y - k)(y - x^2) = 0,\ k \in \mathbb{R}\}.$$

Let $\mathcal{F}_{\Sigma_{k-}}$ be the family of pwlc separated by Σ_k with $k < 0$. In this case, we have the following three regions in the plane:

$$R^1_{\Sigma_{k-}} = \{y > x^2\}, \quad R^2_{\Sigma_{k-}} = \{k < y < x^2\}, \quad R^3_{\Sigma_{k-}} = \{y < x^2, y < k\}.$$

Let $\mathcal{F}_{\Sigma_0}$ be the family of pwlc separated by Σ_k with $k = 0$. Here we have the four regions

$$R^1_{\Sigma_0} = R^1_{\Sigma_{k-}}, \quad R^2_{\Sigma_0} = \{0 < y < x^2,\ x < 0\}, \quad R^3_{\Sigma_0} = R^3_{\Sigma_{k-}}, \quad R^4_{\Sigma_0} = \{0 < y < x^2, x > 0\}.$$

Here we have two types of clc, first clc of type 4 formed by parts of orbits of the four regions, see Figure 1b. Second clc of type 5, see Figure 1c, which intersect $R^1_{\Sigma_0}, R^3_{\Sigma_0}$ and $R^4_{\Sigma_0}$. Let $\mathcal{F}_{\Sigma_{k+}}$ be the family of pwlc separated by Σ_k with $k > 0$, here we have the five regions

$$R^1_{\Sigma_{k+}} = \{k < y < x^2, x > \sqrt{k}\}, \qquad R^2_{\Sigma_{k+}} = \{y > x^2, y > k\},$$
$$R^3_{\Sigma_{k+}} = \{k < y < x^2, x < -\sqrt{k}\}, \qquad R^4_{\Sigma_{k+}} = R^3_{\Sigma_{k-}}, \qquad R^5_{\Sigma_{k+}} = \{x^2 < y < k\}.$$

Here we have six types of clc. First we have clc such that are formed by parts of orbits of $R^1_{\Sigma_{k+}}, R^2_{\Sigma_{k+}}, R^5_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$, or clc formed by parts of $R^2_{\Sigma_{k+}}, R^3_{\Sigma_{k+}}, R^4_{\Sigma_{k+}}$ and $R^5_{\Sigma_{k+}}$, namely clc of type 6^+ and clc of type 6^-, respectively, see Figure 1d. Second we have clc of type 7, see Figure 1e, which intersect $R^2_{\Sigma_{k+}}, R^5_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$. Third we have the clc of type 8, see Figure 1f, which intersect $R^1_{\Sigma_{k+}}, R^2_{\Sigma_{k+}}, R^3_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$. And finally we have the clc such that intersect $R^1_{\Sigma_{k+}}, R^2_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$, or clc formed by parts of orbits of $R^2_{\Sigma_{k+}}, R^3_{\Sigma_{k+}}$ and $R^4_{\Sigma_{k+}}$, namely clc of type 9^+ and clc of type 9^-, respectively, see Figure 1g. In what follows we exhibit the main results and their respective configurations.

Theorem 1. *The following statements hold.*

(i) *There are pwlc in $\mathcal{F}_{\Sigma_{k-}}$ that have 4 clc that intersect Σ_k, see Figure 1a.*

(ii) *There are pwlc in $\mathcal{F}_{\Sigma_0}$ that have 4 clc of type 4, see Figure 1b.*

(iii) *There are pwlc in $\mathcal{F}_{\Sigma_0}$ that have 3 clc of type 5, see Figure 1c.*

(iv) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 5 clc of type 6^+, see Figure 1d.*

(v) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 3 clc of type 7, see Figure 1e.*

(vi) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 4 clc of type 8, see Figure 1f.*

(vii) *There are pwlc in $\mathcal{F}_{\Sigma_{k+}}$ that have 3 clc of type 9^+, see Figure 1g.*

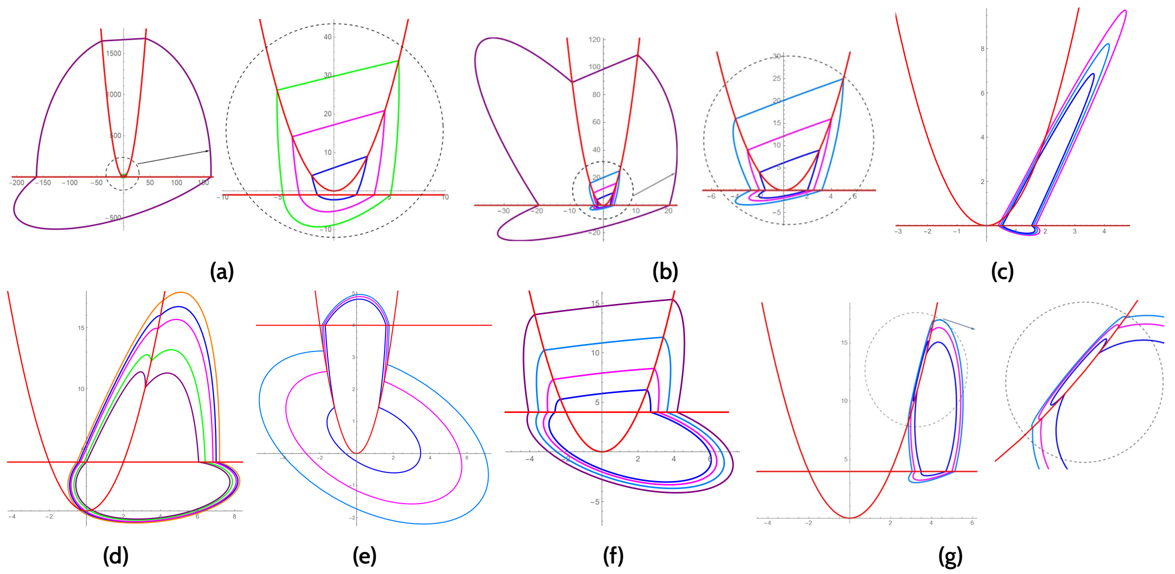

Figure 1: (a) 4 clc with four points on Σ_k. (b) 4 clc of type 4. (c) 3 clc of type 5. (d) 5 clc of type 6^+. (e) 3 clc of type 7. (f) 4 clc of type 8. (g) 3 clc of type 9^+. These limit cycles are traveled in counterclockwise.

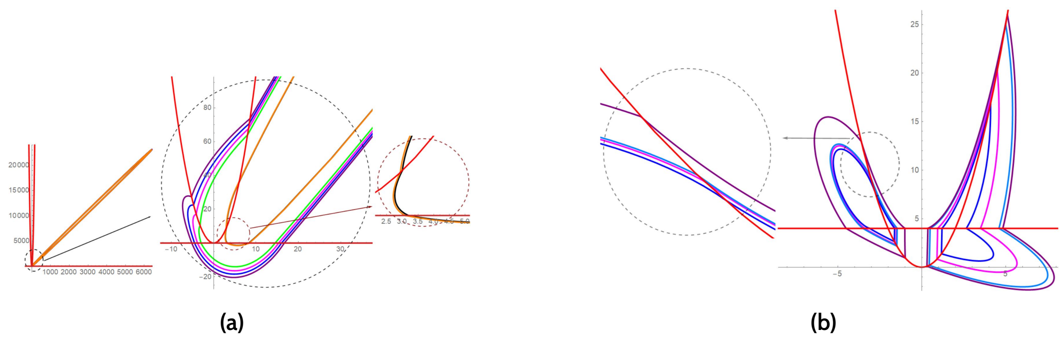

Figure 2: (a) 4 clc of type 4 and 2 clc of type 5. (b) 4 clc of type 6^+ and 4 clc of type 6^-. These limit cycles are traveled in counterclockwise.

In the following theorem, we study the pwlc in the families $\mathcal{F}_{\tilde{\Sigma}_k}$, $k \in \mathbb{R}$, with two and three clc of different types, simultaneously.

Theorem 2. *The following statements hold.*

(i) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_0}$ *that have simultaneously* 4 *clc of type* 4 *and* 2 *clc of type* 5*, see Figure 2a.*

(ii) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 4 *clc of type* 6^+ *and* 4 *clc of type* 6^-*, see Figure 2b.*

(iii) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 4 *clc of type* 6^+ *and* 2 *clc of type* 7*, see Figure 3a.*

(iv) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 3 *clc of type* 6^+ *and* 4 *clc of type* 8*, see Figure 3b.*

(v) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 4 *clc of type* 6^+ *and* 2 *clc of type* 9^+*, see Figure 3c.*

(vi) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 3 *clc of type* 7 *and* 4 *clc of type* 8*, see Figure 3d.*

(vii) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 4 *clc of type* 8 *and* 2 *clc of type* 9^+*, see Figure 3e.*

(viii) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 2 *clc of type* 6^+*,* 2 *clc of type* 7 *and* 4 *clc of type* 8*, see Figure 3f.*

(ix) *There are pwlc in* $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ *that have simultaneously* 4 *clc of type* 6^+*,* 3 *clc of type* 8 *and* 2 *clc of type* 9^+*, see Figure 3g.*

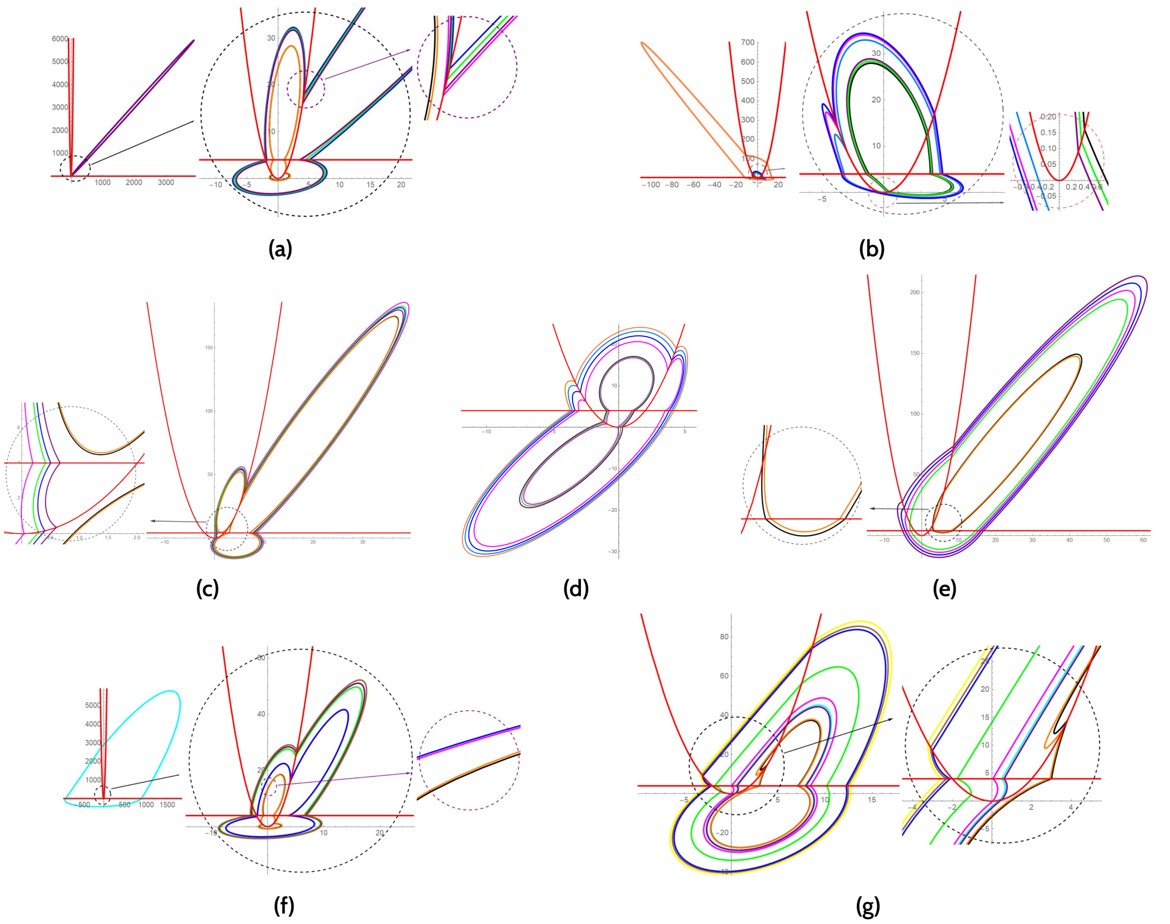

Figure 3: (a) 4 clc of type 6^+ and 2 clc of type 7. (b) 3 clc of type 6^+ and 4 clc of type 8. (c) 4 clc of type 6^+ and 2 clc of type 9^+. (d) 3 clc of type 7 and 4 clc of type 8. (e) 4 clc of type 8 and 2 clc of type 9^+. (f) 2 clc of type 6^+, 2 clc of type 7 and 4 clc of type 8. (g) 4 clc of type 6^+, 3 clc of type 8 and 2 clc of type 9^+. These limit cycles are traveled in counterclockwise.

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Recent results on interpolation by minimal surfaces

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Abstract: Complex analysis and minimal surfaces are strongly connected via the Weierstrass representation formula. This fact has been exploited recently to construct lots of examples of such surfaces with different properties. We would present the first results dealing with interpolation in the setting of minimal surfaces. These results are inspired by classical Weierstrass Interpolation theorem for holomorphic functions and are proved using techniques coming from complex analysis.

More concretely, given an open Riemann surface M, we would construct conformal minimal immersions $M \to \mathbb{R}^n$, $n \geq 3$, such that the values of the immersion at some points of M are prescribed.

Resumen: El análisis complejo y las superficies mínimas están fuertemente relacionados a través de la fórmula conocida como representación de Weierstrass. Esta relación ha permitido recientemente construir muchos ejemplos de tales superficies con diferentes propiedades. A continuación presentamos los primeros resultados sobre interpolación en el ambiente de superficies mínimas. Estos resultados están inspirados en el teorema clásico de interpolación de Weierstrass para funciones holomorfas y se prueban utilizando técnicas provenientes del análisis complejo.

Concretamente, dada una superficie de Riemann abierta M, construiremos inmersiones mínimas conformes $M \to \mathbb{R}^n$, $n \geq 3$, de manera que los valores de la inmersión en algunos puntos de M estén prescritos.

Keywords: minimal surface, Weierstrass theorem, Riemann surface.

MSC2010: 53A10, 32E30, 32H02, 53A05.

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1. Introduction

An immersed surface in the Euclidean space of dimension $n \geq 3$ is called a *minimal surface* if it is locally area-minimizing, that is, small pieces of it are the ones with least area among all the surfaces with the same boundary. Minimal surfaces are usually defined as those surfaces with vanishing mean curvature vector field; which is equivalent to the previous definition. In the classical theory of minimal surfaces in $\mathbb{R}^n$, we may point out the so-called *Enneper-Weierstrass representation formula*. This formula provides any minimal surface in $\mathbb{R}^n$ in terms of holomorphic data defined on an open Riemann surface.

Let M be an open Riemann surface and $X = (X_1, \dots, X_n) \colon M \to \mathbb{R}^n$ a conformal minimal immersion, denoting by ∂ the complex linear part of the exterior differential $d = \partial + \overline{\partial}$ on M (here $\overline{\partial}$ denotes the antilinear part), we have that the 1-form $\partial X = (\partial X_1, \dots, \partial X_n)$, assuming values in $\mathbb{C}^n$, is holomorphic, has no zeros, and satisfies $\sum_{j=1}^n (\partial X_j)^2 = 0$. Furthermore, its real part $\Re(\partial X)$ is an exact 1-form on M.

Conversely, every holomorphic 1-form $\Phi = (\phi_1, \dots, \phi_n)$ with values in $\mathbb{C}^n$, vanishing nowhere on M, satisfying the nullity condition $\sum_{j=1}^n (\phi_j)^2 = 0$ everywhere on M, and whose real part $\Re(\Phi)$ is exact on M, determines a conformal minimal immersion $X \colon M \to \mathbb{R}^n$ by the classical Enneper-Weierstrass (or simply Weierstrass) representation formula:

$$X(p) = x_0 + \int_{p_0}^{p} \Re(\Phi), \quad p \in M,$$

for any fixed base point $p_0 \in M$ and initial condition $X(p_0) = x_0 \in \mathbb{R}^n$. This formula yields minimal surfaces in $\mathbb{R}^n$ from holomorphic 1-forms assuming values in the complex subvariety of $\mathbb{C}^n$ determined by $\mathfrak{A}_* := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 0\} \setminus \{0\}$.

Weierstrass representation formula has provided powerful tools coming from complex analysis in one and several variables to the study of minimal surfaces in $\mathbb{R}^n$. In particular, Runge-Mergelyan theorem for open Riemann surfaces (see [9, 11]) has resulted very useful in the study of minimal surfaces in the Euclidean space. For instance, the pioneer works of Jorge and Xavier [8] or Nadirashvili [10] combined the classical Runge approximation theorem with the Weierstrass formula to refute the belief that hyperbolic Riemann surfaces play a marginal role in the global theory of minimal surfaces. An open Riemann surface is hyperbolic, by definition, if it carries nonconstant negative subharmonic functions.

However, the most recent results that combine complex analysis and Weierstrass representation formula in this setting use methods coming from modern Oka theory. Roughly speaking, Oka manifolds are natural target for holomorphic functions; the key is that the punctured null quadric $\mathfrak{A}_*$ is an Oka manifold and hence Oka theory applies. A detailed explanation may be seen at the survey [3].

2. Interpolation results for conformal minimal immersions

General existence results for minimal surfaces in $\mathbb{R}^n$ have been proved using Oka theory. Further, one may add very interesting global properties to the solutions. In the following sections, we are going to show some of these results concerning interpolation. In particular we show in §2.1 those of interpolation for conformal minimal immersions in $\mathbb{R}^n$, $n \geq 3$. Next, we state in §2.2 the corresponding analogues for minimal surfaces of finite total curvature in $\mathbb{R}^3$. Finally, we show some applications in §2.3 to the construction of examples.

2.1. Results for conformal minimal immersions in any dimension $n \geq 3$

Approximation by holomorphic functions began with the classical Runge Theorem. It gives a topological characterization of those subsets of $\mathbb{C}$ for which any holomorphic function on them may be uniformly approximated by entire functions. Interpolation by holomorphic functions is another main research topic in Complex Analysis. It began with the classical Weierstrass Interpolation Theorem that ensures that one may prescribe the values of an entire function on a discrete subset of $\mathbb{C}$. Both results have been generalized to the framework of maps from Stein manifolds into Oka manifolds, and in particular for functions from open Riemann surfaces. Recall that any open Riemann surface is a Stein manifolds and that the null quadric is an Oka manifold.

Focusing on minimal surfaces, Alarcón, Forstnerič, and López have developed an uniform approximation theory for conformal minimal immersions in $\mathbb{R}^n$, $n \geq 3$ and more general families of holomorphic immersions in $\mathbb{C}^n$; see [4, 5]. Concerning interpolation for conformal minimal immersion the author in collaboration with Alarcón proved the following analogue to the Weierstrass interpolation theorem for conformal minimal immersions in $\mathbb{R}^n$. This result is proved in [1].

Theorem 1. (Weierstrass Interpolation Theorem for conformal minimal surfaces). *Let Λ be a closed discrete subset of an open Riemann surface, M, and let $n \geq 3$ be an integer. Every map $\Lambda \to \mathbb{R}^n$ extends to a conformal minimal immersion $M \to \mathbb{R}^n$.*

The assumptions on Λ in Theorem 1 are necessary since Λ has no accumulation point by the Identity Principle for harmonic maps. We obtain in that paper a much more general result which ensures not only interpolation but also *jet-interpolation of given finite order*, *uniform approximation on Runge compact subsets*, *control on the flux*, and global properties such as completeness and, under natural assumptions, properness and injectivity; see [1, Theorem 1.2] for the detailed statement and the necessary definitions. In addition, an analogue for directed holomorphic curves in $\mathbb{C}^n$ is proved, see [1, Theorem 1.3].

2.2. Conformal minimal immersions of finite total curvature in dimension $n = 3$

One of the main topic of research in the global theory of minimal surfaces in $\mathbb{R}^3$ are complete minimal surfaces with finite total curvature. We recall that a conformal minimal immersion $X: M \to \mathbb{R}^3$ has finite total curvature if

$$TC(X) := \int_M K\, \mathrm{d}s^2 = -\int_M |K|\, \mathrm{d}s^2 > -\infty,$$

here $\mathrm{d}s^2$ is the area element of the surface and K denotes the Gauss curvature of $(M, \mathrm{d}s^2)$. These surfaces have the simplest topological, conformal, and asymptotic geometry. They are intimately related to meromorphic functions and 1-forms on compact Riemann surfaces. Indeed, given an open Riemann surface M and a complete conformal minimal immersion $X: M \to \mathbb{R}^3$ with finite total curvature, there are a compact Riemann surface Σ and a finite subset $\varnothing \neq E \subset \Sigma$ such that M is biholomorphic to $\Sigma \setminus E$.

The author in collaboration with Alarcón and López proved the following interpolation result for complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature. It is proved in [2].

Theorem 2. (Weierstrass Interpolation Theorem for conformal minimal immersions with finite total curvature). *Let Σ be a compact Riemann surface with empty boundary and let $E \neq \varnothing$ and Λ be disjoint finite sets in Σ. Every map $\Lambda \to \mathbb{R}^3$ extends to a complete conformal minimal immersion $\Sigma \setminus E \to \mathbb{R}^3$ with finite total curvature.*

We shall obtain a more general result providing also *uniform approximation*, *jet-interpolation of given finite order*, and *control on the flux*, see [2, Theorem 3.1] for details and definitions.

2.3. Applications and other results

Finally, we finish with some applications to the construction of examples. As we said before, an uniform approximation theory on compact subset have been developed for conformal minimal immersions, analogous to the one of holomorphic functions ([4, 5]). Continuing a natural sequence of approximation results, one may ask whether Carleman approximation theorem holds for minimal surfaces. Carleman theorem for holomorphic functions asserts that one may approximate any continuous function $\mathbb{R} \to \mathbb{C}$ by entire functions better than any given positive function. Next result is an analogue for conformal minimal immersions and it is proved in [7].

Theorem 3 (Carleman Theorem for conformal minimal immersions). *Let M be an open Riemann surface and let $R \subset M$ be a proper embedded curve. Let $f: R \to \mathbb{R}^n$, $n \geq 3$, and $\epsilon: M \to \mathbb{R}_+$ be continuous maps. There exists a complete conformal minimal immersion $X: M \to \mathbb{R}^n$ such that $\|X(p) - f(p)\| < \epsilon(p)$, $p \in M$. Furthermore, if $n \geq 5$, then X may be chosen to be injective.*

Similarly to the previous results, in collaboration with Chenoweth we proved an analogue to holomorphic directed immersions which is stated in [7, Theorem 1.2]. Furthermore, the solutions may be chosen to be complete and proper under natural assumptions, see [7, Theorems 1.3 and 1.4].

On the other hand, the next interpolation result ensures that one may construct minimal surfaces with all coordinates prescribed but two. The theorem is proved in [6].

Theorem 4. *Let M be an open Riemann surface and $n \geq 3$ be an integer. Let $\Lambda \subset M$ be a closed discrete subset and let $h\colon M \to \mathbb{R}^{n-2}$ be a nonconstant harmonic map. For any map $F\colon \Lambda \to \mathbb{R}^2$, there is a complete conformal minimal immersion $X = (X_1, X_2, \dots, X_n)\colon M \to \mathbb{R}^n$ such that $(X_1, X_2)|_\Lambda = F$ and $(X_3, \dots, X_n) = h$.*

As a consequence of the previous result, it is shown on [6] that we may interpolate by minimal surfaces in $\mathbb{R}^n$, $n \geq 3$, whose *generalized Gauss map G_X* is nondegenerate and fails to intersect n hyperplanes in general position. In dimension $n = 3$, we have the following.

Corollary 5. *Let M be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Any map $\Lambda \to \mathbb{R}^3$ extends to a complete nonflat conformal minimal immersion $X\colon M \to \mathbb{R}^3$ whose Gauss map $M \to \mathbb{S}^2$ omits two (antipodal) values of the sphere $\mathbb{S}^2$.*

For the general statement of the previous result and the necessary definitions, see [6, Theorem 1.1].

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Regular origamis with totally non-congruence groups as Veech groups

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Abstract: Veech groups are an important tool to examine translation surfaces and related mathematical objects. Origamis, also known as square-tiled surfaces, form an interesting class of translation surfaces with finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ as Veech groups. We study when Veech groups of origamis with maximal symmetry group are totally non-congruence groups, i.e., when they surject onto $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{Z}_+$. For this, we use a result of Schlage-Puchta and Weitze-Schmithüsen to deduce sufficient conditions on the deck transformation group of the origami. More precisely, we show that origamis with certain quotients of triangle groups as deck transformation groups satisfy this condition. All Hurwitz groups are such quotients.

Resumen: Los grupos de Veech son una herramienta importante para examinar superficies de traslación y objetos matemáticos relacionados. Los origamis, también conocidos como superficies cuadradas, forman una clase interesante de superficies de traslación con subgrupos de índice finito en $\mathrm{SL}(2,\mathbb{Z})$ como grupos de Veech. Estudiamos cuándo los grupos de Veech de los origamis con grupo de simetría máximo son grupos totalmente no congruentes, es decir, cuándo surgen en $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ para cada $n \in \mathbb{Z}_+$. Para ello, utilizamos un resultado de Schlage-Puchta y Weitze-Schmithüsen para deducir condiciones suficientes sobre el grupo de transformación de *deck* del origami. Más concretamente, mostramos que los origamis con ciertos cocientes de grupos triangulares como grupos de transformación de *deck* satisfacen esta condición. Todos los grupos de Hurwitz son tales cocientes.

Keywords: translation surfaces, square-tiled surfaces, origamis, Veech groups, totally non-congruence groups, triangle groups, simple groups.

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1. Introduction

A *translation surface* is a closed Riemann surface with an additional structure which can be described by certain gluing data. We construct such a surface as finitely many polygons in the Euclidean plane with edge identifications along pairs of parallel edges. If all polygons are unit squares one obtains an *origami* (also known as *square-tiled surface*). Each origami naturally defines a torus cover sending each square in the tiling to the torus. We are interested in the case where this cover is normal. Then the symmetry group of the origami is maximal and we call the origami *regular*. A regular origami is completely determined by its deck transformation group G and two deck transformations x and y mapping a fixed square to its right and upper neighbor, respectively (see, e.g., [2]). We denote such an origami by the tuple (G, x, y).

The matrix group $\mathrm{SL}(2,\mathbb{R})$ acts on translation surfaces by sheering the polygons in the Euclidean plane. Sometimes the orbit defines an algebraic curve in the moduli space of complex algebraic curves called *Teichmüller curve*. The stabilizer under this action captures whether this happens and - assuming a positive answer - much of the geometry of the Teichmüller curve. For a translation surface X, the stabilizer is called the *Veech group* of X and is denoted by $\mathrm{SL}(X)$. Origamis define always Teichmüller curves. The Veech groups of *reduced origamis*, i.e., the geodesic segments between singularities span $\mathbb{Z}^2$, are finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$. Here a singularity means a vertex of a square in the tiling with cone angle larger than 2π. Since non-trivial regular orgiamis are reduced, we restrict to study the $\mathrm{SL}(2,\mathbb{Z})$-action. On a regular origami $\mathcal{O} = (G, x, y)$ this action is defined by $S \cdot \mathcal{O} = (G, y^{-1}, x)$ and $T \cdot \mathcal{O} = (G, x, yx^{-1})$, where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For more details, see, e.g., [6] and [7].

We are interested in the following open question: for which origamis are the Veech groups congruence subgroups and for which are they far away from being a congruence subgroup? Weitze-Schmithüsen showed in [6] that almost all congruence groups occur as Veech groups. However, Hubert and Leliévre proved that in the stratum $\mathcal{H}(2)$ all but one of the occurring Veech groups are not congruence groups (see [3]). In [7], Weitze-Schmithüsen introduced the deficiency of finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$. It measures how far the group is from being a congruence subgroup. She also established the notion of totally non-congruence groups. Such a group projects surjectively onto $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{Z}_+$, i.e., no information about the group itself can be recovered from the images under these natural projections. In [5], an infinite family of origamis with totally non-congruence subgroups as Veech groups are constructed for each stratum. These origamis had only few symmetries. In this article, we present sufficient conditions for regular origamis to have totally non-congruence subgroups as Veech groups and introduce a class of regular origamis satisfying this condition.

2. Prerequisites and preliminary results

In this section, we introduce basic concepts and preliminary results, which are used in Section 3. Note that the Euclidean metric on $\mathbb{R}^2$ lifts to a metric on a translation surface. Therefore, notions as directions and geodesics are well-defined on translation surfaces. A *cylinder* on a translation surface is a maximal collection of parallel closed geodesics. Given a cylinder on a translation surface there exist $w, h > 0$ such that the cylinder is isometric to a Euclidean cylinder $\mathbb{R}/w\mathbb{Z} \times (0, h)$. One calls w the *circumference*, h the *height*, and the quotient $\frac{h}{w}$ the *modulus* of the cylinder. If the genus of the translation surface is larger than one, a cylinder is bounded by geodesics between singularities. We call such a geodesic a *saddle connection*. The direction of a saddle connection bounding a cylinder is called the *direction of the cylinder*. A *cylinder decomposition* is a collection of pairwise disjoint cylinders such that the union of their closures covers the whole surface.

The cylinder decompositions of an origami lead to a parabolic element in its Veech group given the following situation. Let $\mathcal{O}$ be an origami, $v \in \mathbb{Z}^2$ be a rational direction, and $A \in \mathrm{SL}(2,\mathbb{Z})$ be a matrix mapping $e_1 = \binom{1}{0}$ to v. If $\mathcal{O}$ decomposes into cylinders $C_1, \ldots, C_k$ with inverse moduli $m_1, \ldots, m_k$ and m is the smallest common integer multiple of all the m_i, then the matrix $A \cdot \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \cdot A^{-1}$ is contained in the Veech group $\mathrm{SL}(\mathcal{O})$ (see, e.g., [7, Section 2.1]).

We conclude this section giving a sufficient condition when finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ are totally non-congruence groups. It is used in the proofs of Proposition 3 and Theorem 6.

Theorem 1 ([5, Theorem 1]). *Let Γ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Suppose that for each prime p there exist matrices $A_1, A_2 \in \mathrm{SL}(2,\mathbb{Z})$ with the following properties:*

(i) *For all $j \in \mathbb{Z}_+$, $A_1 e_1 \neq j \cdot A_2 e_1$ modulo p.*

(ii) *There exist $m_1, m_2 \in \mathbb{Z}_+$ with $A_1 T^{m_1} A_1^{-1}$, $A_2 T^{m_2} A_2^{-1} \in \Gamma$ such that p divides neither m_1 nor m_2.*

Then, Γ is a totally non-congruence group.

3. Application to regular origamis

In this section, we use cylinder decompositions in different directions to construct the matrices occurring in Theorem 1. The following lemma computes the inverse moduli of the cylinders in the directions of interest.

Lemma 2. *Let $\mathcal{O} = (G, x, y)$ be a regular origami. For $m \in \mathbb{Z}_{\geq 0}$, the inverse modulus of all cylinders in direction $\left(\begin{smallmatrix} 1 \\ -m \end{smallmatrix}\right)$ coincides with the order of xy^m.*

Proof. Denote $\left(\begin{smallmatrix} 1 \\ -m \end{smallmatrix}\right)$ by v. Acting with the matrix $A = \left(\begin{smallmatrix} 1 & 0 \\ -m & 1 \end{smallmatrix}\right) = \left(S^3TS\right)^m \in \mathrm{SL}(2,\mathbb{Z})$ maps the horizontal direction to the direction v, i.e., $A \cdot e_1 = v$. The inverse modulus of all horizontal cylinders of the origami $A \cdot \mathcal{O} = (G, xy^m, y)$ coincides with the order of xy^m. Note that acting by matrices in $\mathrm{SL}(2,\mathbb{Z})$ does not change the modulus of a cylinder. Hence, the inverse modulus of the cylinder in direction v of the origami $\mathcal{O}$ coincides with the order of xy^m. ∎

Using Theorem 1 and Lemma 2, we deduce a sufficient condition for regular origamis to have a totally non-congruence group as Veech group.

Proposition 3. *Let $\mathcal{O} = (G, x, y)$ be a regular origami. If for each prime p one of the following holds*

(i) *there exist $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ with $\overline{m}_1 \not\equiv \overline{m}_2 \mod p$ and $\gcd(p, \mathrm{ord}(xy^{-m_1}) \cdot \mathrm{ord}(xy^{-m_2})) = 1$ or*

(ii) *$\gcd(p, \mathrm{ord}(y) \cdot \mathrm{ord}(yx)) = 1$,*

then the Veech group $\mathrm{SL}(\mathcal{O})$ is a totally non-congruence group.

Proof. Fix a prime p. If condition (i) holds, then let m_1, m_2 be natural numbers satisfying condition (i). Define the matrices $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ m_1 & 1 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 1 & 0 \\ m_2 & 1 \end{smallmatrix}\right)$. Since $m_1 \not\equiv m_2 \mod p$, we have $A_1 e_1 \neq j \cdot A_2 e_1$ modulo p for each $j \in \mathbb{Z}_+$.

As $\gcd(p, \mathrm{ord}(xy^{-m_1}) \cdot \mathrm{ord}(xy^{-m_2})) = 1$, set $k_1 = \mathrm{ord}(xy^{-m_1})$ and $k_2 = \mathrm{ord}(xy^{-m_2})$. Using Lemma 2, we conclude that the matrices $A_i T^{k_i} A_i^{-1}$ are contained in the Veech group of the origami $\mathcal{O}$.

If condition (ii) holds, consider the matrices $S^{-1}T^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and $TS^{-1} = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We obtain $S^{-1}T^{-1} \cdot \mathcal{O} = (G, yx, x^{-1})$ and $TS^{-1} \cdot \mathcal{O} = (G, y, x^{-1}y^{-1})$. The moduli of the horizontal cylinders of the regular origamis (G, yx, x^{-1}) and $(G, y, (yx)^{-1})$ are $\mathrm{ord}(yx) =: a$ and $\mathrm{ord}(y) =: b$, respectively. Hence, $S^{-1}T^{-1} \cdot T^a \cdot TS$ and $TS^{-1} \cdot T^b \cdot ST^{-1}$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Moreover, we obtain for each $j \in \mathbb{Z}$ the inequality $S^{-1}T^{-1} \cdot e_1 = \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \neq j \cdot \left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right) = j \cdot TS^{-1} \cdot e_1$ modulo p. By Theorem 1, the claim follows. ∎

In the following corollary, we construct generating sets $\{x, y\}$ of alternating groups A_n satisfying the conditions given in Proposition 3. Consequently, the infinite family of regular origamis (A_n, x, y) have totally non-congruence groups as Veech groups.

Corollary 4. *For each prime $n \geq 5$, the regular origami $(A_n, (1,2,3), (1,2,3,\dots,n))$ has a totally non-congruence group as Veech group.*

Proof. Set $x := (1,2,3)$ and $y := (1,2,3,\dots,n)$. For each prime $p \neq n$, we consider the group elements yx and y. Since the orders of y and yx are equal to n, the prime p does not divide $\mathrm{ord}(y) \cdot \mathrm{ord}(yx)$.

For the prime n, we consider the group elements xy^{n-1} and x, i.e., $m_1 = 1 - n$ and $m_2 = 0$. Note that $1 - n \not\equiv 0 \mod n$. The permutation xy^{n-1} has the fixed point 2 and thus n does not divide the order of xy^{n-1}. Since $\mathrm{ord}(x) = 3 < n$, the prime n does not divide the order of x either. By Proposition 3, the claim follows. ∎

Corollary 4 motivates to examine finite simple groups more generally. Simple groups form an interesting class of 2-generated groups. The natural question, how the orders of generators for a fixed group can be chosen, has been studied intensively (see, e.g., [4] for further information). This question suggests to consider (a,b,c)-groups.

Definition 5. A finite group generated by two elements x, y with $\operatorname{ord}(x) = a$, $\operatorname{ord}(y) = b$, and $\operatorname{ord}(xy) = c$ is called an (a,b,c)-*group*. We call such generators (a,b,c)-*generators*. ◂

Each (a,b,c)-group is a finite quotient of the *triangle group* $T_{(a,b,c)} = \langle x, y, z \mid x^a = y^b = z^c = xyz = 1\rangle$. The following theorem shows that (a,b,c)-groups where a, b, c are chosen pairwise coprime produce regular origamis with a totally non-congruence group as Veech group.

Theorem 6. *Let $a, b, c \in \mathbb{Z}_{\geq 0}$ be pairwise coprime and G be an (a,b,c)-group with (a,b,c)-generators x, y. The Veech group of the regular origami (G, y, x) is a totally non-congruence group.*

Proof. We prove that the assumptions of Theorem 1 are satisfied for the Veech group of the regular origami $\mathcal{O} = (G, y, x)$. Let p be a prime. Since a, b, and c are pairwise coprime, p divides at most one of the numbers a, b, and c. We consider each of the three cases separartely.

If p is coprime to $b \cdot c$, then consider the matrices I and $S^{-1}T^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. We obtain $I \cdot \mathcal{O} = \mathcal{O}$ and $S^{-1}T^{-1} \cdot \mathcal{O} = (G, xy, y^{-1})$. The inverse moduli of the horizontal cylinders of the regular origamis $\mathcal{O}$ and (G, xy, y^{-1}) are $\operatorname{ord}(y) = b$ and $\operatorname{ord}(xy) = c$, respectively. Hence T^b and $S^{-1}T^{-1} \cdot T^c \cdot TS$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Moreover, we obtain $S^{-1}T^{-1} \cdot e_1 = \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \neq j \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ modulo p for each $j \in \mathbb{Z}$.

If p is coprime to $a \cdot c$, then consider the matrices $S^{-1}T^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and $TS^{-1} = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We obtain the regular origamis $S^{-1}T^{-1} \cdot \mathcal{O} = (G, xy, y^{-1})$ and $TS^{-1} \cdot \mathcal{O} = (G, x, y^{-1}x^{-1})$. The moduli of the horizontal cylinders of the regular origamis (G, xy, y^{-1}) and $(G, x, (xy)^{-1})$ are $\operatorname{ord}(xy) = c$ and $\operatorname{ord}(x) = a$, respectively. Hence $S^{-1}T^{-1} \cdot T^c \cdot TS$ and $TS^{-1} \cdot T^a \cdot ST^{-1}$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Moreover, we obtain for each $j \in \mathbb{Z}$ the inequality $S^{-1}T^{-1} \cdot e_1 = \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \neq j \cdot \left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right) = j \cdot TS^{-1} \cdot e_1$ modulo p.

If p is coprime to $a \cdot b$, then consider the matrices I and $S^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We obtain the regular origamis $I \cdot \mathcal{O} = \mathcal{O}$ and $S^{-1} \cdot \mathcal{O} = (G, x, y^{-1})$. The moduli of the horizontal cylinders of the regular origamis $\mathcal{O}$ and (G, x, y^{-1}) are $\operatorname{ord}(y) = b$ and $\operatorname{ord}(x) = a$, respectively. Hence T^b and $S^{-1} \cdot T^a \cdot S$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Moreover, we have $S^{-1} \cdot e_1 = \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \neq j \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ modulo p for each $j \in \mathbb{Z}$. ■

Example 7. A well-studied family of groups satisfying the assumptions in Theorem 6 are $(2,3,7)$-groups, which are also called Hurwitz groups (see, e.g., [1]). ◂

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Subharmonics in a class of planar periodic predator-prey models

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Abstract: This contribution studies the existence of positive subharmonics of arbitrary order in the planar periodic Volterra predator-prey model. When the model is *non-degenerate*, in the sense that the birth rate of the prey intersects the support of the death rate of the predator, as in [8], then the existence of positive subharmonics can be derived from the Poincaré–Birkhoff theorem version [3]. Nevertheless, in the *degenerate* case when these supports do not intersect, then, the Poincaré–Birkhoff theorem fails in general. Still in these degenerate situations, the techniques of [7] provide us with the existence of positive subharmonics of arbitrary order.

This is based on a joint work with Julián López-Gómez (UCM) and Fabio Zanolin (UNIUD).

Resumen: Este trabajo analiza la existencia de subarmónicos positivos de orden arbitrario en el modelo plano periódico de presa y depredador de Volterra. Cuando el modelo es *no degenerado*, en el sentido de que la tasa de natalidad de la presa interseca el soporte de la tasa de mortalidad del depredador, como en [8], entonces la existencia de subarmónicos positivos puede ser derivada mediante un la versión del teorema de Poincaré–Birkhoff que se establece en [3]. Sin embargo, en el caso *degenerado* cuando los soportes no intersecan, el teorema de Poincaré–Birkhoff no puede aplicarse directamente. En estos casos, las técnicas de [7] nos proporcionan la existencia de subarmónicos positivos de orden arbitrario.

Esta colaboración está basada en un trabajo conjunto con Julián López-Gómez (UCM) y Fabio Zanolin (UNIUD).

Keywords: periodic predator-prey model of Volterra type, subharmonic coexistence states, Poincaré–Birkhoff twist theorem, degenerate versus non-degenerate models.

MSC2010: 34C25, 37B55, 37E40.

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1. Introduction

In this contribution, we analyze the existence of positive subharmonics of arbitrary order (nT-periodic coexistence states) of the periodic Volterra predator-prey model

(1) $$\begin{cases} u' = \lambda\alpha(t)u(1-v), \\ v' = \lambda\beta(t)v(-1+u), \end{cases}$$

where $\lambda > 0$ is a real parameter, and, for some $T > 0$, $\alpha(t)$ and $\beta(t)$ are T-periodic real continuous functions. We set

$$A := \int_0^T \alpha(s)\,\mathrm{d}s \quad \text{and} \quad B := \int_0^T \beta(s)\,\mathrm{d}s.$$

They can arise two different cases according to whether, or not, the following condition holds

(2) $$\operatorname{supp}\alpha \cap \operatorname{supp}\beta \neq \emptyset.$$

In this *non-degenerate* situation the existence of subharmonics of arbitrary order can be obtained through an updated version of the celebrated Poincaré–Birkhoff twist theorem for sufficiently large λ. Nevertheless, in the *degenerate* case when the next condition holds

(3) $$\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$$

the Poincaré–Birkhoff theorem is unable to provide, in general, with subharmonics of arbitrary order, unless $\alpha(t)$ and $\beta(t)$ have some special nodal structure.

2. The non-degenerate case

The non-degenerate case when (2) is satisfied has been studied in [8] by adapting some original ideas in [3] (later revised and applied in [2]), where a Poincaré–Birkhoff version for Hamiltonian systems was presented. Through the change of variables

$$x = \log u, \qquad y = \log v,$$

(1) is transformed into the planar Hamiltonian system

(4) $$\begin{cases} x' = -\lambda\alpha(t)(\mathrm{e}^y - 1), \\ y' = \lambda\beta(t)(\mathrm{e}^x - 1). \end{cases}$$

The [3] version of the Poincaré–Birkhoff twist theorem that we will use reads as follows:

Theorem 1 (Poincaré–Birkhoff). *Assume that there exist $0 < \varrho_0 < \varrho_1$ and a positive integer ω such that*

$$\operatorname{rot}_{\varrho_0}[(x_0, y_0); [0, nT]] > \omega \quad \textit{and} \quad \operatorname{rot}_{\varrho_1}[(x_0, y_0); [0, nT]] < \omega,$$

where

$$\operatorname{rot}_\rho[(x_0, y_0); [0, nT]] = \frac{\theta(nT) - \theta(0)}{2\pi}$$

with $\|(x_0, y_0)\| = \rho$, $\theta(t)$ being the angular polar coordinate of the solution starting at (x_0, y_0), $(x(t), y(t))$. Then, (4) admits, at least, two nT-periodic solutions lying in different periodicity classes with rotation number ω.

As a consequence of Theorem 1, we get the next result:

Theorem 2. *Assume (2). Then, for every positive integers ω and n, there exists $\lambda_n^\omega > 0$ such that (4) possesses, at least, two nT-periodic solutions with rotation number ω for every $\lambda > \lambda_n^\omega$.*

Proof. First, we focus attention into the small solutions of (4). There exists $\varepsilon > 0$ such that

$$(\mathrm{e}^{\xi} - 1)\xi \geq \frac{\xi^2}{2} \quad \text{if} \ \ |\xi| < \varepsilon. \tag{5}$$

It can be chosen an initial data $(x(0), y(0)) = (x_0, y_0)$ sufficiently close to $(0, 0)$, say $|(x_0, y_0)| \leq \varrho_0$, so that the solution of (4), $(x(t), y(t))$, satisfy $|(x(t), y(t))| < \varepsilon$ for all $t \in [0, nT]$. This is possible by continuous dependence on the initial conditions. According to (2), there are $\tau \in (0, T)$ and $\delta > 0$ such that $\alpha(t)\beta(t) > 0$ for every $t \in [\tau - \delta, \tau + \delta] \subsetneq [0, T]$. Thus,

$$\zeta := \min_{t\in[\tau-\delta,\tau+\delta]} \{\alpha(t), \beta(t)\} > 0. \tag{6}$$

Consequently, due to (4), (5) and (6), we obtain that, for every $t \in [0, nT]$,

$$\theta'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x^2(t) + y^2(t)} = \frac{\lambda\beta(t)(\mathrm{e}^{x(t)} - 1)x(t) + \lambda\alpha(t)(\mathrm{e}^{y(t)} - 1)y(t)}{x^2(t) + y^2(t)} \geq \frac{\lambda\zeta}{2}. \tag{7}$$

Hence, owing to (7),

$$\operatorname{rot}_{\varrho_0}[(x_0, y_0); [0, nT]] = \frac{\theta(nT) - \theta(0)}{2\pi} = \frac{1}{2\pi}\int_0^{nT} \theta'(s)\,\mathrm{d}s \geq \frac{n}{2\pi}\int_{\tau-\delta}^{\tau+\delta} \theta'(s)\,\mathrm{d}s \geq \frac{n\lambda\zeta 2\delta}{2\pi} > \omega$$

if $\lambda > \frac{\pi\omega}{n\zeta\delta} =: \lambda_n^{\omega}$.

On the other hand, solutions with sufficiently large initial data do not rotate (see, for further details, Theorem 2.2 of [8]). Hence, the hypothesis of Theorem 1 holds for every $\lambda > \lambda_n^{\omega}$, which ends the proof. ∎

3. The degenerate case

To analyze the problem (1) under the condition (3), we suppose that

$$\operatorname{supp}\alpha \subseteq [0, \tfrac{T}{2}] \quad \text{and} \quad \operatorname{supp}\beta \subseteq [\tfrac{T}{2}, T]. \tag{8}$$

In case (8), introduced in [5], we have that, for every $t \in [0, T]$,

$$u(t) = u_0\mathrm{e}^{(1-v_0)\lambda\int_0^t \alpha(s)\,\mathrm{d}s}, \quad v(t) = v_0\mathrm{e}^{(u(T)-1)\lambda\int_0^t \beta(s)\,\mathrm{d}s},$$

Hence, the T-time Poincaré map is

$$(u_1, v_1) := \mathcal{P}_1(u_0, v_0) := (u(T), v(T)) = (u_0\mathrm{e}^{(1-v_0)\lambda A}, v_0\mathrm{e}^{(u_1-1)\lambda B}).$$

Consequently, iterating n times this map, it becomes apparent that

$$\begin{aligned}(u_n, v_n) &:= \mathcal{P}_n(u_0, v_0) = \mathcal{P}_1^{\,n}(u_0, v_0) := (u(nT), v(nT)) = (u_{n-1}\mathrm{e}^{(1-v_{n-1})\lambda A}, v_{n-1}\mathrm{e}^{(u_n-1)\lambda B}) \\ &= (u_0\mathrm{e}^{(n-v_0-v_1-\cdots-v_{n-1})\lambda A}, v_0\mathrm{e}^{(u_1+u_2+\cdots+u_n-n)\lambda B}).\end{aligned} \tag{9}$$

By the uniqueness for the underlying Cauchy problem, the nT-periodic coexistence states of (1) are given by the positive fixed points of $\mathcal{P}_n$. Thus, by (9), we are driven to solve the system

$$\begin{cases} n &= u_0 + u_1 + \cdots + u_{n-1}, \\ n &= v_0 + v_1 + \cdots + v_{n-1}. \end{cases} \tag{10}$$

The next result proves the existence and multiplicity of nT-periodic coexistence states of (1) when $n \geq 2$ in case (8). To get it, we impose the following condition:

$$A = B \quad \text{and} \quad u_0 = v_0 = x. \tag{11}$$

Theorem 3. *Assume (8) and (11). Then, for every $\lambda > \frac{2}{A}$, (1) admits, at least, n coexistence states with period nT if n is even, and $n-1$ coexistence states with period nT if n is odd.*

Proof. By (11), it turns out that, given $\varphi_1(x) = x - 1$,

$$\varphi_n(x) = \varphi_{n-1}(x) - 1 + E_{n-1}(x),$$

is the map whose zeros provide us with the nT-periodic coexistence states of (1), where $E_n(x)$ is a sequence of exponential functions. In order to obtain some information concerning the nT-periodic coexistence states of (1), we analyze the variational equations of these maps at the trivial curve $(\lambda, 1)$,

$$p_n(\lambda) := \frac{\partial \varphi_n}{\partial x}(\lambda, 1).$$

It is easy to prove that $p_n(\lambda)$ is a sequence of polynomials in the indeterminate λ that satisfy the recursive formula

$$p_n(\lambda) = [2 - (-1)^n A\lambda] p_{n-1}(\lambda) - p_{n-2}(\lambda),$$

where $p_1(\lambda) = 1$ and $p_2(\lambda) = 2 - A\lambda$. From this recursive formula, it can be shown that any root of p_n is real and algebraically simple. Thanks to these facts, for any given $r \in p_n^{-1}(0)$, the transversality condition of Crandall-Rabinowitz [1] holds. Thus, for any given $r \in p_n^{-1}(0)$, the algebraic multiplicity of Esquinas and López-Gómez [4] equals one at every point $(r, 1)$. So, according to Crandall and Rabinowitz [1, Th. 1.7], a local bifurcation occurs at every point $(r, 1)$. Moreover, by the unilateral theorem of López-Gómez [6, Th. 6.4.3], the underlying subcomponents of nT-periodic coexistence states are unbounded in λ. As the number of positive roots of $p_n(\lambda)$ equals $\frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd, the result holds. This ends the proof. ∎

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Integrability and rational soliton solutions for gauge invariant derivative nonlinear Schrödinger equations

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Abstract: The present work addresses the study and characterization of the integrability of three famous nonlinear Schrödinger equations with derivative-type nonlinearities in $1 + 1$ dimensions. Lax pairs for these three equations are successfully obtained by means of a Miura transformation and the singular manifold method. After implementing the associated binary Darboux transformations, we are able to construct rational soliton-like solutions for those systems.

Resumen: El presente trabajo aborda el estudio y caracterización de la integrabilidad de tres célebres ecuaciones diferenciales tipo Schödinger no lineal con nolinearidades que incluyen términos en derivadas. Mediante el método de la variedad singular, junto con una trasformación de Miura, se obtienen las expresiones de los pares de Lax para dichas ecuaciones. A través del formalismo de las trasformaciones binarias de Darboux, se consigue construir soluciones solitónicas racionales para estos tres sistemas.

Keywords: integrability, derivative nonlinear Schrödinger equation, Lax pair, rational solitons.

MSC2010: 35C08, 35Q55, 37J35.

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1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most famous integrable equations in soliton theory and mathematical physics. Among the several integrable generalizations of NLS, we are interested in the study of modified NLS systems with derivative-type nonlinearities in $1+1$ dimensions, which are known as derivative nonlinear Schrödinger (DNLS) equations. There exist three celebrated equations of this kind, *i.e.*, the Kaup-Newell (KN) system [4],

(1) $$\mathrm{i}m_t - m_{xx} - \mathrm{i}\left(|m|^2 m\right)_x = 0,$$

the Chen-Lee-Liu (CLL) equation [2],

(2) $$\mathrm{i}m_t - m_{xx} - \mathrm{i}|m|^2 m_x = 0,$$

and the Gerdjikov-Ivanov (GI) equation [3]

(3) $$\mathrm{i}m_t - m_{xx} + \mathrm{i}m^2\overline{m}_x - \frac{1}{2}|m|^4 m = 0,$$

where m is a complex valued function and $\overline{m}$ denotes the complex conjugate of m.

It is already known that these three equations are equivalent via a $U(1)$-gauge transformation [5]. If $m(x,t)$ is a solution of the KN system (1), it is easy to find that the new field $M(x,t)$

(4) $$M(x,t) = m(x,t)\mathrm{e}^{\frac{\mathrm{i}\gamma}{2}\theta(x,t)}, \qquad \text{with} \quad \theta_x = |m|^2, \quad \theta_t = \mathrm{i}(m\overline{m}_x - \overline{m}m_x) + \frac{3}{2}|m|^4,$$

satisfies the CLL equation for $\gamma = 1$, and the GI equation for $\gamma = 2$.

Gauge transformations constitute an useful tool to link integrable evolution equations in soliton theory, since they provide Bäcklund transformations between those equations as well as the relation of their associated linear problems [6]. In this contribution we exploit this gauge invariance property to construct a Lax pair and rational soliton solutions for these three equations. For a detailed analysis and explicit calculations, we refer the reader to [1].

2. Integrability and Lax pair

The Painlevé test [7] has been proved to be a powerful criterion for the identification of integrable partial differential equations (PDEs). A PDE is said to posses the Painlevé property, frequently considered as a proof of integrability, when its solutions are singled-valued about the movable singularity manifolds. This requires the generalized Laurent expansion for the field $m(x,t) = \sum_{j=0}^{\infty} a_j(x,t)\phi(x,t)^{j-\mu}$, where $\phi(x,t)$ is an arbitrary function called the singular manifold and the index $\mu \in \mathbb{N}$ is an integer.

The Painlevé test is unable to check the integrability of any DNLS equation since the leading index is not integer, $\mu = \frac{1}{2}$. This fact allow us to introduce two new real fields $\alpha(x,t)$, $\beta(x,t)$

(5) $$m(x,t) = \sqrt{2\alpha_x}\mathrm{e}^{\frac{\mathrm{i}}{2}\beta(x,t)}, \quad \text{with} \quad \alpha_x = \frac{1}{2}|m|^2, \quad \beta = (2\gamma - 3)\alpha + \int \frac{\alpha_t}{\alpha_x}\,\mathrm{d}x,$$

with $\gamma = 0$ for the KN system, $\gamma = 1$ for the CLL equation and $\gamma = 2$ for the GI equation. This ansatz yields an identical differential equation for α in each case, expressed in the conservative form

(6) $$\left[\alpha_x^2 - \alpha_t\right]_t = \left[\alpha_{xxx} + \alpha_x^3 - \frac{\alpha_t^2 + \alpha_{xx}^2}{\alpha_x}\right]_x.$$

From expression (4), it can be easily seen that the probability density $\theta_x = |m|^2 = |M|^2$ is invariant under a $U(1)$-gauge transformation, indeed it constitutes the first conservation law for these systems. Due to this symmetry, it is straightforward to see that once we obtain a soliton solution for a particular DNLS equation, it is immediate to derive soliton solutions for any DNLS equation linked by a $U(1)$-gauge transformation.

Since $\alpha_x = \frac{\theta_x}{2}$, we may conclude that equation (6) is the representative equation for the probability density of any DNLS equation. Equation (6) passes the Painlevé test, but it possesses two branches of expansion. The best method to overcome this inconvenience requires the splitting of the field α as

$$\alpha = \mathrm{i}(u - \overline{u}), \qquad \alpha_x^2 - \alpha_t = u_{xx} + \overline{u}_{xx}. \tag{7}$$

The combination of equations in (7) yields two Miura transformations for $\{u, \overline{u}\}$ and the condition

$$u_{xx} = \frac{1}{2}\left(\alpha_x^2 - \alpha_t - \mathrm{i}\alpha_{xx}\right), \qquad \overline{u}_{xx} = \frac{1}{2}\left(\alpha_x^2 - \alpha_t + \mathrm{i}\alpha_{xx}\right), \qquad \mathrm{i}u_t + u_{xx} - \mathrm{i}\overline{u}_t + \overline{u}_{xx} + (u_x - \overline{u}_x)^2 = 0, \tag{8}$$

which finally lead to the same nonlocal Boussinesq equation for both $u(x,t)$ and $\overline{u}(x,t)$, of the form

$$\left[u_{tt} + u_{xxxx} + 2u_{xx}^2 - \frac{u_{xt}^2 + u_{xxx}^2}{u_{xx}}\right]_x = 0. \tag{9}$$

Equation (9) has the Painlevé property with an unique branch of expansion. Hence, this equation is conjectured integrable and it is possible to derive an equivalent linear spectral problem associated to the nonlinear equation (9). This aim may be achieved by means of the singular manifold method (SMM). The SMM [7] focuses on solutions which emerge from the truncated Painlevé series, as auto-Bäcklund transformations of the form $u^{[1]} = u^{[0]} + \log(\phi)$. Thus, the singular manifold ϕ is no longer an arbitrary function, since it satisfies the singular manifold equations. The associated linear problem arises from the linearization of these equations, and it can be demonstrated that the Lax pair for u reads [1]

$$\begin{aligned} \psi_{xx} &= \left(\frac{u_{xxx}^{[0]} - \mathrm{i}u_{xt}^{[0]}}{2u_{xx}^{[0]}} - \mathrm{i}\lambda\right)\psi_x - u_{xx}^{[0]}\psi, \qquad \psi_t = \mathrm{i}\psi_{xx} - 2\lambda\psi_x + \mathrm{i}\left(2u_{xx}^{[0]} + \lambda^2\right)\psi, \\ \chi_{xx} &= \left(\frac{u_{xxx}^{[0]} + \mathrm{i}u_{xt}^{[0]}}{2u_{xx}^{[0]}} + \mathrm{i}\lambda\right)\chi_x - u_{xx}^{[0]}\chi, \qquad \chi_t = -\mathrm{i}\chi_{xx} - 2\lambda\chi_x - \mathrm{i}\left(2u_{xx}^{[0]} + \lambda^2\right)\chi, \end{aligned} \tag{10}$$

where $\{\chi, \psi\}$ are two complex conjugated eigenfunctions satisfying $\frac{\psi_x\chi_x}{\psi\chi} + u_{xx}^{[0]} = 0$ and λ is the spectral parameter. From (10), we may compute the Lax pair for the DNLS equations, obtaining

$$\begin{aligned} \chi_{xx} &= \left[\mathrm{i}\lambda - \frac{\mathrm{i}(\gamma-2)}{2}\left|m^{[0]}\right|^2 + \frac{m_x^{[0]}}{m^{[0]}}\right]\chi_x + \frac{1}{2}\left[\mathrm{i}m^{[0]}\overline{m}_x^{[0]} - \frac{\gamma-1}{2}\left|m^{[0]}\right|^4\right]\chi, \\ \chi_t &= \mathrm{i}\chi_{xx} - \left[(\gamma-2)\left|m^{[0]}\right|^2 + \frac{2\mathrm{i}m_x^{[0]}}{m^{[0]}}\right]\chi_x - \mathrm{i}\lambda^2\chi, \end{aligned} \tag{11}$$

and its complex conjugate, for the corresponding value of γ. It is worthwhile to remark that the condition $\frac{\psi_x\chi_x}{\psi\chi} - \frac{\mathrm{i}}{2}m^{[0]}\overline{m}_x^{[0]} + \frac{\gamma-1}{4}\left|m^{[0]}\right|^4 = 0$ allows us to determine an equivalent Lax pair for those systems.

3. Rational soliton solutions

Once the Lax pair have been obtained for a given PDE by means of the SMM, binary Darboux transformations can be constructed in order to obtain iterated solutions for that PDE. We implement the Darboux transformation formalism over the spectral problem (10) so as to provide a general iterative procedure to compute up to the nth iteration for u. By virtue of expressions (4), (5) and (7), solutions for the DNLS equations can be forthrightly established. Thus, soliton solutions for DNLS equations may be derived by considering a suitable choice for the seed solution and the eigenfunctions in the Lax pair.

In the following lines we summarize the main results regarding this procedure, oriented to the obtention of rational soliton solutions. Further details and a general rigorous analysis may be found in [1].

We start from a polynomial seed solution $u^{[0]}$ for (9) and binary exponential eigenfunctions for (10),

$$u^{[0]} = -\frac{j_0^2}{4}\left[j_0^2 z_0^2 x\left(\frac{x}{2} + j_0^2(z_0^2+1)t\right)\mathrm{i}\left(x + j_0^2\left(z_0^2 + \frac{1}{2}\right)t\right)\right], \qquad \chi_\sigma = \mathrm{e}^{\frac{\mathrm{i}}{2}j_0^2 z_0 \sigma\left[x + j_0^2\left(-\frac{\sigma}{2z_0}(z_0^4+7z_0^2+1)+3(z_0^2+1)\right)t\right]}, \tag{12}$$

where $\psi_\sigma = \overline{\chi}_\sigma$, j_0 and z_0 are arbitrary parameters, $\sigma = \pm 1$ and $\lambda_\sigma = \frac{j_0^2}{2}\left(2\sigma z_0 - (z_0^2+1)\right)$. The first and second iterations $u^{[j]}$, $j = 1, 2$ can be performed and the soliton solution profile may be computed as $\left|m^{[j]}\right|^2 = 2\mathrm{i}(u_x^{[j]} - \overline{u}_x^{[j]})$. The results are displayed in Figure 1.

- The first iteration ($j = 1$) provides a rational soliton-like travelling wave along the $x - j_0^2\left(\sigma z_0 - (z_0^2+1)\right)t$ direction and constant amplitude, of expression

$$\left|m_\sigma^{[1]}\right|^2 = j_0^2 - \frac{4}{j_0^2 z_0(\sigma - z_0)\left[\left(x - j_0^2\left(\sigma z_0 - (z_0^2+1)\right)t\right)^2 + \frac{1}{j_0^4 z_0^2(\sigma - z_0)^2}\right]}, \qquad \sigma = \pm 1. \tag{13}$$

- For the second iteration ($j = 2$), we get the two-soliton solution

$$\left|m^{[2]}\right|^2 = j_0^2 + \frac{8\left[\left(x + j_0^2(z_0^2+2)\,t\right)^2 + j_0^4(z_0^2-1)\,t^2 + \frac{1}{j_0^4(z_0^2-1)}\right]}{j_0^2(z_0^2-1)\left[\left(\left(x + j_0^2(z_0^2+1)\,t\right)^2 - j_0^4 z_0^2 t^2 - \frac{1}{j_0^4(z_0^2-1)}\right)^2 + \frac{4\left(x + j_0^2(z_0^2+2)\,t\right)^2}{j_0^4(z_0^2-1)^2}\right]}. \tag{14}$$

  leading to a two asymptotically travelling rational solitons of the form (13) (for $\sigma = 1$ and $\sigma = -1$, respectively) interacting at the origin.

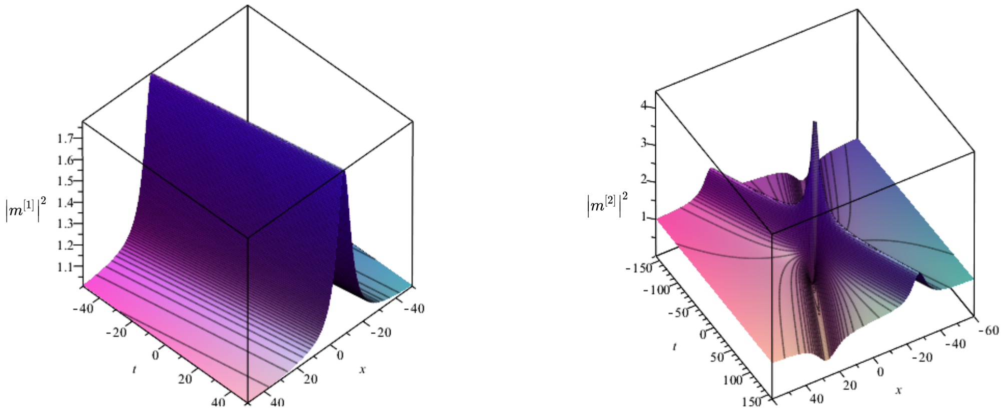

Figure 1: Spatio-temporal plot of $\left|m^{[1]}\right|^2$ and $\left|m^{[2]}\right|^2$ for parameters $\sigma = -1$, $j_0 = 1$, $z_0 = \frac{1}{6}$.

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Character varieties of torus knots

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Abstract: Attached to any topological space X we find its character variety. This is an algebraic variety parametrizing isomorphism classes of representations $\pi_1(X) \to G$ of the fundamental group of X into an algebraic reductive group G. These spaces are particularly useful in classical knot theory, since they provide very subtle invariants of knot $K \subset \mathbb{R}^3$ by taking $X = \mathbb{R}^3 - K$. However, even in the simplest cases a full understanding of these character varieties is an open problem. In this paper, we compute the motif of the irreducible character variety of representations of the fundamental group of the complement of an arbitrary torus knot into $G = \mathrm{SL}_4(k)$. For that purpose, we introduce a stratification of the variety in terms of the type of a canonical filtration attached to any representation. This allows us to reduce the computation of the virtual class to a purely combinatorial problem.

Resumen: Asociado a cada espacio topológico X tenemos su variedad de caracteres. Esta es una variedad algebraica que parametriza las clases de isomorfismo de representaciones $\pi_1(X) \to G$ del grupo fundamental de X en un grupo algebraico reductivo G. Estos espacios resultan especialmente útiles en teoría de nudos clasica, pues proveen de invariates muy sutiles de nudos $K \subset \mathbb{R}^3$ al tomar $X = \mathbb{R}^3 - K$. A pesar de esta importancia, incluso en los casos más simples el entendimiento completo de estas variedades de caracteres es un problema abierto. En este artículo, calculamos el motivo de la variedad de caracteres irreducible de representaciones del grupo fundamental de un nudo toroidal arbitrario en $G = \mathrm{SL}_4(k)$. Para este fin, introducimos una estratificación de la variedad en términos del tipo de una filtración canónica asociada a cada representación. Esto permite reducir el cálculo de la clase virtual a un problema puramente combinatorio.

Keywords: torus knot, character varieties, representations.

MSC2010: Primary: 57K31. Secondary: 14D20, 14C15.

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1. Motivic theory of character varieties

Let Γ be a finitely generated group and let G be a reductive linear algebraic group over an algebraically closed field k. The space $R(\Gamma, G)$ of representations $\rho : \Gamma \to G$ forms an algebraic variety known as the G-*representation variety*. Additionally, consider the open subset $R^{\mathrm{irr}}(\Gamma, G) \subseteq R(\Gamma, G)$ of irreducible representations. By Schur's lemma the adjoint action of G by conjugation on $R^{\mathrm{irr}}(\Gamma, G)$ is closed and its stabilizer at any point is the center of G. Therefore, the orbit space

$$\mathfrak{M}^{\mathrm{irr}}(\Gamma, G) = R^{\mathrm{irr}}(\Gamma, G)/G$$

is an algebraic variety known as the *irreducible* G-*character variety*. These varieties play a prominent role in the topology of 3-manifolds, starting with the foundational work of Culler and Shalen [1] on the study of hyperbolic geometry via $\mathrm{SL}_2(\mathbb{C})$-character varieties. Due to its importance, the algebro-geometric properties of character varieties have been widely studied, particularly regarding their motivic class.

Definition 1. The *Grothendieck ring of algebraic varieties* $K\mathcal{V}ar_k$ is the ring generated by isomorphism classes of algebraic varieties $[X]$, called virtual classes or motives in this context, with the relations $[X_1 \sqcup X_2] = [X_1] + [X_2]$ and $[X_1 \times X_2] = [X_1] \cdot [X_2]$ for any algebraic varieties X_1 and X_2. ◂

Remark 2. Great efforts have been made to compute the virtual classes $[\mathfrak{M}^{\mathrm{irr}}(\Gamma, G)] \in K\mathcal{V}ar_k$. Three approaches are proposed in the literature: the arithmetic viewpoint [4], the geometric perspective [6] and through Topological Quantum Field Theories [3]. ◂

An useful tool for studying the geometry of the character variety is the so- called *semi-simple filtration*. This is the analogue of the composition series or the Harder-Narasimhan filtration in the representation theoretic framework. Working similarly to the Jordan-Hölder theorem, we get the following result.

Proposition 3. *Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation. There exists an unique filtration of Γ-modules*

$$0 = V_0 \subset V_1 \subset \dots \subset V_i \subset \dots \subset V_s = V,$$

such that $\mathrm{Gr}_i(V_\bullet) = V_i/V_{i-1}$ is a maximally semi-simple subrepresentation of V/V_{i-1}.

By restriction, the semi-simple filtration also exists for representations onto any linear group G. Thanks to this filtration, we can decompose the graded pieces of a representation into its isotypic components $\mathrm{Gr}_i(V_\bullet) \cong \bigoplus_{j=1}^{s_i} W_{i,j}^{m_{i,j}}$, with $W_{i,1}, \dots, W_{i,s_i}$ non-isomorphic representations. From this information, we define the *shape* of the representation as the tuple collecting of dimensions and multiplicities of this decomposition $\xi = (\{(\dim W_{i,j}, m_{i,j})\}_j)_i$.

Moreover, we can add spectral information to the shape. For each $\gamma \in \Gamma$, denote by $\sigma_{i,j}(\gamma)$ the collection of eigenvalues of $\rho(\gamma) \in \mathrm{End}(W_{i,j})$ and set $\sigma = (\sigma_{i,j}(\gamma))$. The pair $\tau = (\xi, \sigma)$ is called the *type* of the representation and it is invariant under the adjoint action. Writing $\mathcal{T}(\Gamma, G)$ for the space of possible types arising in representations $\Gamma \to G$, we get a natural map

$$\Phi : R(\Gamma, G) \to \mathcal{T}(\Gamma, G)$$

assigning each representation to its underlying type. Also set $\mathcal{T}^{\mathrm{irr}}(\Gamma, G)$ for the types of irreducible representations, all of which have the same shape. The map Φ restricts to $\Phi : R^{\mathrm{irr}}(\Gamma, G) \to \mathcal{T}^{\mathrm{irr}}(\Gamma, G)$. Notice that if $\mathcal{T}^{\mathrm{irr}}(\Gamma, G)$ is finite, the morphism Φ induces a stratification of $R^{\mathrm{irr}}(\Gamma, G)$.

2. Character varieties of torus knots

Given a knot $K \subset \mathbb{R}^3$, it natural to study the fundamental group of its complement $\pi_1(\mathbb{R}^3 - K)$. An important case arises when $K = K_{n,m}$ is the (n, m)-torus knot $(\gcd(n, m) = 1)$ whose fundamental group of the complement is $\Gamma_{n,m} = \pi_1(\mathbb{R}^3 - K_{n,m}) = \langle x, y \mid x^n = y^m \rangle$. Using the image of the generators x, y to identify a representation, we get that the representation variety is

$$R(\Gamma_{n,m}, G) = \{(A, B) \in G \mid A^n = B^m\}.$$

The G-character varieties of torus knots have been studied for $G = \mathrm{SL}_2(\mathbb{C})$ [5, 8], $G = \mathrm{SL}_3(\mathbb{C})$ [9] and $G = \mathrm{SU}(2)$ [7], among others. However, very little is known in the higher rank case $G = \mathrm{SL}_r(k)$ for $r \geq 4$. A key observation towards this aim is the following.

Proposition 4. *Any irreducible representation $\rho : \Gamma_{n,m} \to G$ lifts, up to rescalling, to a representation $\tilde{\rho} : \mathbb{Z}_n \star \mathbb{Z}_m \to G$.*

Proof. Set $P = A^n = B^m$. Trivially $PA = AP$ and $PB = BP$, so $P^{-1}\rho P = \rho$. Thus, P is a Γ-equivariant automorphism of ρ which, by Schur's lemma, implies that $P = \varpi \mathrm{Id}$ for some $\varpi \in k^*$. ∎

Corollary 5. *$\mathcal{T}^{\mathrm{irr}}(\Gamma_{n,m}, \mathrm{SL}_r(k))$ is finite.*

Proof. In this case the scalling factor $\varpi \in k^*$ of Proposition 4 must satisfy $\varpi^r = 1$, so there are only finitely many posibilities. Thus, it is enough to show that $\mathcal{T}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m, \mathrm{SL}_r(k))$ is finite. If $(A, B) \in R^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m, \mathrm{SL}_r(k))$, then $A \in R(\mathbb{Z}_n, \mathrm{SL}_r(k))$ and $B \in R(\mathbb{Z}_m, \mathrm{SL}_r(k))$ so they are diagonalizable in so far as representations of finite abelian groups. Moreover, $A^n = B^m = \mathrm{Id}$ so the eigenvalues of A and B must be roots of unit. These are finitely many for fixed n, m, implying that $\mathcal{T}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m, \mathrm{SL}_r(k))$ is finite. ∎

From now on, we focus on $G = \mathrm{SL}_r(k)$, $r \geq 1$, as target group so we will omit it from the notation. Fixed a spectrum $\kappa = (\sigma_A, \sigma_B)$ for the matrices of a representation $(A, B) \in R(\Gamma_{n,m})$, let us denote by $\mathcal{T}_\kappa$ the set of types $\tau = (\xi, \sigma) \in \mathcal{T}(\Gamma_{n,m})$ whose spectral data σ are drawn from κ. Set $\mathcal{T}_\kappa^{\mathrm{irr}} = \mathcal{T}_\kappa \cap \mathcal{T}^{\mathrm{irr}}(\Gamma_{n,m})$, $\mathcal{T}_\kappa^{\mathrm{red}} = \mathcal{T}_\kappa - \mathcal{T}_\kappa^{\mathrm{irr}}$, $R_\kappa(\mathbb{Z}_n \star \mathbb{Z}_m) = \Phi^{-1}(\mathcal{T}_\kappa)$ and $R_\kappa^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) = \Phi^{-1}(\mathcal{T}_\kappa^{\mathrm{irr}})$. Then, we have that

$$R^{\mathrm{irr}}(\Gamma_{n,m}) \cong \bigsqcup_\kappa R_\kappa^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) = \bigsqcup_\kappa \Big(R_\kappa(\mathbb{Z}_n \star \mathbb{Z}_m) - \bigsqcup_{\tau \in \mathcal{T}_\kappa^{\mathrm{red}}} X(\tau) \Big), \tag{1}$$

where $X(\tau) = \Phi^{-1}(\tau)$ is the set of (reducible) representations of type τ. The virtual class $[R_\kappa(\mathbb{Z}_n \star \mathbb{Z}_m)] \in K\mathcal{V}ar_k$ can be easily computed as the product of the adjoint orbits of two diagonal matrices. Hence, Equation (1) shows that, to compute the virtual class of $R^{\mathrm{irr}}(\Gamma_{n,m})$, it is enough to compute the virtual classes of $X(\tau)$ for all κ and $\tau \in \mathcal{T}_\kappa^{\mathrm{red}}$. This amounts to a combinatorial problem and the knowledge of $[R^{\mathrm{irr}}(\Gamma_{n,m}, \mathrm{SL}_s(k))]$ for $s < r$, so the computation can be performed recursively. For further details, check [2, Section 3].

2.1. Counting components

Consider partitions $\pi = \{1^{e_1}, 2^{e_2}, \dots, r^{e_r}\}$ and $\pi' = \left\{1^{e'_1}, 2^{e'_2}, \dots, r^{e'_r}\right\}$ of r with $r = \sum_i ie_i = \sum_i ie'_i$. Denote by $M_{n,m,r}^{\pi,\pi'}$ the collection of (unordered) spectra $\kappa = (\sigma_A, \sigma_B)$ where σ_A (resp. σ_B) has e_i (resp. e'_i) collections of i equal eigenvalues for $i = 1, \dots, r$. Notice that, for any $\kappa, \kappa' \in M_{n,m,r}^{\pi,\pi'}$ we have that $\left[\Phi^{-1}(\mathcal{T}_\kappa^{\mathrm{red}})\right] = \left[\Phi^{-1}(\mathcal{T}_{\kappa'}^{\mathrm{red}})\right]$. Hence, we can collect the summands in (1) that contribute equaly to get

$$\left[R^{\mathrm{irr}}(\Gamma_{n,m})\right] = \sum_{\pi,\pi'} |M_{n,m,r}^{\pi,\pi'}| \Big(\left[R_{\kappa(\pi,\pi')}(\mathbb{Z}_n \star \mathbb{Z}_m)\right] - \sum_{\tau \in \mathcal{T}_{\kappa(\pi,\pi')}^{\mathrm{red}}} [X(\tau)] \Big). \tag{2}$$

Here, we have fixed an element $\kappa(\pi, \pi') \in M_{n,m,r}^{\pi,\pi'}$ for every permutations π, π'. The first step towards the calculation of all the terms involved this sum is provided in the following result.

Theorem 6 ([2, Section 6 and Theorem 6.8]). *If $\gcd(n, r) = \gcd(m, r) = 1$ or $r \leq 4$, then we have*

$$|M_{n,m,r}^{\pi,\pi'}| = \frac{r}{nm} \binom{n}{e_1, e_2, \dots, e_r} \binom{m}{e'_1, e'_2, \dots, e'_r} = \frac{r}{nm} \frac{n!}{e_1! \cdots e_r!(n - e_1 - \dots - e_r)!} \frac{m!}{e'_1! \cdots e'_r!(n - e'_1 - \dots - e'_r)!}.$$

Remark 7. It is an open problem whether this formula also holds true for $r \geq 5$ without the awkward hypothesis $\gcd(n, r) = \gcd(m, r) = 1$. ◂

2.2. Counting representations of fixed type

Fix a type τ, let $m_{i,j}$ be the multiplicity of the isotypic piece $W_{i,j}$ of the semi-simple filtration and set $\kappa_{i,j} = (\sigma_{i,j}(x), \sigma_{i,j}(y))$ for the corresponding eigenvalues of these pieces. Then we consider

$$\mathcal{I}(\tau) = \prod_{i=1}^{s} \prod_{j=1}^{s_i} \mathrm{Sym}^{m_{i,j}} \left(R_{\kappa_{i,j}}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) \right), \quad \hat{\mathcal{I}}(\tau) = \prod_{i=1}^{s} \prod_{j=1}^{s_i} \left(R_{\kappa_{i,j}}^{\mathrm{irr}}(\mathbb{Z}_n \star \mathbb{Z}_m) \right)^{m_{i,j}}.$$

There is a map $\mathrm{Gr}_\bullet : X(\tau) \to I(\tau)$ that assigns any representation to its graded complex (its "semi-simplification"). Pulling-back $\mathrm{Gr}_\bullet$ through the quotient map $\hat{\mathcal{I}}(\tau) \to \mathcal{I}(\tau)$, we obtain a morphism $\hat{\mathrm{Gr}}_\bullet : X(\tau) \times_{\mathcal{I}(\tau)} \hat{\mathcal{I}}(\tau) \to \hat{\mathcal{I}}(\tau)$. It is a Zariski locally trivial fibration whose fiber F_ϱ at $\varrho \in \hat{\mathcal{I}}(\tau)$ is the set of ways we can complete the block- diagonal semi-simple representation induced by ϱ with off-diagonal elements.

These calculations of the virtual classes of the fibers F_ϱ can be carried out using Schubert calculus (see [2, Sections 4 and 5], where the calculations for rank $r \leq 4$ are performed). Moreover, if for every $m_{i,j} > 1$ we have that $\dim W_{i,j} = 1$ (i.e. if all the repeated irreducible representations are 1-dimensional) then we have that $\hat{\mathcal{I}}(\tau) = \mathcal{I}(\tau)$ so $[X(\tau)] = [F_\varrho][\mathcal{I}(\tau)]$. These conditions hold for $r \leq 4$ [2, Corollary 4.7 and Proposition 8.1]. Thus performing these calculations for all the possible combinations of permutations and types, we can compute the virtual class of $R^{\mathrm{irr}}(\Gamma_{n,m}, \mathrm{SL}_r(k))$ by means of (2) for $r \leq 4$.

In the case $r = 4$, there are 10 posible partitions such that $\mathcal{T}^{\mathrm{irr}}_{\kappa(\pi,\pi')} \neq \varnothing$ and more than 350 types must be analyzed for these partitions. Carrying out the calculations with a symbolic algebra system, we finally obtain the following result (see [2, Section 8] for further details).

Theorem 8. *The virtual class of the irreducible* $\mathrm{SL}_4(k)$*-character variety of the* (n,m)*-torus knot is*

$$\begin{aligned}
[\mathfrak{M}^{\mathrm{irr}}(\Gamma, \mathrm{SL}_4(k)] = {} & \tfrac{4}{nm}\tbinom{n}{4}\tbinom{m}{4}(q^9 + 6q^8 + 20q^7 + 17q^6 - 98q^5 - 26q^4 + 38q^3 + 126q^2 - 144) \\
& + \tfrac{4}{nm}\tbinom{n}{2,1}\tbinom{m}{2,1}(q^5 + 2q^4 - 10q^3 + 7q^2 + 11q - 17) + \tfrac{4}{nm}\left(\tbinom{n}{4}\tbinom{m}{2} + \tbinom{n}{2}\tbinom{m}{4}\right)(q^5 + 4q^4 - 11q^3 + q^2 + 18q - 18) \\
& + \tfrac{4}{nm}\left(\tbinom{n}{4}\tbinom{m}{1,1} + \tbinom{n}{1,1}\tbinom{m}{4}\right)(q^3 - 4q^2 + 6q - 4) + \tfrac{4}{nm}\left(\tbinom{n}{2,1}\tbinom{m}{2} + \tbinom{n}{2}\tbinom{m}{2,1}\right)(q^3 - 3q^2 + 5q - 4) \\
& + \tfrac{4}{nm}\left(\tbinom{n}{4}\tbinom{m}{2,1} + \tbinom{n}{2,1}\tbinom{m}{4}\right)(q^7 + 5q^6 + 7q^5 - 34q^4 + 34q^2 + 18q - 48),
\end{aligned}$$

where $q = [\mathbb{A}^1_k] \in K\mathcal{V}ar_k$ *is the virtual class of the affine line.*

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On the prime graph associated with class sizes of a finite group

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Abstract: The aim of this paper is to present some current results that investigate the relation between the structure of a finite group G and graph-theoretical properties of the prime graph associated with its conjugacy class sizes.

Resumen: En este trabajo presentamos algunos resultados recientes que estudian la relación existente entre la estructura de un grupo finito G y las propiedades del grafo primo asociado a los tamaños de sus clases de conjugación.

Keywords: finite groups, conjugacy classes, prime graph.

MSC2010: 20E45.

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1. Introduction

Hereafter, only finite groups will be considered. A well-established area of research within finite group theory is the study of the connection between the structure of a group G and the arithmetical properties of certain sets of positive integers associated to it. In particular, the set $cs(G) = \{|G : \mathbf{C}_G(x)| \ : \ x \in G\}$ of conjugacy class sizes of G has been thoroughly analysed. For instance, a classical result within this research line states that a group G has a central Sylow p-subgroup, for a given prime p, if and only if p does not divide any number in $cs(G)$. Indeed, non-divisibility properties have been studied since many decades ago, as the next theorem due to N. Itô in 1953 (see the paper [5]): if p and q are two distinct prime numbers that divide two distinct class sizes of a group G, but pq does not divide any number in $cs(G)$, then G has a normal p-complement and abelian Sylow p-subgroups.

A useful tool that is gaining an increasing interest for studying the arithmetical structure of the set $cs(G)$ is the (complement) prime graph. In general, if X is a set of positive integers, then the *prime graph* $\Delta(X)$ is defined as the simple undirected graph whose vertex set $V(X)$ consists of the prime divisors of the numbers in X, and whose edge set $E(X)$ contains $\{p, q\} \subseteq V(X)$ whenever pq divides some element in X. Further, the *complement prime graph* $\overline{\Delta(X)}$ is the graph with the same vertex set $V(X)$, and two primes p and q are adjacent in $\overline{\Delta(X)}$ if and only if they are not adjacent in $\Delta(X)$. In this paper, we consider the prime graph $\Delta(G)$ built on the set $cs(G)$, with vertex and edge set $(V(G), E(G))$, respectively; and in particular, we will present some current results in collaboration with S. Dolfi, E. Pacifici, and L. Sanus.

Two natural questions that arise in this context are the following ones:

- What can be said about the structure of G if some properties of $\Delta(G)$ are known?
- What graphs can occur as $\Delta(G)$ for some finite group G?

2. Features of $\Delta(G)$

Both classical results stated in the first paragraph can be framed within the first question, since they have the next transcription in terms of $\Delta(G)$.

Lemma 1. *Let G be a group, and let $p, q \in V(G)$ with $p \neq q$. Then we have:*

(i) *$p \notin V(G)$ if and only if $P \leqslant \mathbf{Z}(G)$, for some Sylow p-subgroup P of G.*

(ii) *If $\{p, q\} \notin E(G)$, then G has a normal p-complement and abelian Sylow p-subgroups.*

In the context of the second question above, those graphs that possess "few" edges cannot occur as $\Delta(G)$ for a group G. This is due to the following result of S. Dolfi, which we call the "Three-Vertex Theorem".

Theorem 2. [2, Theorem A] *Let G be a group. Then for every choice of three vertices in $\Delta(G)$, there exists at least an edge that joins two of them.*

Indeed, this result is an improvement of [1, Theorem 16], where Dolfi proved the soluble version of the Three-Vertex Theorem. As a direct consequence, we obtain the next result, which actually was known to be true even before the existence of the Three-Vertex Theorem (see the paper [1]).

Corollary 3. *Let G be a group. Then we have:*

(i) *If $\Delta(G)$ is connected, then its diameter is at most 3.*

(ii) *If $\Delta(G)$ is non-connected, then it is the union of two complete subgraphs.*

In fact, a group G has non-connected prime graph $\Delta(G)$ if and only if $G = AB$ with A and B abelian Hall subgroups of coprime order, and $G/\mathbf{Z}(G)$ is a Frobenius group with kernel $A\mathbf{Z}(G)/\mathbf{Z}(G)$ (see [1, Theorem 4]). Further, the set of prime divisors of $|A\mathbf{Z}(G)/\mathbf{Z}(G)|$ and the one of $|B\mathbf{Z}(G)/\mathbf{Z}(G)|$ are two *cliques* of $\Delta(G)$, so they form the two complete connected components of $\Delta(G)$. In general, a subset of vertices of a graph Δ is called a *clique* if their induced subgraph in Δ is complete.

3. The complement prime graph

Recall that the complement prime graph $\overline{\Delta(G)}$ has the same vertex set $V(G)$, and there is and edge between two primes p and q whenever they are not adjacent in $\Delta(G)$. Observe that the Three-Vertex Theorem can be expressed in terms of the complement prime graph as follows: for every finite group G, the graph $\overline{\Delta(G)}$ does not contain any cycle of length 3. This fact suggests the study of the (non-)existence of cycles within $\overline{\Delta(G)}$ of length larger than 3.

Example 4. Let $N = C_{31} \times C_{61} = C_{1891}$ and $H = C_3 \times C_5 = C_{15}$. Then H acts on N fixed-point-freely, and we can consider the semidirect product $G = N \rtimes H$, which is a Frobenius group. It follows that $cs(G) = \{1, 15, 1891\}$ and that $\Delta(G)$ is the union of two complete connected components, which are the prime divisors of N and H, respectively. So $\overline{\Delta(G)}$ is a cycle of length 4. ◂

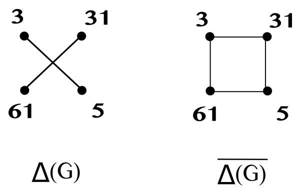

Figure 1: an illustration of $\Delta(G)$ and $\overline{\Delta(G)}$ where $G = C_{1891} \rtimes C_{15}$.

In view of the above example, the next natural step would be to study the case of a cycle of length 5 in $\overline{\Delta(G)}$. Nevertheless, this is also impossible. Indeed, this fact is more general, as the main theorem of the paper [3] shows.

Theorem 5. [3, Theorem A] *Let G be a group. Then the graph $\overline{\Delta(G)}$ does not contain any cycle of odd length.*

In other words, this means that $\overline{\Delta(G)}$ is a bipartite graph, i.e., a graph where the vertex sex can be partitioned into two disjoint subsets A and B such that every edge connects a vertex in A to another one in B. As an immediate consequence, we obtain the next result.

Corollary 6. [3, Corollary B] *Let G be a group. Then the vertex set of $\Delta(G)$ can be partitioned into two subsets of pairwise adjacent vertices.*

We have previously commented that if $\Delta(G)$ is disconnected for some group G, then $V(G)$ is the union of two cliques. But from the above corollary, this property turns out to hold in full generality.

Therefore, at least half of the vertices of $\Delta(G)$ are pairwise adjacent, for any group G. So denoting by $w(G)$ the clique number (i.e., the maximum size of a clique) of $\Delta(G)$, we obtain what follows.

Corollary 7. [3, Corollary C] *Let G be a group. Then, the inequality $|V(G)| \leq 2w(G)$ holds.*

It is not difficult to see that this bound is best possible, as the group in Example 4 shows. We close this section with another illustrating example.

Example 8. Let $G = A\Gamma(11^3) = ((C_{11} \times C_{11} \times C_{11}) \rtimes C_{11^3-1}) \rtimes C_3$ be an affine semilinear group. Then $V(G) = \{2, 3, 5, 7, 11, 19\}$, and it follows that $\Delta(G)$ is the union of the clique $V(G) \setminus \{3\}$ and the edge $\{3, 11\}$ (see Figure 2). ◂

4. Cut vertices

The last example has the following interesting property: if we remove the vertex 11 from the graph and all the edges adjacent to 11 from $\Delta(G)$, then the resulting graph is disconnected. Let us define this "almost non-connectedness" feature of $\Delta(G)$ in general: $r \in V(G)$ is called a *cut vertex* of $\Delta(G)$ if the subgraph

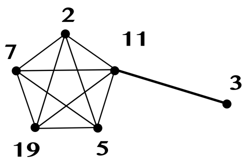

Figure 2: the prime graph $\Delta(G)$ for $G = A\Gamma(11^3)$.

induced by $V(G) \setminus \{r\}$ in $\Delta(G)$ (i.e., the graph $\Delta(G) - r$ obtained by removing the vertex p and all edges incident to r from $\Delta(G)$) has more connected components than $\Delta(G)$.

Example 9. There is an easy way of obtaining groups G such that $\Delta(G)$ has a cut vertex r. It is enough to consider $G = R \times (A \rtimes B)$ where $A \rtimes B$ is a Frobenius group with A and B abelian, and R is a non-abelian r-group such that r does not divide the order of $A \rtimes B$. ◂

The next result states, among other facts, that $\Delta(G)$ can have at most two cut vertices.

Theorem 10. [4, Theorem A] *Let G be a group such that $\Delta(G)$ has a cut vertex r. Then, the following conclusions hold.*

(i) *G is soluble with Fitting height at most 3, and its Sylow p-subgroups are abelian for all primes $p \neq r$.*

(ii) *$\Delta(G) - r$ is a graph with two connected components, that are both complete graphs.*

(iii) *If r is a complete vertex of $\Delta(G)$, then it is the unique complete vertex and the unique cut vertex of $\Delta(G)$. If r is non-complete, then $\Delta(G)$ is a graph of diameter 3, and it can have at most two cut vertices; moreover, G is metabelian with abelian Sylow subgroups.*

Example 11. Let $R = C_{31}$, $A = C_{11} \times C_{61}$, $B_0 = C_2 \times C_3$, and $B_1 = C_5$, and consider $G = (A \times R) \rtimes (B_0 \times B_1)$, where there is a Frobenius action of $B_0 \times B_1$ on R, another Frobenius action of B_1 on A, and B_0 acts trivially on A. It is not difficult to show that $\Delta(G)$ is the union of the two cliques $\{11, 31, 61\}$ and $\{2, 3, 5\}$ together with the edge $\{5, 31\}$, so 5 and 31 are cut vertices of $\Delta(G)$. ◂

Moreover, Theorem 3.3 and Theorem C of [4] completely characterise the structure of G (and the corresponding one of $\Delta(G)$) in both the cases when $\Delta(G)$ has either one or two cut vertices, respectively. In particular, these results yield a classification of those groups G such that $\Delta(G)$ is acyclic, i.e., that it has no cycle as an induced subgraph (see [4, Corollary 3.4]).

In addition, there is a necessary and sufficient condition for a graph that possesses a cut vertex to occur as $\Delta(G)$ for a suitable group G.

Theorem 12. [4, Theorem D] *Let Δ be a graph having a cut vertex. Then there exists a finite group G such that $\Delta = \Delta(G)$ if and only if Δ is connected and the vertex set of Δ can be partitioned in two subsets of pairwise adjacent vertices.*

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Constructing normal integral bases of Hopf Galois extensions

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Abstract: A Hopf Galois extension is an extension of fields that has attached a Hopf algebra together with an action on the top field, called a Hopf Galois structure. Every Galois extension is Hopf Galois, but the converse does not hold. For an extension of local or global fields, we recall the definition of associated order in a Hopf Galois structure and use it to generalize the concept of normal integral basis of a Galois extension to Hopf Galois extensions. We shall present a method to construct effectively a normal integral basis of a Hopf Galois extension and apply it to the Hopf Galois non-Galois extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$.

Resumen: Una extensión Hopf Galois es una extensión de cuerpos que tiene asociada un álgebra de Hopf junto con una acción en el cuerpo superior, llamado una estructura Hopf Galois. Toda extensión de Galois es Hopf Galois, pero el recíproco no es cierto. Para una extensión de cuerpos locales o globales, recordamos la definición de orden asociado y la usamos para generalizar el concepto de base normal entera de una extensión de Galois a extensiones Hopf Galois. Presentamos un método para construir de manera efectiva una base normal entera de una extensión y la aplicamos a la extensión Hopf Galois no Galois $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$.

Keywords: Hopf Galois extension, associated order, normal integral basis.

MSC2010: 11R32, 12F10, 16T05.

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1. Introduction: Hopf Galois extensions

Galois theory establishes a connection between field theory and group theory: for extensions of fields L/K with the property that every polynomial $f \in K[X]$ with a root in L has $\deg(f)$ roots in L (the so called Galois extensions), it is possible to read these extensions algebraically in terms of their Galois group $G := \mathrm{Gal}(L/K)$, the group of all K-automorphisms of L. This theory was introduced by French mathematician Évariste Galois to characterize the solvability by radicals of polynomial equations, and it has been proved as an essential tool in modern algebraic number theory.

The theory of Hopf Galois extensions establishes a similar link between the theory of fields and the one of Hopf algebras. Let L/K be a finite extension of fields and and let G be a group that acts on L by automorphisms. There is a natural group representation of G

$$\begin{array}{rccl} \rho_G\colon & G & \longrightarrow & \mathrm{Aut}_K(L) \\ & \sigma & \longmapsto & y \mapsto \sigma(y). \end{array}$$

Now, we can extend this map by K-linearity to a map $\rho_{K[G]}\colon K[G] \to \mathrm{End}_K(L)$ which is a linear representation of the K-group algebra $K[G]$. But $K[G]$ is a K-Hopf algebra and this structure is compatible with its action on L (concretely, L is a $K[G]$-module algebra, see the book [4] for a definition).

Let L/K be a finite extension and assume that H is a K-Hopf algebra that endows L with left H-module algebra structure. Similarly we have a linear representation

$$\begin{array}{rccl} \rho_H\colon & H & \longrightarrow & \mathrm{End}_K(L) \\ & h & \longmapsto & x \mapsto h \cdot x \end{array}$$

of the K-Hopf algebra H. We can construct a canonical map $(1, \rho_H)\colon L \otimes_K H \to \mathrm{End}_K(L)$ defined by sending each $x \otimes h \in L \otimes_K H$ to the endomorphism $y \mapsto x(h \cdot y)$.

Definition 1. Let L/K be a finite extension of fields. A *Hopf Galois structure* of L/K is a pair $(H, \cdot)$ where H is a K-Hopf algebra and $\cdot\colon H \otimes_K L \to L$ is a K-linear action of H on L which endows it with K-module algebra structure and such that $(1, \rho_H)$ is an isomorphism of K-vector spaces. A *Hopf Galois extension* is an extension L/K that admits some Hopf Galois structure. If $(H, \cdot)$ is a Hopf Galois structure of L/K, we will also say that L/K is H-Galois. ◂

By construction, every Galois extension is Hopf Galois, because the K-group algebra $K[G]$ of its Galois group together with the evaluation action is a Hopf Galois structure. However, the converse does not hold: for instance, the extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is a Hopf Galois extension that is not Galois. This example was used by Greither and Pareigis in their article [3] to nicely illustrate the notion of Hopf Galois extension.

2. The theory of Hopf Galois modules

If F is a number (resp. p-adic) field, we will denote by $\mathcal{O}_F$ its ring of integers, i.e, the elements of K which are roots of monic polynomials with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Z}_p$). From now on, we will deal only with extensions of number or p-adic fields L/K such that $\mathcal{O}_K$ is a principal ideal domain (actually, this is always satisfied when the fields are p-adic). Under this hypothesis, it follows that $\mathcal{O}_L$ is free as $\mathcal{O}_K$-module, and any basis of that module is called an integral basis of L.

One of the applications of Galois theory is the theory of Galois modules, which studies the structure of $\mathcal{O}_L$ as module over its associated order $\mathfrak{A}_{L/K}$ in $K[G]$, where $G = \mathrm{Gal}(L/K)$. This is defined as the maximal $\mathcal{O}_K$-order in $K[G]$ such that its evaluation action on L leaves $\mathcal{O}_L$ invariant. The notion of associated order can be easily generalized to the setting of Hopf Galois theory.

Definition 2. Let L/K be an H-Galois extension of fields as above. The *associated order* of $\mathcal{O}_L$ in H is defined as

$$\mathfrak{A}_H = \{h \in H \mid h \cdot x \in \mathcal{O}_L \; \forall x \in \mathcal{O}_L\}.$$ ◂

The associated order is indeed an $\mathcal{O}_K$-order in H, and in particular it is free as $\mathcal{O}_K$-module. Let $V = \{v_i\}_{i=1}^n$ be an $\mathcal{O}_K$-basis of $\mathfrak{A}_H$. If in addition $\mathcal{O}_L$ is $\mathfrak{A}_H$-free of rank one with generator β, then $\{v_i \cdot \beta \mid 1 \le i \le n\}$ is an $\mathcal{O}_K$-basis of $\mathcal{O}_L$, called a *normal integral basis*. Thus, once computed a basis of $\mathfrak{A}_H$, the key point turns to whether $\mathcal{O}_L$ is free over $\mathfrak{A}_H$. We present a constructive method to answer both questions.

3. The reduction method

Let L/K be an H-Galois extension of number or p-adic fields. The reduction method provides a way to find effectively an $\mathcal{O}_K$-basis of $\mathcal{O}_L$ and determine whether $\mathcal{O}_L$ is $\mathfrak{A}_H$-free. The idea behind the method is the same as in representation theory: instead of working with the elements of the Hopf algebra, we use the matrices representing them. We present the main definitions and results (see the paper [2] for more details).

Definition 3. Let $W = \{w_i\}_{i=1}^n$ and $B = \{\gamma_j\}_{j=1}^n$ be K-bases of H and L respectively. Given $1 \leq j \leq n$, we denote

$$M_j(H,L) = \begin{pmatrix} | & | & \cdots & | \\ w_1 \cdot \gamma_j & w_2 \cdot \gamma_j & \cdots & w_n \cdot \gamma_j \\ | & | & \cdots & | \end{pmatrix} \in \mathcal{M}_{n^2 \times n}(K),$$

that is, $M_j(H,L)$ is the matrix whose i-th column is the column vector of the coordinates of $w_i \cdot \gamma_j$ with respect to the basis B. Then, the *matrix of the action* is defined as

$$M(H,L) = \left(\begin{array}{c} M_1(H,L) \\ \hline \cdots \\ \hline M_n(H,L) \end{array}\right).$$

◂

The key step of the reduction method is to reduce $M(H,L)$ to an invertible matrix by linear transformations, but preserving the integral structure (i.e., multiplying by an unimodular matrix). This can be achieved by considering the Hermite normal form of $M(H,L)$, whose definition is well known for matrices with coefficients in $\mathbb{Z}$ and can be consulted in the book [1] for a general PID. However, $M(H,L)$ may have coefficients out of the ring, but in any case in its field of fractions. What we do in practice is to drop out of the matrix the least common multiple of the denominators and consider as Hermite normal form of $M(H,L)$ the fractional part times the Hermite normal form of the matrix with integral coefficients. In the language of polynomials, the first part would be the content and the second one, the principal part.

Theorem 4. *Assume that $B = \{\gamma_j\}_{j=1}^n$ is an integral basis of L. Let D be the Hermite normal form of the matrix $M(H,L)$ and call $D^{-1} = (d_{ij})_{i,j=1}^n$. Then, the elements*

$$v_i = \sum_{l=1}^n d_{li} w_l$$

form an $\mathcal{O}_K$-basis of $\mathfrak{A}_H$. Moreover, a given element $\beta = \sum_{j=1}^n \beta_j \gamma_j$ is a free generator of $\mathcal{O}_L$ as $\mathfrak{A}_H$-module if and only if the matrix

$$M_\beta(H,L) = \sum_{j=1}^n \beta_j M_j(H,L) D^{-1}$$

is unimodular.

4. An example of application

We apply Theorem 4 to study the example of the extension $L/\mathbb{Q}$ with $L = \mathbb{Q}(\omega)$, where $\omega = \sqrt[3]{2}$. Let c and s be the $\mathbb{Q}$-endomorphisms of L defined by the relations

$$c(1) = 1, \qquad c(\omega) = -\frac{1}{2}\omega, \qquad c(\omega^2) = -\frac{1}{2}\omega^2,$$
$$s(1) = 0, \qquad s(\omega) = \frac{1}{2}\omega, \qquad s(\omega^2) = -\frac{1}{2}\omega^2.$$

In the aforementioned article [3], it is shown that $L/\mathbb{Q}$ has an unique Hopf Galois structure, given by the Hopf algebra

$$H = \mathbb{Q}(c, s)/\langle 3s^2 + c^2 - \mathrm{Id}_L, (2c + \mathrm{Id}_L)s, (2c + \mathrm{Id}_L)(c - \mathrm{Id}_L)\rangle$$

together with the evaluation action on L. Taking $\{\mathrm{Id}_L, c, s\}$ as $\mathbb{Q}$-basis of H and $\{1, \omega, \omega^2\}$ as $\mathbb{Q}$-basis of L, the blocks of the matrix of the action $M(H, L)$ are

$$M_1(H,L) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2(H,L) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3(H,L) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Joining these blocks as in Definition 3 gives the matrix of the action $M(H, L)$. Its Hermite normal form and inverse are

$$D = \frac{1}{2}\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad D^{-1} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Following Theorem 4, the associated order $\mathfrak{A}_H$ has $\mathbb{Z}$-basis

$$V = \left\{\mathrm{Id}_L, \frac{\mathrm{Id}_L + 2c}{3}, \frac{-\mathrm{Id}_L + c + 3s}{3}\right\}.$$

Let us check whether $\mathcal{O}_L$ is $\mathfrak{A}_H$-free or not. For $\beta = \beta_1 + \beta_2\omega + \beta_3\omega^2$, we have

$$M_\beta(H,L) = \begin{pmatrix} \beta_1 & \beta_1 & 0 \\ \beta_2 & 0 & 0 \\ \beta_3 & 0 & -\beta_3 \end{pmatrix},$$

whose determinant is $\beta_1\beta_2\beta_3$. Then, taking $\beta = 1 + \omega + \omega^2 \in \mathcal{O}_L$, the determinant is 1, and then $M_\beta(H, L)$ is unimodular. Thus, $\mathcal{O}_L$ is $\mathfrak{A}_H$-free of rank one and β is a generator. Consequently, $\mathcal{O}_L$ has a normal integral basis:

$$\left\{\mathrm{Id}_L(\beta), \frac{\mathrm{Id}_L + 2c}{3}(\beta), \frac{-\mathrm{Id}_L + c + 3s}{3}(\beta)\right\}.$$

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Higher derivations with Lie structure of associative rings

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Abstract: In this paper, we suppose R is a prime ring with the centre $Z(R)$, $D = (d_i \neq 0)_{i\in\mathbb{N}}$ is a higher derivation of R and L is a Lie ideal of R, this gives under certain conditions R has a weak zero-divisor or a weakly semiprime ideal.

Resumen: En este artículo, suponemos que R es un anillo principal con centro $Z(R)$, $D = (d_i \neq 0)_{i\in\mathbb{N}}$ es una derivación de R y L es un ideal de Lie de R. Bajo ciertas condiciones, resulta que R tiene un divisor de cero débil o un ideal semiprimo débil.

Keywords: weakly semiprime ideal, weak zero-divisor, Lie ideal, derivation, prime ring.

MSC2010: 16W25, 47B47.

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1. Introduction

One of the earliest results on Lie derivations of associative rings is dur to Martindale [10], who proved that every Lie derivation on a primitive ring can be written as a sum of derivation and an additive mapping of ring to its center that maps commutators into zero, *i.e.*, Lie derivation has the standard form. In 1993, Brešar [5] gave a characterization of Lie derivations of prime rings. This result together with other results initiated the theory of functional identities on rings. Actually, study behaviour of a derivation on the whole ring with many of the results achieved by extending the other ones proven previously. For a full account on the theory of functional identities and zero Lie product we refer the reader to the paper of Brešar [6]. Lie derivations, as well as other Lie maps, have been active research subjects for a long time (see, e.g., [1], Benkovič [3] and Brešar [6]). Also, Cheung [8] gave a characterization of linear Lie derivations on triangular algebras. Qi and Hou [11] discussed additive ξ-Lie derivations on nest algebras. The most interesting result on additive Lie derivations of prime rings was obtained by Brešar [6].

Throughout the article, R will represent an a commutative ring with identity $1 \neq 0$. The center of R is denoted by $Z(R)$. The symbols $[x, y]$ stand for the commutator $xy - yx$, and $x \circ y$ stands for the anticommutator $xy + yx$, for any $x, y \in R$. A ring R is called a prime if $xRy = 0$ implies either $x = 0$ or $y = 0$. Suppose L is an additive subgroup of R, L is said to be a Lie ideal of R if for every $u \in L$, $r \in R$ then the commutator $[u, r] = ur - ru \in L$. Any ordinary, two-sided ideal of R is automatically a Lie ideal of R. Let $n > 1$ be an integer; then, a ring R is said to be n-torsion free, in case $nx = 0$ implies that $x = 0$ for any $x \in R$.

The idea of a weakly semiprime ideal is due to Badawi [2]. He introduced that the ideal L is a weakly semiprime ideal of R such that R is a commutative ring with identity $1 \neq 0$ and L is a proper ideal of R. If $a \in R$ and $0 \neq a^2 \in L$ then $a \in L$. While the concept of a weak zero-divisor of a ring R introduced by Burgess, Lashgari, and Mojiri [7], where the authors defined an element $a \in R$ is called a weak zero-divisor. If there is $r, s \in R$ with $ras = 0$ and $rs \neq 0$. A derivation d is an additive mapping $d : R \to R$ satisfies the Leibniz's formula $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Moreover, D is said to be a higher derivation of U into R if for every $n \in \mathbb{N}$, we conclude that $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in L$ and $D = (d_i \neq 0)$ for all $i \in \mathbb{N}$ is the family of additive mappings of R such that $d_0 = id_R$ and $\mathbb{N}$ is set of a positive integers, where L is a Lie ideal of R.

By the above facts, it is fascinating to study weakly semiprime ideals and weak zero-divisors on a Lie ideal of a prime ring R via a higher derivation $D = (d_i \neq 0)_{i\in\mathbb{N}}$. This is our main motivation for this paper. The following lemmas are also going to be applied:

Lemma 1 (Bergen, Herstein, and Kerr [4], Lemma 4). *Suppose R is a prime ring with characteristic not two and $a, b \in R$. If L is a non-central Lie ideal of R such that $aUb = 0$, then either $a = 0$ or $b = 0$.*

Lemma 2 (Herstein [9], Lemma 1.8). *Let R be a semiprime ring, and $a \in R$ be a centralizer of all commutators $[x, y]$, $x, y \in R$. Then, $a \in Z(R)$.*

2. The main results

Theorem 3. *Let R be a prime ring with the centre $Z(R)$ and L be a Lie ideal of R. Suppose that $D = (d_i \neq 0)_{i\in\mathbb{N}}$ is a higher derivation of U into R. If d satisfy one of the following relations*

(i) $[a, d_i(u)] \in Z(R)$ *for all* $u \in L$, $a \in R$, *then* L *is a weakly semiprime ideal.*

(ii) $[d_i(L), d_i(L)] \subseteq Z(R)$, *then either* L *is a weakly semiprime ideal of* R *or* $d_n(L)$ *is a weak zero-divisor of* R.

(iii) $[a, d_i(u)] \in Z(R)$ *and* $d_i(Z(R)) \neq 0$ *for all* $u \in L$, $a \in R$, *then* $[a, [L, R]] \subseteq Z(R)$.

Based on Theorem 3, we can easily prove the following theorem.

Theorem 4. *Let R be a prime ring with the centre $Z(R)$ and L be a Lie ideal of R. Suppose that $D = (d_i \neq 0)_{i\in\mathbb{N}}$ is a higher derivation of U into R. If d satisfy one of the following relations*

(i) $[d_i(L), d_i(L)] \subseteq Z(R)$ *and* $d_i(Z(R)) \neq 0$, *then R has a weakly semiprime ideal.*

(ii) $d_i^2(L) \subseteq Z(R), d_i(Z(R)) \neq 0$ *and* $d_i d_j(L) \subseteq Z(R)$, $i, j \in \mathbb{N}$, *then either $d_n(L)$ is a weak zero-divisor of R or R has a weakly semiprime ideal, where R is 2-torsion free.*

(iii) $ad_i(L) \subseteq Z(R)$ *and* $d_i(Z(R)) \neq 0$, *then either a is a weak zero-divisor of R or R has a weakly semiprime ideal, where $a \in R$.*

In the following theorem, R not to be a commutative ring with identity $1 \neq 0$.

Theorem 5. *For any fixed integers $n, q > 1$, let R be prime ring with the centre $Z(R)$ and D be a derivation on R. If D satisfy one of the following relations*

(i) $D^n(xoy) \mp [x, y] \in Z(R)$;

(ii) $D^n(x \circ y) \pm D^q(x \circ y) \mp [x, y] \in Z(R)$ *and R is 2-torsion free;*

(iii) $D^n([x, y]) \pm D^q([x, y]) \mp (x \circ y) \in Z(R)$ *and R is 2-torsion free*

for all $x, y \in R$, then R has a weak zero-divisors.

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Efimov K-theory of diamonds

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Abstract: Motivated by Scholze and Fargues's geometrization of the Local Langlands Correspondence using perfectoid diamonds and Clausen and Scholze's work on the K-theory of adic spaces using condensed mathematics, we introduce the Efimov K-theory of diamonds. We propose a large stable $(\infty, 1)$-category of diamonds $\mathcal{D}^{\diamond}$, a diamond spectra and chromatic tower, and a localization sequence for diamond spectra. Commensurate with the localization sequence, we detail three potential applications of the Efimov K-theory of $\mathcal{D}^{\diamond}$: to quantum gravity and reconstructing the holographic principle using diamonds and Scholze's six operations in the étale cohomology of diamonds; to post-quantum diamond cryptography in the form of programming AI with Efimov K-theory of $\mathcal{D}^{\diamond}$; and to nonlocality in perfectoid quantum physics.

Resumen: Motivados por la geometrización de Scholze y Fargues de la Correspondencia Local de Langlands usando diamantes perfectoides y el trabajo de Clausen y Scholze con la K-teoría de espacios ádicos usando matemáticas condensadas, nosotros introducimos la K-teoría de diamantes de Efimov. Proponemos una $(\infty, 1)$-categoría de diamantes $\mathcal{D}^{\diamond}$; un espectro de diamantes y una torre cromática, y una secuencia de localización del espectro de un diamante. Acorde con esta secuencia de localización, detallamos tres potenciales aplicaciones de la K-teoría de $\mathcal{D}^{\diamond}$ de Efimov: a gravedad cuántica y la reconstrucción del principio holográfico usando diamantes y las seis operaciones de Scholze en la cohomología étale de diamantes; a criptografía de diamante postcuántica, en forma de programación de IA con K-teoría de $\mathcal{D}^{\diamond}$ de Efimov, y a no localidad en física perfectoide cuántica.

Keywords: perfectoid spaces, Efimov K-theory, diamonds, Fargues-Fontaine curve, geometric Langlands, $(\infty, 1)$-topoi.

MSC2010: 11F77, 11S70, 19D06, 19E08, 19E20.

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1. Introduction

K-theory is defined on the category of small stable ∞-categories which are idempotent, complete, and where morphisms are exact functors. A certain category of large compactly generated stable ∞-categories is equivalent to this small category. In Efimov K-theory the idea is to weaken to ‘dualizable’ the condition of being compactly generated so that K-theory is still defined. A category $\mathcal{C}$ being dualizable implies that $\mathcal{C}$ fits into a localization sequence $\mathcal{C} \to \mathcal{S} \to \mathcal{X}$ with $\mathcal{S}$ and $\mathcal{X}$ compactly generated. The Efimov K-theory should be the fiber of the K-theory in the localization sequence. As Clausen and Scholze propose [6]:

Proposition 1. *Let Nuc_R be a full subcategory of solid modules [1]. Nuc_R is a presentable stable ∞-category closed under all colimits and tensor products. If R is a Huber ring, then Nuc_R is dualizable, making its K-theory well-defined. Nuc_R embeds into Mod_R. Let $\mathcal{X}$ be a Noetherina formal scheme and X the torsion perfect complexes of modules over R. Then,*

Theorem 2. *$(R, R^+) \to Nuc_R$ satisfies descent over $Spa(R)$ and so does its Efimov K-theory. There exists a localization sequence $K(X) \to K^{Efimov}(\mathcal{X}) \to K^{Efimov}(\mathcal{X}^{rig})$ [2].*

We introduce the Efimov K-theory of diamonds.

2. Main conjectures

Conjecture 3. *There exists a large, stable, presentable $(\infty, 1)$-category of diamonds $\mathcal{D}^\diamond$ with spatial descent datum. $\mathcal{D}^\diamond$ is dualizable. Therefore, the Efimov K-theory is well defined.*

Conjecture 4. *Let S be a perfectoid space, $\mathcal{D}^\diamond$ a stable dualizable presentable category, and R a sheaf of E_1-ring spectra on S. Let $\mathcal{T}$ be a stable compactly generated $(\infty, 1)$-category and $F\colon \mathrm{Cat}_{St}^{idem} \to \mathcal{T}$ a localizing invariant that preserves filtered colimits. Then, $F_{cont}(Shv(\mathbb{S}^n, \mathcal{D}^\diamond)) \simeq \Omega^n F_{cont}(\mathcal{D}^\diamond)$.*

Conjecture 5. *Let $\mathcal{D}_\diamond$ be the complex of v-stacks of locally spatial diamonds. Let $\mathcal{Y}_{(R,R^+),E} = Spa(R, R^+)$ x_{SpaF_q} $SpaF_q[[t]]$ be the relative Fargues-Fontaine curve. Let $(\mathcal{Y}^\diamond_{S,E})$ be the diamond relative Fargues-Fontaine curve. There exists a localization sequence $K(\mathcal{D}_\diamond) \to K^{Efimov}(\mathcal{Y}^\diamond_{S,E}) \to K^{Efimov}(\mathcal{Y}_{(R,R^+),E})$.*

Conjecture 6. *$\mathcal{D}^\diamond$ admits a topological localization, in the sense of Grothendieck-Rezk-Lurie $(\infty, 1)$-topoi.*

Conjecture 7. *There exists a diamond chromatic tower $\mathcal{D}^\diamond \to \ldots \to L_n\mathcal{D}^\diamond \to L_{n-1}\mathcal{D}^\diamond \to \ldots \to L_0\mathcal{D}^\diamond$ for L_n a topological localization for $K\mathcal{D}^\diamond$ the K-theory spectrum that represents the étale cohomology of diamonds.*

Conjecture 8. *The $(\infty, 1)$-category of perfectoid diamonds is an $(\infty, 1)$-topos.*

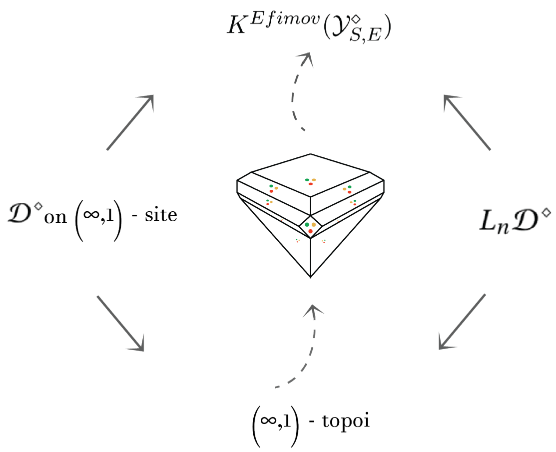

Figure 1: Efimov K-theory and Diamond Chromatic Tower.

3. Efimov K-theory of diamonds

Our terminology and exposition of Efimov K-theory follows Hoyois [4]

∞-categories are called categories. Let $\mathcal{P}r$ denote the category of presentable categories and colimit-preserving functors. Let $\mathcal{P}r^{dual} \subset \mathcal{P}r$ denote the subcategory of dualizable objects and right-adjointable morphisms (with respect to the symmetric monoidal and 2-categorical structures of $\mathcal{P}r$). Let $\mathcal{P}r^{cg} \subset \mathcal{P}r$ be the subcategory of compactly generated categories and compact functors. Compact functors are functors whose right adjoints preserve filtered colimits. Let $\mathcal{P}r^{\star}_{St}$ denote the corresponding full subcategories consisting of stable categories.

Definition 9. A functor $F\colon \mathcal{P}r^{dual}_{St} \to \mathcal{T}$ is called a localizing invariant if it preserves final objects and sends localization sequences to fiber sequences. ◂

Definition 10. Let $C \in \mathcal{P}r$ be stable and dualizable. The continuous K-theory of $\mathcal{C}$ is the space $\mathrm{K}_{cont}(\mathcal{C}) = \Omega$ $\mathrm{K}(Calk(C)^{\omega})$. ◂

Lemma 11. *If $\mathcal{C}$ is compactly generated, then $K_{cont}(\mathcal{C}) = K(\mathcal{C}^{w})$. Proof. The localization sequence is Ind of the sequence $\mathcal{C}^{w} \hookrightarrow \mathcal{C} \to Calk(\mathcal{C})^{w}$. Since $K(\mathcal{C}) = 0$, the result follows from the localization theorem.*

Definition 12. A functor $F\colon \mathcal{P}r^{dual}_{St} \to \mathcal{T}$ is called a localizing invariant if it preserves final objects and sends localization sequences to fiber sequences. ◂

Theorem 13 (Efimov). *Let $\mathcal{T}$ be a category. The functor $Fun(\mathcal{P}r^{dual}_{St}, \mathcal{T}) \longrightarrow Fun(Cat^{idem}_{St}, \mathcal{T})$, $F \mapsto F \circ Ind$, restricts to an isomorphism between the full subcategories of localizing invariants, with inverse $F \mapsto F_{cont}$. In particular, if $\mathcal{C} \in \mathcal{P}r^{cg}_{St}$, then $F_{cont}(\mathcal{C}) = F(C^{\omega})$. Proof. See* [4, Theorem 10].

Theorem 14 (Efimov*). *Let X be a locally compact Hausdorff topological space, $\mathcal{C}$ a stable dualizable presentable category, and R a sheaf of E_1-ring spectra on X. Suppose that $Shv(X)$ is hypercomplete (i.e., X is a topological manifold). Let $\mathcal{T}$ be a stable compactly generated category and $F\colon \mathrm{Cat}^{idem}_{St} \to \mathcal{T}$ a localizing invariant that preserves filtered colimits. Then, $F_{cont}(Mod_R(Shv(X, \mathcal{C}) \simeq \Gamma_c(X, F_{cont}(Mod_R(\mathcal{C})))$. Specifically, $F_{cont}(Shv(\mathbb{R}^n, \mathcal{C})) \simeq \Omega^n F_{cont}(\mathcal{C})$. Proof. See* [4, Theorem 15].

Remark 15. The main goal is to develop a Waldhausen S-construction to obtain the K-theory spectrum $K\mathcal{D}^{\diamond}$ on the $(\infty,1)$-category of diamonds $\mathcal{D}^{\diamond}$. In parallel, to construct a topology on the $(\infty,1)$-category of diamonds, we must first construct the $(\infty,1)$-site on the $(\infty,1)$-category of diamonds. Recall, the definition of an $(\infty,1)$-site. ◂

Definition 16. The $(\infty,1)$-site on an $(\infty,1)$-category $\mathcal{C}$ is the data encoding an $(\infty,1)$-category of $(\infty,1)$-sheaves $Sh(\mathcal{C}) \hookrightarrow PSh(\mathcal{C})$ inside the $(\infty,1)$-category of $(\infty,1)$-presheaves on $\mathcal{C}$ [5]. ◂

Remark 17. $\mathcal{D}^{\diamond}$ admits a topological localization. Recall equivalence classes of topological localizations are in bijection with Grothendieck topologies on $(\infty,1)$-categories C. Topological localizations are appropos because in passing to the full reflective sub-$(\infty,1)$-category, objects and morphisms have reflections in the category, just as geometric points have reflections in the profinitely many copies of $Spa(\mathcal{C})$. ◂

Remark 18. Recall, the category of sheaves on a (small) site is a Grothendieck topos. Lurie discusses the structure needed for our construction. Recall the following [5]. ◂

Definition 19. An $(\infty,1)$-category of $(\infty,1)$-sheaves is a reflective sub-$(\infty,1)$-category $Sh(C) \overset{\overset{L}{\leftarrow}}{\hookrightarrow} PSh(C)$ of an $(\infty,1)$-category of $(\infty,1)$-presheaves such that the following equivalent conditions hold:

(i) L is a topological localization.

(ii) There is the structure of an $(\infty,1)$-site on C such that the objects of $Sh(C)$ are precisely those $(\infty,1)$-presheaves A that are local objects with respect to the covering monomorphisms $p\colon U \to j(c)$ in $PSh(C)$ in that $A(c) \simeq PSh(j(c), A) \xrightarrow{PSh(p,A)} PSh(U, A)$ is an $(\infty,1)$-equivalence in ∞Grpd.

(iii) The $(\infty,1)$-equivalence is the descent condition and the presheaves satisfying it are the $(\infty,1)$-sheaves. ◂

4. Diamonds

The construction of diamonds imitates that of algebraic spaces in taking the quotient of a scheme by an étale equivalence relation. Our terminology and exposition follows [6].

Definition 20. Let *Perfd* be the category of perfectoid spaces and *Perf* be the subcategory of perfectoid spaces of characteristic p. A *diamond* is a pro-étale sheaf $\mathcal{D}$ on *Perf* which can be written as the quotient X/R of a perfectoid space X by a pro-'etale equivalence relation $R \subset X \times X$. ◂

The diamond quotient lives in a category of sheaves on the site of perfectoid spaces with pro-étale covers.

Examples of diamonds are the following:

$SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$. $SpdQ_p$ is the coequalizer of $\underline{Z_p^\times} \times Spa(Q_p^{cycl})^\flat \rightrightarrows Spa(Q_p^{cycl})^\flat$. $SpdQ_p$ attaches to any perfectoid space S of characteristic p the set of all untilts $S\#$ over Q_p. The moduli space of shtukas for $(\mathcal{G}, b, \{\mu_1, ..., \mu_m\})$ fibered over $SpaQ_p \times SpaQ_p ... \times_m SpaQ_p$; the diamond relative Fargues-Fontaine Curve: $\mathcal{Y}_{S,E}^\diamond = S \times (Spa\mathcal{O}_E)^\diamond$; any $\diamond$ product: $SpdQ_p \times_\diamond SpdQ_p$.

Definition 21. Let C be an algebraically closed affinoid field and $\mathcal{D}$ a diamond. A *geometric point* $Spa(C) \to \mathcal{D}$ is "visible" by pulling it back through a quasi-pro-étale cover $X \to \mathcal{D}$, resulting in profinitely many copies of $Spa(C)$. The geometric point $Spa(C) \to \mathcal{D}$ is a mathematical minerological impurity. ◂

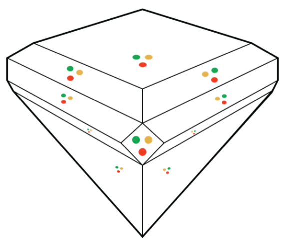

Figure 2: Diamond $SpdQ_p = Spd(Q_p^{cycl})/\underline{Z_p^\times}$ with geometric point $Spa(C) \to \mathcal{D}$.

Remark 22. For a detailed discussion of the author's applications of the six operations and diamonds to quantum gravity, post-quantum diamond cryptography, and nonlocality of perfectoid quantum physics, see [3]. Recall, a perfectoid space is an adic space covered by affinoid adic spaces of the form $Spa(R, R^+)$ where R is a perfectoid ring. Any completion of an arithmetically profinite extension is perfectoid. A nice source of APF extensions is p-divisible formal group laws [6]. ◂

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Why use topological and analytical methods in aggregation of fuzzy preferences?

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Abstract: In this work, we expose our last results in the aggregation of fuzzy preferences. Comparing these results with the literature, we notice a tendency towards using similar methods than those in classical models and, consequently, also leading to impossibility results. We propose using topological or analytical tools to obtain new possibility results.

Resumen: En este trabajo se exponen nuestros últimos resultados obtenidos sobre la posibilidad de agregar preferencias fuzzy bajo diferentes modelos. Comparando esos resultados con la literatura, observamos una tendencia en usar métodos similares a los propios de los modelos clásicos, y por lo tanto con resultados de imposibilidad. Proponemos el uso de herramientas topológicas o analíticas para obtener nuevos resultados de posibilidad.

Keywords: Arrow's impossibility theorem, topology, social choice, fuzzy set theory.

MSC2010: 91B14, 91-10.

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1. Introduction

Arrow's Impossibility Theorem [1] states that there is no function fusing individual preferences into a social one satisfying certain properties of "common sense". On the contrary, in some of the fuzzy extensions of the Arrovian model, possibility arises [5, 6].

In previous works [7], we developed a technique which has been able to prove new impossibility results in the fuzzy approach. Here, we will explain the grounds of this technique and in which models we can apply it.

This technique is based on controlling the aggregation of fuzzy preferences through some aggregation functions of dichotomic preferences. For each fuzzy aggregation function, we get a family of dichotomic aggregation functions. Studying this family, we obtain information about the initial aggregation function. We will discuss why the fuzzy Arrovian models in which we can apply this technique are, in some sense, "less fuzzy". Moreover, we will expose why we should use topological and analytical methods in the fuzzy models out of the scope of our technique.

2. Classic Arrovian model and the theorem of impossibility

Let X be the set including all alternatives involved in a decision. They can be ordered by using binary relations satisfying certain properties. Particularly, in the Arrovian model, these binary relations are total preorders (reflexive, transitive and complete binary relations). Moreover, to give a total preorder on X is equivalent to give a ranking with ties on X.

Every binary relation $\succsim$ factorizes into the relations $\succ$ and $\sim$ defined as: $x \succ y \Leftrightarrow x \succsim y \wedge \neg(y \succsim x)$ and $x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x$. These binary relations are the strict preference (or asymmetric part) and the indifference (or symmetric part) of $\succsim$. If $x \succsim y$ we say that x is at least as good as y, if $x \succ y$ that x is better than y, and if $x \sim y$ that x and y are equally preferred.

Arrow [1] proved that, given a finite set of agents $N = \{1, \dots, n\}$, each one expressing their preferences over a set of alternatives X with total preorders, there is no "fair" rule which aggregates all individual preferences obtaining a social one. Formally, if the set of all total preorders on X is denoted by $\mathcal{O}_X$:

Theorem 1 (Arrow's Impossibility Theorem). *There is no function $f : {\mathcal{O}_X}^n \to \mathcal{O}_X$ on a set of alternatives with $|X| \geq 3$ satisfying, for every $x, y \in X$ and profiles $\succsim, \succsim' \in {\mathcal{O}_X}^n$, the following conditions:*

(i) *Paretian:* $\forall i \in N \; x \succ_i y \Rightarrow x \succ_{f(\succsim)} y$.

(ii) *Independence of irrelevant alternatives (IIA):*

$$\left[\forall i \in N \;\; \succsim_{i|\{x,y\}} = \succsim'_{i|\{x,y\}}\right] \Rightarrow f(\succsim)_{|\{x,y\}} = f(\succsim')_{|\{x,y\}}.$$

(iii) *Non dictatoriship:* $\nexists\, k \in N \,[x \succ_k y \Rightarrow x \succ_{f(\succsim)} y]$.

Given this result, many researchers looked for alternative ways to aggregate preferences. We will focus on using fuzzy sets to find new aggregation methods.

3. Extending the Arrovian model to the fuzzy setting

Studying the Arrovian model in the fuzzy framework consists in generalizing the objects and the properties from the previous section, and checking if the aggregation of preferences is possible in the new framework. All these properties can be generalized in different manners. So, a huge number of fuzzy Arrovian models is obtained.

In the fuzzy setting, a preference is a fuzzy binary relation $R : X \times X \to [0, 1]$. There are many generalizations of the crisp strict preference $\succ$ (of $\succsim$) to the fuzzy strict preference P_R (of R). For every fuzzy Arrovian model, we have to set a method of factorization to obtain P_R.

The properties of preferences $\succsim$ can be generalized to the fuzzy setting in different ways. For instance, the transitivity can be extended saying that R is T-transitive (with T a t-norm) if, $\forall x, y, z \in X$, $R(x, z) \geq T(R(x, y), R(y, z))$. However, it also may be generalized to the weak transitivity defined as $R(x, y) \geq R(y, x) \wedge R(y, z) \geq R(z, y) \Rightarrow R(x, z) \geq R(z, x)$. The completeness can be generalized to being S-connected (with S a t-conorm) as $\forall x, y \in X\ S(R(x, y), R(y, x)) = 1$.

Let $\mathcal{FP}$ be a set of fuzzy preferences on X. An aggregation fuzzy rule is a function $f : \mathcal{FP}^n \to \mathcal{FP}$. Arrow axioms can also be generalized in various ways. For instance, the Paretian property may be generalized to the weakly (resp. strongly) Paretian property as $\forall x, y \in X\ P_{R_i}(x, y) > 0 \Rightarrow P_{f(\mathbf{R})}(x, y) > 0$ (resp. $P_{f(\mathbf{R})}(x, y) \geq \min_{i \in N} P_{R_i}(x, y)$). The dictatorship may be extended to the weak (resp. strong) dictatorship as $\exists k \in N\ P_{R_k}(x, y) > 0 \Rightarrow P_{f(\mathbf{R})}(x, y) > 0$ (resp. $\forall t \in [0, 1]\ P_{R_k}(x, y) > t \Rightarrow P_{f(\mathbf{R})}(x, y) > t$. And the IIA may be generalized to $\forall x, y \in X\ [\forall i \in N\ R_i \approx_{\{x,y\}} R'_i \Rightarrow f(\mathbf{R}) \approx_{\{x,y\}} f(\mathbf{R}')]$, where $\approx_{\{x,y\}}$ can be defined as, $R \approx^1_{\{x,y\}} R' \Leftrightarrow R_{]\{x,y\}} = R'_{]\{x,y\}}$, $R \approx^2_{\{x,y\}} R' \Leftrightarrow \operatorname{supp}(R_{]\{x,y\}}) = \operatorname{supp}(R'_{]\{x,y\}})$ or $R \approx^3_{\{x,y\}} R' \Leftrightarrow R \approx^2_{\{x,y\}} R' \wedge [\forall \overline{z}, \overline{z}' \in \{x, y\}^2\ R(\overline{z}) > R(\overline{z}') \Leftrightarrow R'(\overline{z}) > R(\overline{z}')]$, among others (see [8]).

4. Studying fuzzy aggregation using crisp preferences

In this section, we draft a strategy to study fuzzy aggregation functions using the Arrovian theorem and other combinatorial techniques from the crisp model.

Consider a set of fuzzy preferences $\mathcal{FP}$ were all its preferences are reflexive and satisfy one type of fuzzy transitivity and one type of fuzzy connectedness. Then, we define a projection p from $\mathcal{FP}$ to a set of crisp preferences $\mathcal{B}$ on X. These projections are interpreted as collapsing the fuzzy preferences into its qualitative factor (a crisp binary relation). Some examples of projections are:

(i) If R is a weak transitive and S-connected preferences, $\succsim^1_R$ defined as $x \succsim^1_R y \Leftrightarrow R(x, y) \geq R(y, x)$ is a total preorder.

(ii) If R is a T-transitive and max-connected preference, $\succsim^2_R$ defined as $x \succsim^2_R y \Leftrightarrow R(x, y) = 1$ is a total preorder.

(iii) If R is a min-transitive and S-complete preference, $\succsim^3_R$ defined as $x \succsim^3_R y \Leftrightarrow R(x, y) \geq R(y, x)$ is a quasi-transitive binary relation.

The second step is finding the same but applied to aggregation functions. Here, given a fuzzy aggregation function f and n embeddings $\iota_i : \mathcal{B} \to \mathcal{FP}$, we define $f_\iota := p \circ f \circ (\iota_i \times \cdots \times \iota_n)$. We have to choose the right embeddings in order for f_ι to be an Arrovian aggregation function. Then, each f_ι is dictatorial. However, they may have different dictators. When all of them have the same dictator, and the image of all embeddings covers $\mathcal{FP}$, we can ensure that f is dictatorial.

Let $\mathcal{P}$ be the set of weak transitive and S-connected fuzzy preferences on X. Using the strategy above, we proved in [7] the following theorem:

Theorem 2. *Let $f : \mathcal{P}^n \to \mathcal{P}$ be a fuzzy aggregation function satisfying IIA defined by $\{\approx^3_{\{x,y\}}\}_{x,y \in X}$ and weakly Paretian. Then, f is dictatorial.*

The theorem above is an example illustrating that we can reduce the study of a fuzzy model to the study of a family of crisp functions from the Arrovian model (and we obtain an impossibility result), then the fuzziness of the model is an illusion.

In the next section, we will see the relation of some aggregation functions with the projections exposed in the beginning of the present section.

5. Aggregation functions using ordinal expressions

These *illusory fuzziness* arises when we study the fuzzy Arrovian aggregation functions in the literature. We can consider some of these expressions. In [5] there is an aggregation function defined as $f(\mathbf{R})(x, y) = 1$ if $\forall i \in N\ R_i(x, y) > R_i(y, x)$, and $f(\mathbf{R})(x, y) = 0.5$ otherwise. In [6] we find an aggregation function defined as

$f(\mathbf{R})(x, y) = \frac{1}{n} \sum_{i \in N} R_i(x, y)$. Finally, in [4] we find $f(\mathbf{R})(x, y) = \text{median}\{\min_i\{R_i(x, y)\}, h, \max_i\{R_i(x, y)\}\}$, where $T(h, h) = 0$ for any $h \in (0, 1)$.

Notice that in the first and the third functions, the same expression we used in $\succsim_R^1$ and $\succsim_R^3$ is employed, and the second is the well-known arithmetic mean. These three examples represent the present situation in the existing literature. All functions are built using the reasoning based on crisp binary relations or testing pre-existing well-known algebraic expressions as means.

If we look for functions capturing the vagueness, we should think out of the box of crisp binary relations. Moreover, testing the functions with an algebraic expression we know does not seem a suitable method. For these reasons, we stand up for the methods explained in the next section.

6. Conclusions and future research

In order to get more satisfactory results and classify the fuzzy Arrovian models, we cannot rely on functions built as algebraic expressions or close to binary relations. We need a richer framework able to express the vagueness, and it cannot be constrained by human dichotomic thinking.

We propose using topological or analytical tools to build this general framework. Using the fact that the degrees of a preference are in $[0, 1]$, we can interpret a preference as a point in the cube $[0, 1]^{X^2}$, the spaces of preferences as topological subspaces of $[0, 1]^{X^2}$, and the aggregation functions as continuous functions (see [2] for an extended discussion). Using this framework, we expect to find suitable aggregation functions with no need to write them explicitly. For example, using differential equations.

It is important to remark that our approach is different from the topological models proposed by Chichilnisky [3]. We depart from a model with no topological structure, whereas Chichilnisky built her models using a topological background.

Considering our conclusions, we are working on finding a general framework to create suitable binary relation form fuzzy preferences and use them to study fuzzy aggregation functions. Furthermore, we will continue the study initiated in [2] about how fuzzy Arrovian models can be translated to differential equations.

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Model theory and metric approximate subgroups

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Abstract: In combinatorics, approximate subgroups are objects similar to subgroups up to a constant error. In 2011, using model theory, a connexion was found between finite approximate subgroups and Lie groups. This result, known as the Lie model theorem, was the starting point used to finally give a complete classification of finite approximate subgroups.

This short essay is a partial summary of a joint paper with Ehud Hrushovski (in current development) in which we prove a generalization of the Lie model theorem to the case of metric groups.

Resumen: En combinatoria, los subgrupos aproximados son objetos semejantes a subgrupos salvo un error constante. En 2011, usando teoría de modelos, se encontró una relación entre subgrupos aproximados finitos y grupos de Lie. Este resultado, conocido como el teorema de modelos de Lie, fue el punto de partida para establecer finalmente una clasificación completa de los subgrupos aproximados finitos.

Este breve ensayo es un resumen parcial de un artículo conjunto con Ehud Hrushovski (en desarrollo) en el que obtenemos una generalización del teorema de modelos de Lie al caso de grupos métricos.

Keywords: additive combinatorics, approximate subgroups, metric groups, model theory.

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1. The Lie model theorem

Approximate subgroups are basic combinatorial structures modeling objects similar to subgroups up to a constant error. Although first definitions where given in the abelian setting by Freiman in 1973 and Ruzsa in 1994, the current definition was definitely established by Tao [4].

Notation 1. Here, for subsets of a group $X, Y \subseteq G$, we write $XY := \{xy : x \in X, y \in Y\}$ and $X^2 := XX$. ◂

Definition 2. A k*-approximate subgroup* of a group G is a symmetric subset $X \subseteq G$ containing the identity such that $X^2 \subseteq \Delta X$ for some $\Delta \subseteq G$ with $|\Delta| < k$.

We say that two subsets $X, Y \subseteq G$ are k*-commensurable* if there is $\Delta \subseteq G$ such that $X \subseteq \Delta Y$ and $Y \subseteq \Delta X$ with $|\Delta| \leq k$. ◂

Example 3 (geometric progressions). Let G be abelian and $u_1, \dots, u_m \in G$. The set of words $w(\bar{u})$ in G with at most N_i occurrences of u_i is a 2^m-approximate subgroup. ◂

Example 4 (nilprogressions). Let G be nilpotent of nilpotent length s and $u_1, \dots, u_m \in G$. The set of words $w(\bar{u})$ with at most N_i occurrences of u_i is an $k(s, m)$-approximate subgroup. ◂

Using model theory, Hrushovski [2] found a connexion between approximate subgroups and Lie groups. This result, known as the Lie model theorem, was the starting point used to finally give a complete classification of finite approximate subgroups by Breuillard, Green and Tao.

To explain Hrushovski's result, we need to introduce the model theoretic notion of ultraproduct. The idea is to construct models by taking "limits". Formally, we consider a sequence $(\mathfrak{M}_m)_{m\in\mathbb{N}}$ of structures (e.g., groups, graphs, fields, linear orders) and an non-atomic measure $\mathfrak{u} : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$. The *ultraproduct* $\widehat{\mathfrak{M}} = \prod \mathfrak{M}_m/\mathfrak{u}$ is the set of sequences $x \in \prod \mathfrak{M}_m$ modulo the equivalence relation $x = x'$ almost surely.

A fundamental theorem by Łoś says that the ultraproduct satisfies all first-order properties which are almost surely satisfied by the factors. More generally, Łoś's theorem says us that, for any first-order formula $\varphi(x)$ in $\widehat{\mathfrak{M}}$ (possibly with parameters), the definable subset $\widehat{A} = \varphi(\widehat{\mathfrak{M}}) := \{a : a \text{ satisfies } \varphi \text{ in } \widehat{\mathfrak{M}}\}$ of $\widehat{\mathfrak{M}}$ can be written as the ultraproduct $\widehat{A} = \prod A_m/\mathfrak{u}$ where $A_m = \varphi(\mathfrak{M}_m)$. The topology generated by taking as clopen subsets the definable subsets is called the logic topology.

Example 5 (non-standard analysis). When $\mathfrak{M}_m = \mathbb{R}$, we get a model of the hyperreal numbers $\widehat{\mathbb{R}}$. ◂

Morally, the Lie model theorem says that every finite approximate subgroup is "in the limit" commensurable to a compact neighbourhood of the identity of some Lie group.

Theorem 6 (Hrushovski's Lie model theorem). *Let $\widehat{G}$ be an ultraproduct of groups and $\widehat{X} \subseteq \widehat{G}$ an ultraproduct of respective finite k-approximate subgroups. Then, there exists a Lie model of $\widehat{X}$, i.e., a surjective group homomorphism $\pi : H \leq G \to L$ where*

(i) *L is a connected Lie group,*
(ii) *$X^8 \cap H$ is an approximate subgroup, generates H and is commensurable to X,*
(iii) *$K := \ker \pi \subseteq X^4$, and*
(iv) *π is continuous and closed from the logic topology (with enough parameters).*

Using this result, Breuillard, Green, and Tao [1] concluded that every finite approximate subgroup is commensurable to a nilprogression modulo some normal subgroup.

Theorem 7 (Breuillard-Green-Tao Classification Theorem). *In the Lie model theorem, L is nilpotent and K could be made definable taking enough structure. Hence, we get the following result:*

Let G be a group and X a finite k-approximate subgroup. Then, there are $H \leq G$ and $K \trianglelefteq H$ such that

(i) *$H \cap X^8$ is $C(k)$-commensurable to X and generates H,*
(ii) *$K \subseteq X^4$, and*
(iii) *H/K is a nilpotent group of $s(k)$ nilpotent length.*

2. The metric Lie model theorem

Approximate subgroups could be further generalized to the context of metric groups. By a *metric group* we understand a group G together with a metric d invariant under translations.

Notation 8. Write $\mathbb{D}_r(X) = \{y : \text{there is } x \in X\, d(x,y) < r\}$. Note that $\mathbb{D}_r(X) = X\mathbb{D}_r(1) = \mathbb{D}_r(1)X$. ◂

Remark 9. We assume invariance under two-side translations to simplify the statements. In fact, it is possible to assume only that d is invariant under left (alternatively right) translations and some local Lipschitz condition for the right (alternatively left) translations. ◂

Definition 10. A δ-metric k-approximate subgroup $X \subseteq G$ is a symmetric subset $1 \in X^{-1} = X$ such that $X^2 \subseteq \Delta\mathbb{D}_\delta(X)$ with $|\Delta| \le k$.

We say that two subsets $X, Y \subseteq G$ are *δ-metrically k-commensurable* if there is $\Delta \subseteq G$ such that $X \subseteq \Delta\mathbb{D}_\delta(Y)$ and $Y \subseteq \Delta\mathbb{D}_\delta(X)$ with $|\Delta| \le k$. ◂

We do no longer assume finiteness, instead we assume that using the metric we can find nice discretizations. An *r-entropy discretization* of a set X is an r-separated finite subset $Z \subseteq X$ of maximal size. Write $N^{\mathrm{ent}}_r(X) = \sup\{|Z| : Z \subseteq X\ r\text{-separated}\}$. If there are arbitrary large r-separated finite sets, write $N^{\mathrm{ent}}_r(X) = \infty$.

Remark 11. N^{ent} is subadditive and decreasing on r. ◂

Our aim is to generalize Hrushovski's Lie model theorem to the case of metric approximate subgroups. In our case, the ultraproduct of metric groups is then a *non-standard metric group*, i.e., a group $\widehat{G}$ together with a function $\hat{d} : \widehat{G} \times \widehat{G} \to \widehat{\mathbb{R}}$ into the hyperreal numbers that satisfies the usual properties of a metric.

For a sequence $r = (r_n)_{n\in\mathbb{N}}$ of non-standard positive numbers in $\widehat{\mathbb{R}}$ with $2r_{n+1} < r_n$, we define the *r-infinitesimal thickening* of X by

$$o_r(X) \coloneqq \bigcap_{n=0}^{\infty} \mathbb{D}_{r_n}(X) = \{g \in \widehat{G} : \forall n \in \mathbb{N}\ \exists x \in X\ d(g,x) < r_n\}.$$

It follows that $o_r(1_G) \trianglelefteq \widehat{G}$. Also, $o_r(X) = Xo_r(1_G) = o_r(1_G)X$.

Now, we quotient out by $o_r(1)$ and check that the original arguments done by Hrushovski with $\widehat{X}$ could be adapted to $\widehat{X}/o_r(1)$. This requires to generalize various model theoretic techniques to the context of piecewise hyperdefinable sets as it was done in [3].

Theorem 12 (metric Lie model [2]). *Let (G_m, X_m, r^m) be a sequence such that*

1. *G_m is a metric group,*
2. *X_m is a symmetric subset containing the identity,*
3. *$r^m = (r^m_1, \dots, r^m_m)$ satisfies $r^m_i \ge 2r^m_{i+1}$ and*

$$N^{\mathrm{ent}}_{r^m_i/2}(X^9_m) \le C \cdot N^{\mathrm{ent}}_{9r^m_i/2}(X_m) \in \mathbb{R} \text{ for each } i.$$

Let $\widehat{G} = \prod_{m\in\mathbb{N}} G_m/\mathfrak{u}$, $\widehat{X} = \prod_{m\in\mathbb{N}} X_m/\mathfrak{u}$ and $r = r^m/\mathfrak{u}$ be ultraproducts. Then, there is a Lie model of $o_r(\widehat{X})$, i.e., a surjective group homomorphism $\pi : H \le G \to L$ where

(i) *L is a connected Lie group,*

(ii) *$H \cap o_r(\widehat{X}^{12})$ is a C-approximate subgroup, generates H and is commensurable to $o_r(\widehat{X})$,*

(iii) *$K = \ker \pi \subseteq o_r(\widehat{X}^8)$ and $o_r(1_G) \le K$,*

(iv) *π is continuous and closed from the logic topology (with enough parameters).*

Hence, as a corollary of the metric Lie model theorem we get the following result for metric approximate subgroups.

Corollary 13. *Fix constants $k \in \mathbb{R}_{>0}$, $C \in \mathbb{R}_{>1}$, $\delta \in \mathbb{R}_{>0}$ and $N, p, q \in \mathbb{N}$. Take $\alpha \geq 144^2$. There are $e := e(k, C)$ and $m := m(k, C, N, p)$ such that the following holds.*

Let G be a metric group and X a δ-metric k-approximate subgroup such that

$$N^{\mathrm{ent}}_{\delta}(X) \leq C^m N^{\mathrm{ent}}_{\alpha^m \delta}(X) \in \mathbb{R}_{>0}.$$

Then, there is a sequence $X_N \subseteq \cdots \subseteq X_1 \subseteq X^8$ satisfying the following properties:

(i) *X^2 and X_1 are $2^{-p}\alpha^m$ δ-metrically e-commensurable.*

(ii) *$X_{n+1}X_{n+1} \subseteq \mathbb{D}_{2^{-p}\alpha^m\delta}(X_n)$.*

(iii) *X_n is covered by e right cosets of $\mathbb{D}_{2^{-p}\alpha^m\delta}(X_{n+1})$.*

(iv) *$X_{n+1}^{X_1} \subseteq \mathbb{D}_{2^{-p}\alpha^m\delta}(X_n)$.*

(v) *$[X_{n_1}, X_{n_2}] \subseteq \mathbb{D}_{2^{-p}\alpha^m\delta}(X_n)$ whenever $n < n_1 + n_1$.*

(vi) *$\{x \in X_1 : x^2, x^4 \in X_1$ and $x^8 \in X_n\} \subseteq X_{n+1}$.*

(vii) *If $x, y \in X_1$* with *$x^2 = y^2$, then $y^{-1}x \in \mathbb{D}_{2^{-p}\alpha^m\delta}(X_N)$.*

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Principles of least action in geometric mechanics

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Abstract: A path c is said to be a solution of Hamilton's least action principle if it is a critical point of the action functional. Here, the action is the integral of a Lagrangian function L along c. This principle describes many physical theories, and has applications in other fields (optimal control, Riemannian geometry). Its solutions have a nice geometric characterization: they are integral curves of a Hamiltonian vector field on a symplectic manifold.

We introduce a generalization of this principle: the so-called *Herglotz's principle*. Here the Lagrangian not only depends on the positions and velocities, but also on the action itself. Hence, the action is no longer the integral of the Lagrangian, but it is the solution of a non-autonomous ODE. Herglotz's principle allows us to model new problems, such as some dissipative systems in mechanics (where energy is lost), thermodynamics, and some modified optimal control systems. This principle is also related to Hamiltonian systems, but switching symplectic by contact geometry. We will compare both principles, their applicability and the geometric properties of their solutions.

Resumen: Un camino c es una solución del principio de mínima acción de Hamilton si es un punto crítico del funcional de acción. En este caso, la acción es la integral de una función lagrangiana L a lo largo de c. Este principio describe numerosas teorías físicas y tiene aplicaciones en otros campos (control óptimo, geometría riemmaniana). Sus soluciones tienen una interesante caracterización geométrica: son las curvas integrales de un campo Hamiltoniano en una variedad simpléctica.

Proponemos una generalización the este principio: el *principio de Herglotz*. Ahora, el lagrangiano depende de la propia acción, además de las posiciones y velocidades. Aquí, la acción ya no es la intregral del lagrangiano, sino la solución a una EDO no autónoma. El principio de Herglotz nos permite modelizar nuevos problemas, como algunos sistemas disipativos en mecánica (con pérdidas de energía), termodinámica y algunos problemas de contol óptimo. Este principio también está relacionado con los sistemas Hamiltonianos, pero cambiando la geometría simpléctica por geometría de contacto. Compararemos ambos principios, sus aplicaciones y las propiedades geométricas de sus soluciones.

Keywords: variational principles, Herglotz principle, contact Hamiltonian systems, Lagrangian mechanics.

MSC2010: 70H30, 37J55.

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1. Principles of least action

In the 17th century, Fermat formulated the laws of geometric optics in the following way: "light travels between two given points along the path of shortest time". This is known as *the principle of least time* or *Fermat's principle*. Knowing the velocity of light at every point of space, one can use this principle to compute the trajectories of the light rays, obtaining the laws of refraction and reflection.

Many principles such as this one were introduced in mechanics, by Maupertuis, Euler, Lagrange and Hamilton. Although they have different physical interpretation, all these principles (including Fermat's) fit on the same mathematical framework. Given a Lagrangian function $L: TQ \times \mathbb{R} \to \mathbb{R}$[1], let Ω be the space[2] of curves $c: [0, T] \to Q$ with fixed endpoints (say, $c(0) = q_0$, $c(T) = q_1$). We define the action $\mathcal{A}: \Omega \to \mathbb{R}$ of any curve c as

$$\mathcal{A}(c) = \int_{t_0}^{t_1} L(c(t), \dot{c}(t))\, \mathrm{d}t.$$

The principle of least action states that a path c will be followed by the system if and only if c is a critical point of $\mathcal{A}$ among all paths in Ω. The solutions of this principle are precisely the paths that satisfy the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}(c(t), \dot{c}(t))\right) - \frac{\partial L}{\partial q^i}(c(t), \dot{c}(t)) = 0.$$

Picking as a Lagrangian the inverse of the velocity of light in the media, we retrieve Fermat's principle. If we instead pick as the Lagrangian the kinetic minus the potential energy, $L = T - V$, we obtain *Hamilton's principle* for conservative mechanical systems (where energy remains constant), whose solutions satisfy Newton's Second Law[3].

1.1. Why variational principles?

There are many mathematical and physical[4] reasons to study variational principles. In physics, it has been found that the least action principle (sometimes with extensions) can model a wide range of phenomena, including field theory and general relativity. Furthermore, developments on this principle lead to quantum field theory (through Feynman path integral). Outside of physics, least action principles appear in control theory (optimal control problems) and characterize geodesics in Riemannian and Finsler geometry. If we are working with a second order ODE that is the Euler-Lagrange equation of some Lagrangian also provides access to useful mathematical tools.

- The problem is framed in "*generalized coordinates*", i.e., the Euler-Lagrange equation is the same on every coordinate system[5]. This does not hold with Newton's equation, where new terms appear when we work in non-cartesian coordinates or in non-inertial frames.
- Presence of *symplectic geometry* [1, 3]. A (regular) Lagrangian provides a symplectic form $\omega_L = \mathrm{d}q^i \wedge \mathrm{d}(\partial L/\partial \dot{q}^i)$ which is preserved by the evolution of the system. Knowledge on the topology and geometry of symplectic manifolds provides a better understanding on the dynamics of the system.
- It allows to prove *Noether theorems* relating symmetries and conserved quantities.
- It can be used to construct *variational integrators* [12], that preserve the geometry of the system and have better long term behavior than methods for more general ODEs, such as Runge-Kutta.

[1] Here, TQ is the tangent bundle of the configuration manifold TQ, i.e., the space of positions and velocities, with coordinates $(q^i, \dot{q}^i)$.

[2] This space is an infine-dimensional manifold locally modeled on a space of functions $[0, 1] \to \mathbb{R}^n$. We recommend the interested reader [1] and the references therein.

[3] In cartesian coordinates x^i, if $L = \frac{1}{2} m \sum_i (\dot{x}^i)^2 - V(q^i)$, then the equation of motion is $m\ddot{x}^i = -\frac{\partial V}{\partial x^i} = F_i$, where F_i is the force.

[4] Some natural philosophers, such as Maupertuis, were interested in these principles on metaphysical grounds, since they express that nature "acts by the simplest means"[13]. These arguments would now probably be considered unscientific.

[5] Now, within the framework of geometric mechanics, modern differential geometric language is used and the dynamics can be described without the use of coordinates.

2. Herglot's variational principle

There are, however, many interesting systems that cannot be modeled with Hamilton's principle. For example, all mechanical systems that do not preserve the energy, such as the damped harmonic oscillator:

$$\ddot{q}^2 + q = -\gamma \dot{q}. \tag{1}$$

A simple extension is to allow that the Lagrangian depends explicitly on time, however this is not enough on many situations. In 1930, Herglotz [11] proposed a more general formulation, the so-called *Herglotz's variational principle*. Here the Lagrangian not only depends on the positions and velocities, but also on the action itself. Hence, the action is no longer the integral of the Lagrangian, but it is the solution of a non-autonomous ODE. This will allow us to model a wider class of systems.

2.1. Herglotz's principle and Herglotz's equations

Let $L\colon TQ \times \mathbb{R} \to \mathbb{R}$ be the Lagrangian function, where the last coordinate will be denoted by z[6]. In order to formalize the idea of an "action dependent Lagrangian", we will define the action through a non-autonomous ODE, instead of an integral. First we fix the *initial action* $z_0 \in \mathbb{R}$, and we define the Herglotz action $\mathcal{A}\colon \Omega \to \mathbb{R}$ as follows. Given $c \in \Omega$, we solve the Cauchy problem $\dot{z}_c = L(c, \dot{c}, z_c)$ with initial condition $z_c(0) = z_0$. Now we define the Herglotz action[7] $\mathcal{A}$ as

$$\mathcal{A}(c) = z_c(T) - z_0 = \int_0^T L(c(t), \dot{c}(t), z_c(t))\,\mathrm{d}t.$$

In this case, c is a critical point of $\mathcal{A}\,:\,\Omega \to \mathbb{R}$ if and only if $(c, \dot{c}, z_c)$ satisfies Herglotz's equations [6]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}.$$

We note that the energy $E_L = L - \dot{q}^i \dfrac{\partial L}{\partial \dot{q}_i}$ is dissipated along the solutions χ of Herglotz equations at a rate $\partial L/\partial z$. Indeed, if we pick $L = \frac{1}{2}(\dot{q})^2 - q - \gamma z$, the Herglotz equation is the equation of motion of the damped harmonic oscillator (1), and we have $\mathrm{d}E_L/\,\mathrm{d}t = -\gamma E_L$.

3. Further topics

In the recent years a considerable amount of new results related to the Herglotz principle and Lagrangian contact mechanics have been published. We list some of the topics on which there is active research.

- *Contact geometry* is to Herglotz's principle [7] as symplectic geometry is to Hamilton's principle. A contact form $\eta_L = \mathrm{d}z - \partial L/\partial \dot{q}^i \mathrm{d}q^i$ is preserved by the flow of the system.
- *Noether theorems* [8] also exist in this context. However, symmetries do not correspond to conserved, but to *dissipated quantities*, that is, quantities that decay at the same rate as the energy.
- *Variational integrators* can be constructed through the Herglotz principle [15, 16].
- Herglotz's principle and some related variational principles can be applied to the description of *thermodynamic processes* [14] and mechanical systems with dissipation [2], among others.

[6] We can also use Hamilton's principle for explicitly time dependent Lagrangians $L\colon TQ \times \mathbb{R} \to \mathbb{R}$, where we think of the $\mathbb{R}$ coordinate as "time" t. The corresponding Euler-Lagrange equations have the same form as in Hamilton's principle. These should not be confused with contact Hamiltonian systems and Herglotz's variational principle, where the extra coordinate represents the "action".

[7] We remark that this action coincides with the Euler-Lagrange action when L does not depend on z. It is also important to note that the action functional does not only depend on the Lagrangian, like in Hamilton's principle, but also on the initial action z_0.

- *Higher order systems* can be considered [4]. Lagrangians depend not only on positions and velocities, but also on higher order derivatives.
- Constraints can be added to the motion of these systems. They can be either *vakonomic*, that is, implemented on the variations, or *nonholonomic* [5], on the infinitesimal variations. The first ones are useful for optimal control theory [9], while the second ones appear on mechanical systems.
- We can also study *the inverse problem.* Given a second order ODE, does there exist a Lagrangian such that the ODE is its Euler-Lagrange/Herglotz equation?
- Contact Lagrangian mechanics can be extended to *noncorservative field theories* [10].

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On Grünbaum type inequalities

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Abstract: Given a compact set $K \subset \mathbb{R}^n$ of positive volume, and fixing a hyperplane H passing through its centroid, we find a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\,\mathrm{vol}(K)$, depending on the concavity nature of the function that gives the volumes of cross-sections (parallel to H) of K, where K^- denotes the intersection of K with a halfspace bounded by H. When K is convex, this inequality recovers a classical result by Grünbaum. To this respect, we also show that the log-concave case is the limit concavity assumption for such a generalization of Grünbaum's inequality.

Resumen: Dado un conjunto compacto $K \subset \mathbb{R}^n$ y un hiperplano H pasando por su centroide, encontramos una cota inferior óptima para el cociente $\mathrm{vol}(K^-)/\,\mathrm{vol}(K)$, dependiendo de la concavidad de la función que nos da el volumen de las secciones (paralelas a H) de K, donde K^- denota la intersección de K con el semiespacio delimitado por H. Cuando K es convexo, esta desigualdad recupera un resultado clásico de Grünbaum. Además, veremos que el caso log-cóncavo es la mínima concavidad exigible para este tipo de generalización de la desigualdad de Grünbaum.

Keywords: centroid, convex body, Grünbaum, inequality.

MSC2010: 52A40, 52A38, 52A20.

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1. Introduction

Let $K \subset \mathbb{R}^n$ be a compact set with positive volume $\mathrm{vol}(K)$ (i.e., with positive n-dimensional Lebesgue measure). The centroid of K is the affine-covariant point

$$\mathrm{g}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x \,\mathrm{d}x.$$

Furthermore, if we write $[\cdot]_1$ for the first coordinate of a vector with respect to the basis, by Fubini's theorem, we get

$$[\mathrm{g}(K)]_1 = \frac{1}{\mathrm{vol}(K)} \int_a^b t f(t) \,\mathrm{d}t. \tag{1}$$

The classical Grünbaum inequality, originally proven in [2], states that if $K \subset \mathbb{R}^n$ is a convex body with centroid at the origin, then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n, \tag{2}$$

where $K^- = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 0\}$ and $K^+ = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \geq 0\}$ represent the parts of K which are split by the hyperplane $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$, for any given $u \in \mathbb{S}^{n-1}$. Equality holds, for a fixed $u \in \mathbb{S}^{n-1}$, if and only if K is a cone in the direction u, i.e., the convex hull of $\{x\} \cup (K \cap (y + H))$, for some $x, y \in \mathbb{R}^n$.

The underlying key fact in the original proof of (2) (see [2]) is the following classical result (see, e.g., [1, Section 1.2.1] and also [4, Theorem 12.2.1]).

Theorem 1 (Brunn's concavity principle). *Let $K \subset \mathbb{R}^n$ be a non-empty compact and convex set and let H be a hyperplane. Then, the function $f : H^\perp \to \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is $(1/(n-1))$-concave.*

In other words, for any given hyperplane H, the cross-sections volume function f to the power $1/(n-1)$ is concave on its support, which is equivalent (due to the convexity of K) to the well-known *Brunn-Minkowski inequality*.

Although this property cannot be in general enhanced, one can easily find compact convex sets for which f satisfies a stronger concavity, for a suitable hyperplane H. Thus, on the one hand, it is natural to wonder about a possible enhanced version of Grünbaum's inequality (2) for the family of those compact convex sets K such that (there exists a hyperplane H for which) f is p-concave, i.e., f to the power p is concave, with $1/(n-1) < p$. On the other hand, one could expect to extend this inequality to compact sets K, not necessarily convex, for which f is p-concave (for some hyperplane H), with $p < 1/(n-1)$.

Observing that the equality case in Grünbaum's inequality (2) is characterized by cones, that is, those sets for which f is $(1/(n-1))$-affine (i.e., such that $f^{1/(n-1)}$ is an affine function), the following sets of revolution, associated to p-affine functions, arise as natural candidates to be the extremal sets, in some sense, of these inequalities.

Definition 2. Let $p \in \mathbb{R}$ and let $c, \gamma, \delta > 0$ be fixed. Then:

(i) If $p \neq 0$, let $g_p : I \to \mathbb{R}_{\geq 0}$ be the non-negative function given by $g_p(t) = c(t+\gamma)^{1/p}$, where $I = [-\gamma, \delta]$ if $p > 0$ and $I = (-\gamma, \delta]$ if $p < 0$.

(ii) If $p = 0$, let $g_0 : (-\infty, \delta] \to \mathbb{R}_{\geq 0}$ be the non-negative function defined by $g_0(t) = c\mathrm{e}^{\gamma t}$.

Let $u \in \mathbb{S}^{n-1}$ be fixed. By C_p we denote the set of revolution whose section by the hyperplane $\{x \in \mathbb{R}^n : \langle x, u\rangle = t\}$ is an $(n-1)$-dimensional ball of radius $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ with axis parallel to u. (We warn the reader that, in the following, we will use the word "radius" to refer to such a generating function $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ of the set C_p, for short.) ◂

In this short paper we discuss the above-mentioned problem and show that it has a positive answer in the full range of $p \in [0, \infty]$ (in the following, $\sigma_{H^\perp}$ denotes the *Schwarz symmetrization* with respect to $H^\perp$).

2. Main results

As mentioned in the introduction, the sets C_p associated to (cross-sections volume) functions that are p-affine (see Definition 2) seem to be possible extremal sets of such expected inequalities. So, we start by showing the precise value of the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$ for the sets C_p.

Lemma 3 ([3]). *Let $p \in (-\infty,-1) \cup [0,\infty)$ and let H be a hyperplane with unit normal vector $u \in \mathbb{S}^{n-1}$. Let g_p and D_p, with axis parallel to u, be as in Definition 2, for any fixed $c,\gamma,\delta > 0$. If C_p has centroid at the origin, then*

$$\frac{\mathrm{vol}\big(C_p^-\big)}{\mathrm{vol}(C_p)} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{3}$$

where, if $p=0$, the above identity must be understood as

$$\frac{\mathrm{vol}\big(C_0^-\big)}{\mathrm{vol}(C_0)} = \lim_{p\to 0^+}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p} = \mathrm{e}^{-1}. \tag{4}$$

Before showing the general case, we have that if the cross-sections volume function f associated to a compact set K is increasing in the direction of the normal vector of H, then the minimum of the ratios $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ and $\mathrm{vol}(K^+)/\mathrm{vol}(K)$ is attained at $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, independently of the concavity nature of f.

Proposition 4 ([3]). *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane, with unit normal vector $u \in \mathbb{S}^{n-1}$, such that the function $f : H^\perp \to \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}(K \cap (x+H))$ is quasi-concave with $f(bu) = \max_{x\in H^\perp} f(x)$, where $[au,bu] = K|H^\perp$. Then,*

$$\frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)} \geq \frac{1}{2}.$$

Our main result reads as follows:

Theorem 5 ([3]). *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane such that the function $f : H^\perp \to \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}(K \cap (x+H))$ is p-concave, for some $p \in [0,\infty)$. If $p > 0$, then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{5}$$

with equality if and only if $\sigma_{H^\perp}(K) = C_p$. If $p = 0$, then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \mathrm{e}^{-1}. \tag{6}$$

The inequality is sharp, that is, the quotient $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ comes arbitrarily close to e^{-1}.

Note that the "limit case" $p=\infty$ in Theorem 5 is also trivially fulfilled. Indeed, if f is ∞-concave, then f is constant on $[a,b]$, and thus $0 = [\mathrm{g}(K)]_1 = b + a$ (see (1)), which yields that $a=-b$ and, hence,

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{1}{2} = \lim_{p\to\infty}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

Finally, we show that Theorem 5 cannot be extended to the range of $p \in (-\infty,-1)$. In fact, we have a more general result:

Proposition 6 ([3]). *Let $p \in (-\infty,-1)$. There exists no positive constant β_p such that*

$$\min\left\{\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}, \frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}\right\} \geq \beta_p$$

for all compact sets $K \subset \mathbb{R}^n$ with non-empty interior and with centroid at the origin, for which there exists H such that $f(x) = \mathrm{vol}_{n-1}(K \cap (x+H))$, $x \in H^\perp$, is p-concave.

We conclude this work by discussing that the statement of Theorem 5 cannot be extended to the range of $p \in (-1/2, 0)$ either. Therefore, this fact (jointly with the case in which $p \in (-\infty, -1)$, collected in Proposition 6) gives that $[0, \infty]$ is the largest subset of the real line (with respect to set inclusion) for which C_p provides us with the infimum value for the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$, among all compact sets with (centroid at the origin and) p-concave cross-sections volume function.

Note 7. The results presented in this contribution were originally proven in [3]. ◂

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Convergence of manifolds with totally bounded curvature

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Abstract: When we study the topological consequences of the curvature, one of the most successful tools is the Gromov-Hausdorff distance. It allows us to study the convergence of manifolds under some metric constrictions. In this paper, we will focus on the convergence with totally bounded sectional curvature. We will explain some of the most important results in the area (Cheeger-Gromov, Fukaya, Naber & Tian). We will use a result due to S. Roos explaining the collapse with totally bounded curvature with codimension 1, to show our current work: we are trying to generalize it to every codimension using the Uryson k-widths instead of the injectivity radius.

Resumen: Al intentar estudiar las consecuencias topológicas de la curvatura seccional en las variedades riemannianas, la distancia Gromov-Hausdorff es una herramienta muy útil. Gracias a ella podemos estudiar la convergencia de variedades con ciertas restricciones métricas. En este artículo, nos centraremos en las variedades con curvatura seccional totalmente acotada. Explicaremos algunos de los resultados más significativos (Cheeger-Gromov, Fukaya, Naber y Tian). A raíz de un resultado de S. Roos sobre el colapso con codimensión 1 y curvatura totalmente acotada, mostraremos nuestra línea de trabajo actual en pos de generalizar dicho resultado para cualquier codimensión y usando los Uryson k-widths en vez del radio de inyectividad.

Keywords: Riemannian geometry, collapse.

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The main purpose of this study is to understand the geometry of the limit space X, in the Gromov-Hausdorff sense, of a sequence of n-Riemannian manifolds $\{M_i^n\}$ with $|\sec_{M_i}| \leq 1$ and $\operatorname{diam}(M_i) \leq D$, for all $i \in \mathbb{N}$. This has certain implications in some areas where the sequence is formed by Kähler or Einstein manifolds.

We are trying to generalize, for every dimension, a result due to S. Roos. We will be using the Hausdorff dimension on the limit space because, on the majority of the situations, it will be a metric space.

Theorem 1 ([13]). *Let $\{(M_i^n, g_i)\}$ be a sequence of n-Riemannian manifolds such that $\operatorname{diam}(M_i) \leq D$ and $|\sec_{M_i}| \leq 1$, for all i. We have that $M_i \xrightarrow{GH} X$, where X is a metric space. Then, these two are equivalent:*

- $\dim_H X \geq (n-1)$.
- *For every $r > 0$, there exist $C(n, r, X) > 0$ such that*

$$C \leq \frac{\operatorname{vol}(B_r^{M_i}(x))}{\operatorname{inj}_{M_i}(x)}, \text{ for every } x \in M_i \text{ and } i \in \mathbb{N}.$$

We are interested in generalizing this theorem to every dimension on the limit X. In this situation we will work with the Uryson k-widths instead of the injectivity radius. This approach allows us to obtain Roos's result as a corollary. We will also obtain results of Gromov [10] and Perelman [12] as corollaries.

For that purpose we are working with a commuting diagram developed by Fukaya [6, 7],

$$\begin{tikzcd} FM_i \arrow[r, "\widetilde{\eta}_i"] \arrow[d, "\pi_i"'] & \widetilde{X} \arrow[d, "\pi"] \\ M_i \arrow[r, "\eta_i"] & X \end{tikzcd}$$

which relates a sequence of n-Riemannian manifolds $\{(M_i, g_i)\}$ with totally bounded curvature and bounded diameter, its Gromov-Hausdorff limit X, the frame bundles FM_i of each manifold of the sequence and the $C^{1,\alpha}$ limit $\widetilde{X}$ of those frame bundles. We will like to relate the widths of the fibres of these maps to the widths of the manifolds and, with that, generalize Roos's result.

1. Gromov-Hausdorff Distance

The Gromov-Hausdorff distance allows us to define a convergence for sequences of metric spaces. In general, the limit space does not need to conserve any regularity properties if the items of the sequence are manifolds. There can appear topological and metric singularities.

Gromov extended the notion of Hausdorff distance involving two different metric spaces. It is known as Gromov-Hausdorff distance:

Definition 2. Let X, Y be metric spaces. We define the *Gromov-Hausdorff distance* between X and Y as

$$\mathrm{d}_{GH}(X, Y) = \inf_Z\{\mathrm{d}_H(f(X), g(Y)\},$$

where the infimum is taken between all the isometric embeddings $f: X \to Z$ and $g: Y \to Z$ in the same ambient metric space Z. ◂

To be precise, d_{GH} is a distance in the set of compact metric spaces after identifying isometric pairs.

2. Collapse of Riemannian manifolds with $|\sec_{M_i}| \leq 1$

We are going to show some of the most relevant results of the convergence of Riemannian manifolds with totally bounded curvature. We will use the Hausdorff dimension in the limit space and the usual one on the manifolds of the sequence.

The first case is when the limit space has the same dimension as the manifolds of the sequence. Cheeger showed that we can extract a subsequence of manifolds which converges to a manifold:

Theorem 3 ([2, 3]). *Let $\{(M_i^n, g_i)\}$ be a sequence of compact Riemannian manifolds such that $|\sec_{M_i}| \le 1$, $\operatorname{diam}(M_i) \le D$ and $\operatorname{vol}(M_i) > v$, for all i. Then, there exists a subsequence $\{M_j\} \subset \{M_i\}$ such that $M_j \xrightarrow{GH} N$, where N is a $C^{1,\alpha}$ Riemannian manifold with $0 < \alpha < 1$.*

Now, our aim is working with limit spaces with less dimension than the manifolds of the sequence.

Definition 4 (collapse). Let $\{M_i^n\}$ be a sequence of Riemannian manifolds such that $M_i \xrightarrow{GH} X$, where X is a metric space. We say that there exists *collapse* if $\dim_H X < \dim M_i = n$. ◂

One of the most important results on the field is the *almost flat theorem* of Gromov.

Definition 5. An *infranil manifold* N/Γ, is a quotient manifold where N is a simply connect nilpotent Lie group and Γ is a discrete cocompact subgroup of $\operatorname{Aut}(N) \ltimes N$. ◂

Definition 6 (almost flat manifold). We say that M^n is an almost flat Riemannian manifold if there exists a set of metrics g_ϵ such that $|\sec_{M_\epsilon}| \le 1$, $\operatorname{diam}(M_\epsilon) \le \epsilon$, for all $\epsilon > 0$.

For example, every flat manifold is almost flat. ◂

Theorem 7 ([8]). *A Riemannian manifold M^n is almost flat if and only if it is infranil. In other words, if $M_\epsilon \xrightarrow{GH} \{pt\}$, we have that M_ϵ is diffeomorphic to one which is infranil.*

Later on, Cheeger and Gromov worked on *F-structures* [4]. They defined them as actions of torus sheaves on normal coverings of the manifolds on the sequence. This action gives orbits on the manifolds which are going to collapse to points. They proved that if a manifold admits such structure, we can construct a family of metrics g_δ such that the manifold collapses when $\delta \to 0$ while the curvature is totally bounded. In [5], they constructed the converse of the above result.

At present, the most up-to-date results are due to Naber and Tian [11]. In that paper, they try to understand all the geometry besides Fukaya's diagram. They built two fibre bundles $V^T, V^{ad} \to \widetilde{X}$ which show the unwrapped limit geometry above the limit space X.

3. k-dimensional Uryson width

We begin with the definition of the k-width:

Definition 8 (Uryson k-width [12]). The *k-dimensional Uryson width* $w_k(X)$ of a metric space X is defined as the exact lower bound of those $\delta > 0$ for which there exists a k-dimensional space P and a continuous map $f : X \to P$ all of whose inverse images have diameters at most δ. ◂

Remark 9. Let M^n be an n-Riemannian manifold. Then,

- $w_0(M) = \operatorname{diam}(M)$.
- $w_i(M) = 0$, for all $i \ge n$. ◂

Using these metric invariants, Gromov and later Perelman proved some inequalities involving the volume of a Riemannian manifold and the product of all k-widths:

Theorem 10 ([10, 12]). *Let M be an almost flat n-Riemannian manifold. Then, there exists $c > 0$ such that*

$$c^{-1} \cdot \operatorname{vol}(M) \le \prod_{i=0}^{n-1} w_i(M) \le c \cdot \operatorname{vol}(M). \tag{1}$$

Let M be a closed n-Riemannian manifold nonnegatively curved. Then, there exists $c > 0$ such that (1) holds.

Due to this result, we can conjecture the following taking into account that the fibres of our collapse are infranil manifolds:

Conjecture 11. *Let* $\{(M_i^n, g_i)\}$ *be a sequence of* n*-Riemannian manifolds such that* $\operatorname{diam}(M_i) \leq D$ *and* $|\sec_{M_i}| \leq 1$*, for all* i*. We have that* $M_i \xrightarrow{GH} X$*, where* X *is a metric space. Then, these two are equivalent:*

- $\dim_H X \geq (n-k)$.
- *For every* $r > 0$*, there exists* $C(n, r, X) > 0$ *such that*

$$C \leq \frac{\operatorname{vol}(B_r^{M_i}(x))}{\Pi_{j=0}^{k-1} w_{n-k+j}(M_i)}, \text{ for every } x \in M_i \text{ and } i \in \mathbb{N}.$$

Remark 12. If $k = 1$,

$$w_{n-1}(M_i) = w_0(F_i^p) = \operatorname{diam}(F_i^p) = 2\operatorname{inj}(M_i),$$

where F_i^p is de fibre in the Fukaya's map $f : M_i \to X$. Therefore, our conjecture implies Theorem 1. ◂

Remark 13. Suppose $M_i \xrightarrow{GH} \{pt\}$ with totally bounded curvature (Theorem 7). Then, our conjecture implies Theorem 10, because it is our desired result with that kind of collapse. ◂

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Inverse Turán numbers

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Abstract: For given graphs G and F, the Turán number $\mathrm{ex}(G, F)$ is defined to be the maximum number of edges in an F-free subgraph of G. Foucaud, Krivelevich and Perarnau and later independently Briggs and Cox introduced a dual version of this problem wherein for a given number k, one maximizes the number of edges in a host graph G for which $\mathrm{ex}(G, H) < k$.

Addressing a problem of Briggs and Cox, we determine the asymptotic value of the inverse Turán number of the paths of length 4 and 5 and provide an improved lower bound for all paths of even length. Moreover, we obtain bounds on the inverse Turán number of even cycles giving improved bounds on the leading coefficient in the case of C_4. Finally, we give multiple conjectures concerning the asymptotic value of the inverse Turán number of C_4 and P_ℓ, suggesting that in the latter problem the asymptotic behavior depends heavily on the parity of ℓ.

Resumen: Para dos grafos G y F, el número de Turán $\mathrm{ex}(G, F)$ se define como el número máximo de aristas en un subfrafo F-libre de G. Foucaud, Krivelevich y Perarnau, e independientemente Briggs y Cox, introdujeron una versión dual de este problema en la que, dado un número k, se maximiza el número de aristas en un grafo G tal que $\mathrm{ex}(G, F) < k$.

Abordando un problema de Briggs y Cox, determinamos el valor asintótico del número de Turán inverso de los caminos de longitud 4 y 5, y proporcionamos una cota inferior mejorada para todos los caminos de longitud par. Además, obtenemos cotas para el número de Turán inverso de los ciclos pares, dando cotas mejoradas para el término dominante en el caso de C_4. Por último, planteamos múltiples conjeturas sobre el número de Turán inverso de C_4 y P_ℓ, sugiriendo que en el segundo caso el comportamiento asintótico depende fuertemente de la paridad de ℓ.

Keywords: Turán number, extremal combinatorics, paths, cycles.

MSC2010: 05C35, 05D99.

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1. Introduction

This is an extended abstract of manuscript [12].

Turán's theorem [14] asserts that the maximum number of edges in a subgraph of the complete graph K_n on n vertices with no subgraph isomorphic to the complete graph on r vertices is attained by the complete r-partite graph with parts of size $\lfloor n/r \rfloor$ and $\lceil n/r \rceil$. This graph is referred to as the Turán graph and is denoted by $T(n,r)$.

Since Turán's seminal result, the problem of maximizing the number of edges in an n-vertex graph not containing a fixed graph F as a subgraph has been investigated for a variety of graphs F. A graph G containing no member of $\mathcal{F}$ as a subgraph is said to be $\mathcal{F}$-free, and for $\mathcal{F} = \{F\}$ we say that such a graph is F-free. The Turán number $\mathrm{ex}(n,\mathcal{F})$ is defined to be the maximum number of edges in an $\mathcal{F}$-free subgraph of K_n. The classical Turán problem was settled asymptotically for all finite families of graphs $\mathcal{F}$ of chromatic number at least three by Erdős, Stone and Simonovits [7, 8]. However, for most bipartite graphs F, the Turán problem remains open.

More generally for a given host graph G, the Turán number $\mathrm{ex}(G,\mathcal{F})$ is defined to be the maximum number of edges in an $\mathcal{F}$-free subgraph of G (so $\mathrm{ex}(n,\mathcal{F}) = \mathrm{ex}(K_n,\mathcal{F})$). Common alternative host graphs include the complete bipartite graph $K_{m,n}$ (the so-called Zarankiewicz problem), the hypercube Q_n [4], a random graph [10], as well as the class of n-vertex planar graphs [3].

In this paper, we will be concerned with a dual version of Turán's extremal function introduced by Foucaud, Krivelevich, and Perarnau [9] and later (in a different but equivalent form which we will use) by Briggs and Cox [1]. The number of vertices and edges in a graph G are denoted by $v(G)$ and $e(G)$, respectively. The *inverse Turán number* is defined as follows.

Definition 1. For a given family of graphs $\mathcal{F}$, $\mathrm{ex}^{-1}(k,\mathcal{F}) = \sup\{e(G) : G \text{ is a graph with } \mathrm{ex}(G,\mathcal{F}) < k\}$. For $\mathcal{F} = \{F\}$, we write $\mathrm{ex}^{-1}(k,\{F\}) = \mathrm{ex}^{-1}(k,F)$. ◂

Note that $\mathrm{ex}^{-1}(k,\mathcal{F})$ may be infinite. However, Briggs and Cox [1] observed that $\mathrm{ex}^{-1}(k,F)$ is finite whenever F is not a matching or a star. An equivalent formulation of the problem is that we must find the maximum number of edges in a graph G such that any subgraph of G with k edges contains a copy of some $F \in \mathcal{F}$. Observe that if F_1 is a subgraph of F_2, then $\mathrm{ex}^{-1}(k,F_1) \geq \mathrm{ex}^{-1}(k,F_2)$. Throughout this paper, when discussing inverse Turán numbers, the asymptotic notation O and Ω indicates that k tends to infinity, and constants involving other parameters may be hidden.

Briggs and Cox [1] gave upper and lower bounds on the inverse Turán number of C_4 of the form $\Omega(k^{4/3})$ and $O(k^{3/2})$, respectively. Unknown to Briggs and Cox at the time, this problem was considered earlier in a different form by Foucaud, Krivelevich, and Perarnau [9] where a bound was proved that was sharp up to a logarithmic factor. Even more, according to Perarnau and Reed [13] the problem was already proposed by Bollobás and Erdős at a workshop in 1966 (see [6] for a related problem about union-free families from 1970). More generally a recursive bound on the inverse Turán number of $\mathrm{ex}^{-1}(k,\{C_4, C_6, \dots, C_{2t}\})$ was also obtained in [9], which is also tight up to a logarithmic factor.

For graphs F with chromatic number at least 3, Foucaud, Krivelevich, and Perarnau [9] and Briggs and Cox [1] determined the inverse Turán number asymptotically. Moreover, Briggs and Cox [1] determined the inverse Turán number of the complete graph precisely as well as the union of a path of length 1 and a path of length 2. They also settled the case of paths of length 3 and proposed a conjecture about the inverse Turán number of a path of length 4.

In Section 2, we will investigate the inverse Turán problem for paths, resolving a conjecture of Briggs and Cox asymptotically and providing a new lower bound for paths of any even length. In Section 3 we will determine the order of magnitude of the inverse Turán number of any complete bipartite graph resolving another conjecture of Briggs and Cox about the order of magnitude of $\mathrm{ex}^{-1}(k,C_4)$. We note however, that this conjecture already follows directly from an unpublished preprint of Conlon, Fox, and Sudakov [2] which preceded the paper of Briggs and Cox [1], but we provide a proof in the formulation introduced by Briggs and Cox for completeness. In the case of C_4, we give improved bounds on the leading coefficient and conjecture that the lower bound is optimal. Additionally, we give some estimates on the inverse Turán number of an arbitrary even cycle. Finally in Section 4, we present some conjectures and directions for future work.

2. Inverse Turán numbers of paths

In this section, we investigate the inverse Turán problem for paths. We begin by recalling a well-known result of Erdős and Gallai.

Theorem 2 (Erdős, Gallai [5]). *For all $n \geq t$, $\mathrm{ex}(n, P_t) \leq (t-1)n/2$, and equality holds if and only if t divides n and G is the disjoint union of cliques of size t.*

Theorem 3 (Briggs, Cox [1]). *For all $t \geq 3$,*

$$\mathrm{ex}^{-1}(k, P_t) \geq \binom{\left\lfloor \frac{2k}{t-1} \right\rfloor - 1}{2}.$$

The bound in Theorem 3 comes from taking a complete graph of the appropriate size and applying Theorem 2. In the case of $t = 3$, Briggs and Cox [1] proved that a complete graph gives the optimal bound for $\mathrm{ex}^{-1}(k, P_3)$. Briggs and Cox also noted that for P_4 one can do better by considering a complete bipartite base graph and using a result of Gyárfás, Rousseau, and Schelp [11] on the extremal number of P_t in such graphs. However, starting with a clique is superior to a complete bipartite graph for P_t, $t \neq 4$. We will improve the lower bound on $\mathrm{ex}^{-1}(k, P_{2t})$ in general by considering balanced complete multipartite graphs. Note that, since the inverse Turán number is non-decreasing when considering supergraphs, it follows that the inverse Turán number of any path of length at least 3 is $\Theta(k^2)$.

Proposition 4. *Among the Turán graphs $T(n,r)$ with $\mathrm{ex}(T(n,r), P_{2t}) < k$, the one with the maximum number of edges is obtained by $r = t$ and $n = \left\lfloor \frac{k-1}{t-1} \right\rfloor + O(k)$. In particular, for $t \geq 2$,*

$$\mathrm{ex}^{-1}(k, P_{2t}) \geq e\left(T\left(\left\lfloor \frac{k-1}{t-1} \right\rfloor, t\right)\right) = \frac{(k-1)^2}{2t(t-1)} + O(k).$$

Theorem 5. *We have $\mathrm{ex}^{-1}(k, P_4) = k^2/4 + O(k^{3/2})$.*

Theorem 6. *We have $\mathrm{ex}^{-1}(k, P_5) = k^2/8 + O(k)$.*

3. Inverse Turán number of complete bipartite graphs and even cycles

While the classical Turán number $\mathrm{ex}(n, K_{s,t})$ is not known, Conlon, Fox, and Sudakov [2] determined the asymptotics of $\mathrm{ex}^{-1}(k, K_{s,t})$.

Theorem 7. *Let s, t be integers with $1 < s \leq t$. Then, $\mathrm{ex}^{-1}(k, K_{s,t}) = \Theta(k^{1+1/s})$.*

In the case of C_4, we give a more precise calculation to prove upper and lower bounds within a factor of $\frac{3\sqrt{3}}{2\sqrt{2}} < 2$.

Theorem 8. $\lfloor \sqrt{2k/3} \rfloor \lfloor 2k/3 - 1 \rfloor \leq \mathrm{ex}^{-1}(k, C_4) \leq k^{3/2} + o(k^{3/2})$.

In the following theorem we offer some bounds for the inverse Turán number of even cycles.

Theorem 9. *Let $t \geq 2$. Then,*

$$\mathrm{ex}^{-1}(k, C_{2t}) = \begin{cases} O\left(k^{2-\frac{2}{3t-3}}\right) & \text{if } t \text{ is odd,} \\ O\left(k^{2-\frac{2}{3t-2}}\right) & \text{if } t \text{ is even,} \end{cases}$$

and

$$\mathrm{ex}^{-1}(k, C_{2t}) = \begin{cases} \Omega\left(k^{2-\frac{2}{t+1}}\right) & \text{if } t \text{ is odd,} \\ \Omega\left(k^{2-\frac{2}{t+2}}\right) & \text{if } t \text{ is even.} \end{cases}$$

4. Remarks and open questions

We pose two conjectures about the inverse Turán number of the path depending on the parity of its length. In agreement with the intuition of Briggs and Cox [1], we believe that the inverse Turán number of a path with odd length is attained by a clique. On the other hand, we believe that a balanced multi-partite graph of t parts is optimal for forcing a path of length $2t$.

Conjecture 10. *The inverse Turán number of a path of length* $2t+1$ *is attained asymptotically by a complete graph. Therefore, for every* t*,* $\mathrm{ex}^{-1}(k, P_{2t+1}) = \binom{\lfloor \frac{k}{t} \rfloor}{2} + o(k^2)$.

Conjecture 11. *The inverse Turán number of a path of length* $2t$ *is attained asymptotically by a balanced, complete* t*-partite graph. Therefore, for every* t*,* $\mathrm{ex}^{-1}(k, P_{2t}) = \frac{k^2}{2(t-1)^2}\left(1-\frac{1}{t}\right) + o\left(k^2\right)$.

We have given upper and lower bounds for the value of $\mathrm{ex}^{-1}(k, C_4)$, and we conjecture that the lower bound is asymptotically sharp.

Conjecture 12. *We have* $\mathrm{ex}^{-1}(k, C_4) = \frac{2\sqrt{2}k^{3/2}}{3\sqrt{3}} + o(k^{3/2})$.

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3rd *BYMAT Conference* (2020)

Multiquadratic rings and oblivious linear function evaluation

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Abstract: The Ring Learning with Errors (RLWE) problem has been widely used for the construction of new quantum-resistant cryptographic primitives. Most of the existing RLWE-based schemes make use of power-of-two cyclotomic rings due to their good performance and simplicity. This talk explores the replacement of power-of-two cyclotomic rings by multiquadratics. We show that for polynomials with n coefficients, the cost of the polynomial operations can be reduced from $\mathcal{O}(n \log n)$ multiplications to $\mathcal{O}(n)$ multiplications and $\mathcal{O}(n \log n)$ additions. Finally, we discuss the benefits that these rings can bring about when implementing the OLE (Oblivious Linear Function Evaluation) primitive, which is a basic block used in many Secure Multiparty Computation (MPC) protocols.

Resumen: El problema *Ring Learning with Errors* (RLWE) ha sido utilizado ampliamente para la construcción de nuevas primitivas criptográficas resistentes a ataques por parte de un ordenador cuántico. La mayoría de los esquemas existentes basados en RLWE hacen uso de anillos ciclotómicos de orden potencia de dos, debido a su buen comportamiento y sencillez. Esta charla explora el reemplazo de los anillos ciclotómicos potencia de dos por anillos multicuadráticos. Se muestra que, para polinomios con n coeficientes, el coste de las operaciones polinomiales puede ser reducido de $\mathcal{O}(n \log n)$ multiplicaciones a $\mathcal{O}(n)$ multiplicaciones y $\mathcal{O}(n \log n)$ sumas. Finalmente, se discuten los beneficios que estos anillos introducen al implementar la primitiva OLE (*Oblivious Linear Function Evaluation*), que es un bloque básico utilizado en muchos protocolos de *Secure Multiparty Computation* (MPC).

Keywords: ring learning with errors, multiquadratic rings, Walsh-Hadamard transform, oblivious linear function evaluation.

MSC2010: 11T71, 68P25, 94A60.

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1. Introduction

This extended abstract corresponds to a talk given in the BYMAT 2020 conference, and covers some of the results previously introduced in [3] and [4]. Due to space constraints, our main aim here is to provide a high-level overview of the most important aspects highlighted in the presentation. We refer the reader to [3, 4] for further technical details.

Notation. We first introduce the notation used in this work. Vectors and matrices are represented by boldface lowercase and uppercase letters. Polynomials are denoted with regular lowercase letters, omitting the polynomial variable (i.e., a instead of $a(z)$) when there is no ambiguity. We follow a recursive definition for multivariate quotient rings: $R_q[z] = \mathbb{Z}_q[z]/f(z)$ denotes the polynomial quotient ring in the variable z modulo $f(z)$ with coefficients belonging to $\mathbb{Z}_q$. In general, $R_q[x_1, \dots, x_l]$ (resp. $R[x_1, \dots, x_l]$) represents the multivariate quotient polynomial ring with coefficients in $\mathbb{Z}_q$ (resp. $\mathbb{Z}$) and reduced modulo $f_i(x_i)$ for $1 \leq i \leq l$. The polynomial a can also be denoted by a column vector $\boldsymbol{a}$ whose components are the corresponding polynomial coefficients. Finally, the Hadamard (resp. Kronecker) product of two matrices is $\boldsymbol{A} \circ \boldsymbol{B}$ (resp. $\boldsymbol{A} \otimes \boldsymbol{B}$), and $[l]$ denotes the set $\{1, 2, \dots, l\}$. ◂

1.1. Preliminaries: Ring Learning with Erros

The security of modern *homomorphic encryption* (HE) schemes [1] relies on the hardness of the *Ring Learning with Errors* (RLWE) problem [6], where power-of-two cyclotomic rings as $R_q = \mathbb{Z}_q[z]/(1 + z^n)$ are usually considered. An informal definition of RLWE is included in Figure 1, where we can see how the hardness relies on the computational indistinguishability between (a_i, b_i) and (a_i, u_i), where $\chi[z]$ generates polynomials in R_q, whose coefficients are independent and follow a Gaussian distribution.

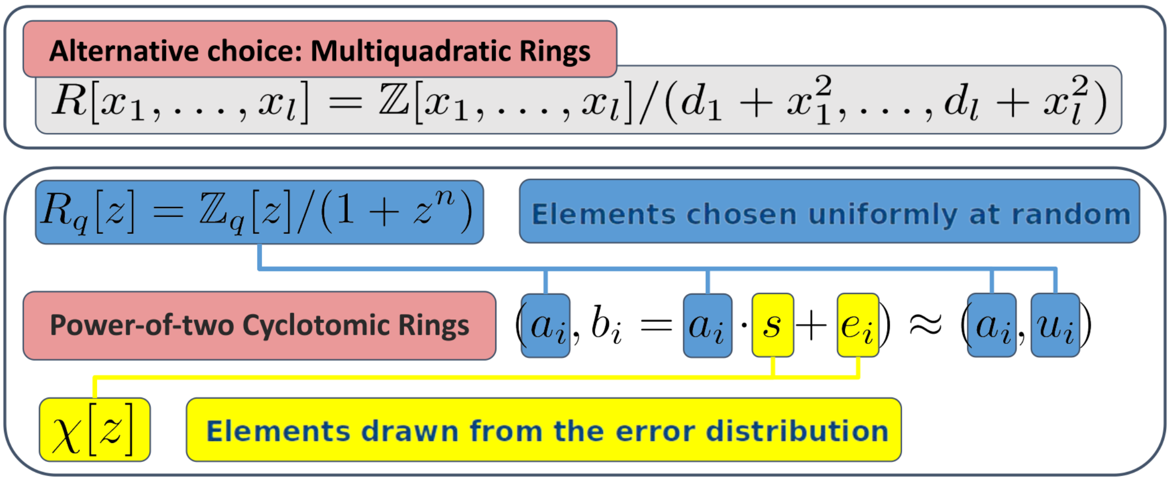

Figure 1: Sketch of the RLWE problem.

The use of RLWE provides two important advantages for the construction of encryption schemes:

- RLWE is believed to be difficult to solve by quantum computers.
- Polynomial arithmetic can be done very efficiently with Number Theoretic Transforms (NTTs) [5].

1.2. NTT representation

Instead of directly working with the coefficient representation, current HE libraries [1] accelerate computation by making use of a double CRT (Chinese Remainder Theorem) and NTT representation (see Figure 2). In particular, by considering power-of-two cyclotomics, a negacyclic NTT is used which introduces an overhead of $\mathcal{O}(n \log n)$ multiplications. Consequently, motivated by this overhead, in [3, 4] we explored the substitution in RLWE of the conventional power-of-two cyclotomics by multiquadratic rings (see Figure 1).

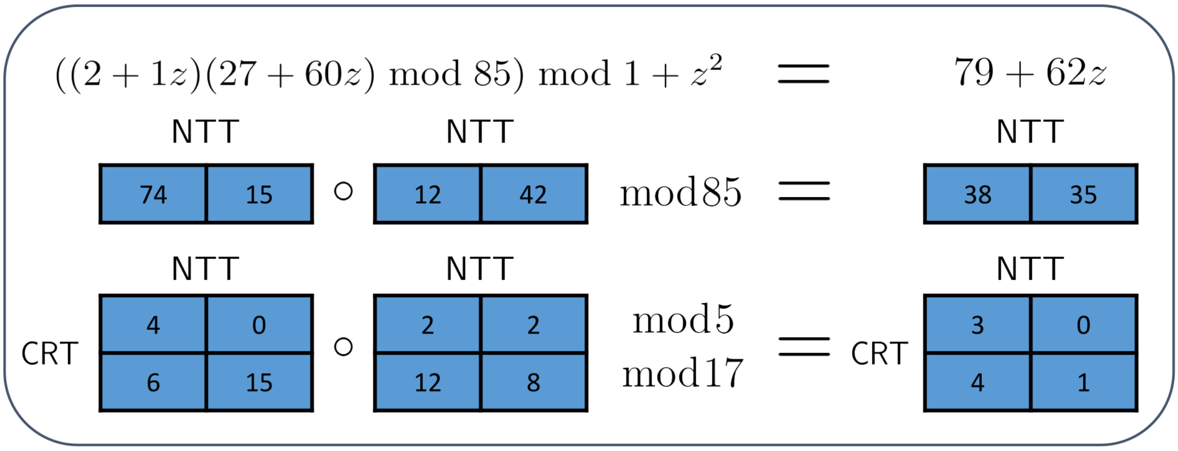

Figure 2: Toy example of the CRT-NTT representation.

2. Multiquadratic Rings and faster arithmetic

Multiquadratic quotient rings as $R_q[x_1, \dots, x_l] = \mathbb{Z}_q[x_1, \dots, x_l]/(d_1+x_1^2, \dots, d_l+x_l^2)$ can satisfy a convolution property with a variant of the Walsh-Hadamard transform that we call $\boldsymbol{\alpha}$-generalized WHT in [3, 4]. $\boldsymbol{W}_l$ and $\boldsymbol{W}_l^{-1}$ denote, respectively, the direct and inverse transform matrices associated to $R_q[x_1, \dots, x_l]$.

Figure 3 includes the matrix expressions for both transforms of length $n = 2^l$, where, in order to the ring R_q factors into linear terms [5], $d_j = -\alpha_j^{-1} \bmod q$ and the square-roots of α_j must exist in R_q for all j.

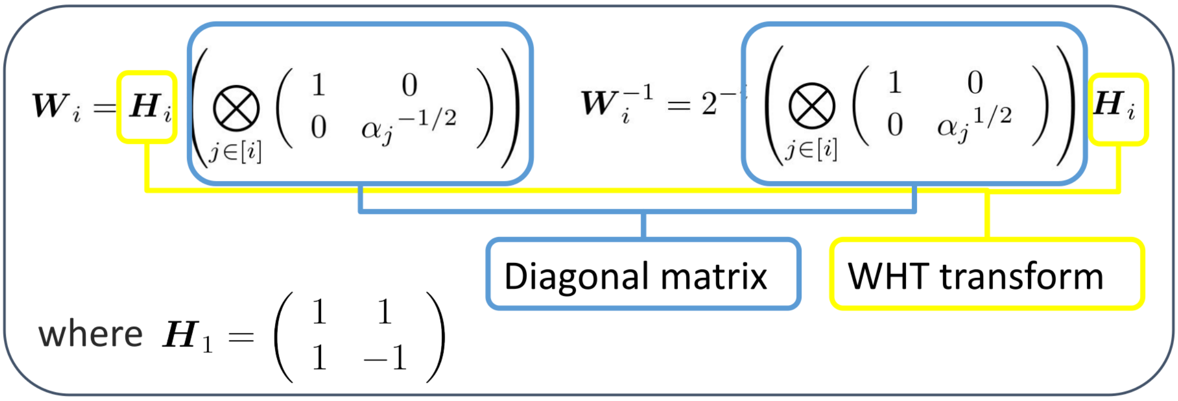

Figure 3: Generalized Walsh-Hadamard Transform.

This transform can be very efficiently computed by decomposing it into two different matrices:

- A diagonal matrix which can be computed with a cost of n products.
- A Walsh-Hadamard matrix $\boldsymbol{H}_l$ which can be computed with a cost of only $\mathcal{O}(n \log n)$ additions.

Hence, comparing to the more conventional negacyclic NTT used in the RLWE problem, the use of multiquadratic rings reduces the multiplicative cost of polynomial multiplications by a factor of $\log_2 n$.

3. OLE applications

The OLE (Oblivious Linear function Evaluation) primitive is a very important building block in many MPC (Secure Multiparty Computation) protocols [2], and consequently, any achieved improvement on its efficiency brings about important benefits on a wide variety of applications.

An informal description of the OLE primitive can be seen in Figure 4 (we refer to [2, 4] for a formal definition). It considers a set $\mathcal{P} = \{\mathcal{P}_R, \mathcal{P}_S\}$ with two different parties:

- The receiver $\mathcal{P}_R$, which holds an input x and learns the output $f(x) = ax + b$, but nothing more about a and b than can be inferred from both x and $f(x)$.
- The sender $\mathcal{P}_S$, which holds inputs a and b, and learns nothing regarding x.

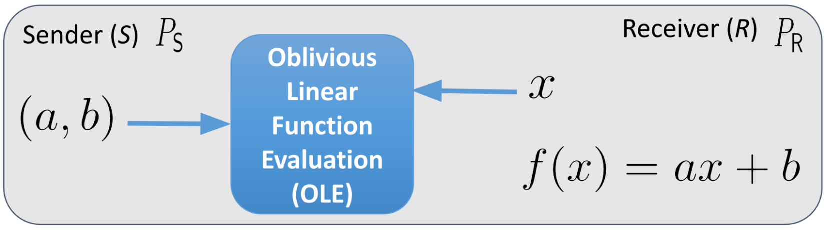

Figure 4: OLE primitive.

3.1. AHE-based OLE

The OLE primitive from Figure 4 can be implemented with additively homomorphic encryption (AHE):

- $\mathcal{P}_R$ sends $\mathsf{E}(x)$ to $\mathcal{P}_S$. Note that $\mathsf{E}(\cdot)$ represents the encryption functionality.
- $\mathcal{P}_S$ homomorphically calculates $a \cdot \mathsf{E}(x) + b = \mathsf{E}(ax + b)$.
- $\mathcal{P}_R$ receives $\mathsf{E}(ax + b)$ and decrypts it to obtain $f(x)$.

We instantiated in [4] an AHE scheme based on the RLWE problem with both multiquadratic and power-of-two cyclotomic rings. A very brief summary of the obtained results with 128 bits of security is:

- Multiquadratic-based OLE is at least two times faster than its power-of-two cyclotomic counterpart.
- Multiquadratic-based OLE has higher storage needs (requires around 1.7 times more bits).

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3rd *BYMAT Conference* (2020)

Neumann p-Laplacian problems with a reaction term on metric spaces

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Abstract: We use a variational approach to study existence and regularity of solutions for a Neumann p-Laplacian problem with a reaction term on metric spaces equipped with a doubling measure and supporting a Poincaré inequality. Trace theorems for functions with bounded variation are applied in the definition of the variational functional and minimizers are shown to satisfy De Giorgi type conditions.

Resumen: Utilizamos un enfoque variacional para estudiar la existencia y regularidad de soluciones para un problema de Neumann p-Laplaciano con un término de reacción en espacios métricos dotados de una medida de duplicación y que permiten una desigualdad de Poincaré. Se aplican teoremas de traza para funciones con variación acotada en la definición del funcional variacional y se demuestra que los minimizadores satisfacen condiciones de tipo De Giorgi.

Keywords: p-Laplacian operator, measure metric spaces, minimal p-weak upper gradient, minimizer.

MSC2010: 31E05, 30L99, 46E35.

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1. Introduction

We extend existence and regularity results for a Neumann boundary value problem valid on the Euclidean setting and, more generally, in Riemannian manifolds (see Nastasi [8]) to the general setting of metric spaces. Applying variational methods such as those based on De Giorgi classes [2], we study a Neumann boundary value problem as in Lahti, Malý, and Shanmugalingam [5] and Malý and Shanmugalingam [6], but the new feature is that we include a reaction term (see Nastasi [9]). Under appropriate conditions on the reaction term, we prove existence and boundedness properties of solutions with a reaction term in a metric space equipped with a doubling measure and supporting a Poincaré inequality and thus extending the corresponding results in Kinnunen and Shanmugalingam [3] and Malý and Shanmugalingam [6].

2. Mathematical background

Let (X, d, μ) be a metric measure space, where μ is a Borel regular measure. Let $B(x, \rho) \subset X$ be a ball with center $x \in X$ and radius $\rho > 0$.

Definition 1 ([1, Section 3.1]). A measure μ on X is said to be doubling if there exists a constant K, called the doubling constant, such that $0 < \mu(B(x, 2\rho)) \leq K\mu(B(x, \rho)) < +\infty$ for all $x \in X$ and $\rho > 0$. ◂

The following notion of upper gradient has been introduced in order to satisfy the lack of a differentiable structure.

Definition 2 ([1, Definition 1.13]). A non negative Borel measurable function g is said to be an upper gradient of function $u : X \to [-\infty, +\infty]$ if, for all compact rectifiable arc lenght parametrized paths γ connecting x and y, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s \tag{1}$$

whenever $u(x)$ and $u(y)$ are both finite and $\int_\gamma g \, \mathrm{d}s = +\infty$ otherwise. ◂

We note that, if g is an upper gradient of function u and ϕ is a non negative Borel measurable function, then $g + \phi$ is still an upper gradient of u. In order to overcome this aspect, we use the following notions that will lead to the definition of the minimal p-weak upper gradient of u.

Definition 3 ([1, Definition 1.33]). Let $p \in [1, +\infty[$. Let Γ be a family of paths in X. We say that $\inf_\phi \int_X \phi^p \, \mathrm{d}\mu$ is the p-modulus of Γ, where the infimum is taken among all non negative Borel measurable functions ϕ satisfying $\int_\gamma \phi \, \mathrm{d}s \geq 1$, for all rectifiable paths $\gamma \in \Gamma$. ◂

Definition 4 ([1, Definition 1.32]). If (1) is satisfied for p-almost all paths γ in X, that is, the set of non constant paths that do not satisfy (1) is of zero p-modulus, then g is said a p-weak upper gradient of u. ◂

The family of weak upper gradients satisfy a result concerning the existence of a minimal element g_u, that is called the minimal p-weak upper gradient of u.

Definition 5 ([1, Definition 4.1]). Let $p \in [1, +\infty[$. A metric measure space X supports a $(1, p)$-Poincaré inequality if there exist $K > 0$ and $\lambda \geq 1$ such that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq Kr \left(\frac{1}{\mu(B(x, \lambda r))} \int_{B(x,\lambda r)} g_u^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

for all balls $B(x, r) \subset X$ and for all $u \in L^1_{loc}(X)$, where $u_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, \mathrm{d}\mu$. ◂

Let X be a complete metric space equipped with a doubling measure supporting a $(1, p)$-Poincaré inequality. We recall the concept of Newtonian space, which is based on the notion of minimal p-weak upper gradient.

Definition 6. Let X be a complete metric space equipped with a doubling measure supporting a $(1, p)$-Poincaré inequality, $p \in [1, +\infty]$. The Newtonian space $N^{1,p}(X)$ is defined by $N^{1,p}(X) = V^{1,p}(X) \cap L^p(X)$, where $V^{1,p}(X) = \{u : u \text{ is measurable and } g_u \in L^p(X)\}$. We consider $N^{1,p}(X)$ equipped with the norm

$$\|u\|_{N^{1,p}(X)} = \|g_u\|_{L^p(X)} + \|u\|_{L^p(X)}.$$

We denote with $N^{1,p}_*(X) = \{u \in N^{1,p}(X) : \fint_X u \, \mathrm{d}x = 0\}$. ◂

The Newtonian space $N^{1,p}(X)$ defined above is a complete normed vector space, which generalizes the Sobolev space $W^{1,p}(\Omega)$ to a metric setting.

Definition 7 (see [7]). A Borel set $E \subset X$ is said to be of finite perimeter if there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $N^{1,1}(X)$ such that $u_n \to \chi_E$ in $L^1(X)$ and $\liminf_{n\to+\infty} \fint_X g_{u_n} \, \mathrm{d}\mu < \infty$. The perimeter $P_E(X)$ of E is the infimum of the above limit among all sequences $\{u_n\}$ as above. For an open set $U \subset X$, the perimeter of E in U is

$$P_E(U) = \inf\left\{\liminf_{n\to+\infty} \int_X g_{u_n} \, \mathrm{d}\mu : \{u_n\}_{n\in\mathbb{N}} \subset N^{1,1}(U), u_n \to \chi_{E\cap U} \text{ in } L^1(U)\right\}. \quad ◂$$

From now on, we consider a bounded domain (non empty, connected open set) Ω in X with $X \setminus \Omega$ of positive measure such that Ω is of finite perimeter with perimeter measure P_Ω. Let $f : \partial\Omega \to \mathbb{R}$ be a bounded P_Ω-measurable function with $\fint_{\partial\Omega} f \, \mathrm{d}P_\Omega = 0$. We make the following assumptions on Ω:

(H_1) There exists a constant $K \geq 1$ such that, for all $y \in \Omega$ and $0 < \rho \leq \operatorname{diam}(\Omega)$, we have

$$\mu(B(y,\rho) \cap \Omega) \geq \frac{1}{K}\mu(B(y,\rho)).$$

(H_2) (Ahlfors codimension 1 regularity of P_Ω) For all $y \in \partial\Omega$ we have that

$$\frac{1}{K\rho}\mu(B(y,\rho)) \leq P_\Omega(B(y,\rho)) \leq \frac{K}{\rho}\mu(B(y,\rho)),$$

where K and ρ are as in (H_1).

(H_3) $(\Omega, d_{|\Omega}, \mu_{|\Omega})$ admits a $(1, p)$-Poincaré inequality with $\lambda = 1$, where $p \in]1, +\infty[$.

Definition 8 ([4, Definition 4.1]). Let $\Omega \subset X$ be an open set and let u be a μ-measurable function on Ω. A function $Tu : \partial\Omega \to \mathbb{R}$ is the trace of u if for $\mathcal{H}$-almost every $y \in \partial\Omega$ we have

$$\lim_{\rho\to 0^+} \frac{1}{\mu(\Omega \cap B(y,\rho))} \int_{\Omega\cap B(y,\rho)} |u - Tu(y)| \, \mathrm{d}\mu = 0. \quad ◂$$

For the existence theorem of the trace operator see Malý and Shanmugalingam [6] and references therein.

Given a Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G, we associate the following functional

$$J(u) = \int_\Omega g_u^p \, \mathrm{d}\mu - \int_\Omega G(u) \, \mathrm{d}\mu + \int_{\partial\Omega} Tuf \, \mathrm{d}P_\Omega \quad \text{for all } u \in N^{1,p}(\Omega).$$

Definition 9. A function $u_0 \in N^{1,p}_*(\Omega)$ is a p-harmonic solution to the Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G if

$$\begin{aligned} J(u_0) &= \int_\Omega g_{u_0}^p \, \mathrm{d}\mu - \int_\Omega G(u_0) \, \mathrm{d}\mu + \int_{\partial\Omega} Tu_0 f \, \mathrm{d}P_\Omega \\ &\leq \int_\Omega g_v^p \, \mathrm{d}\mu - \int_\Omega G(v) \, \mathrm{d}\mu + \int_{\partial\Omega} Tvf \, \mathrm{d}P_\Omega = J(v) \end{aligned}$$

for every $v \in N^{1,p}_*(\Omega)$, where g_{u_0}, g_v are the minimal p-weak upper gradients of u_0 and v in Ω, respectively, and Tu_0 and Tv are the traces of u_0 and v on $\partial\Omega$, respectively. ◂

Later on, in considering the trace Tu of u we will omit T and just write u.

Here, we assume that $G\colon \Omega \to \mathbb{R}$ is defined as $G(u) = c - |u|^\gamma$ for all $u \in N^{1,p}(\Omega)$, for some $c > 0$ and $1 < \gamma < p^* = \frac{ps}{s-p}$ if $p < s$ and $1 < \gamma < +\infty$ otherwise.

In the metric setting, we will look for a minimizer of J in the Newtonian space $N^{1,p}_*(\Omega)$.

3. Existence of a solution and a weaker uniqueness result

The existence of a nontrivial solution to the Neumann boundary value problem with non zero boundary data f and reaction term G is an immediate consequence of the following theorem which shows that J has a minimizer.

Theorem 10. *J has a minimizer in $N^{1,p}_*(\Omega)$. If $u_1, u_2 \in N^{1,p}_*(\Omega)$ are two minimizers of J, then $g_{u_1} = g_{u_2}$ a.e. in Ω.*

4. Boundedness property

We show that minimizers are locally bounded near the boundary under appropriate hypothesis on the boundary data f. In order to do so, the following De Giorgi type inequality plays a key role.

Lemma 11. *Let $u \in N^{1,p}_*(\Omega)$ be a minimizer of J and $f \in L^\infty(\partial\Omega)$. If $y \in \partial\Omega$, $0 < \rho < R < \frac{\mathrm{diam}(\Omega)}{10}$ and $\alpha \in \mathbb{R}$, then there is $K \geq 1$ such that the following De Giorgi type inequality*

$$\int_{\Omega\cap B(y,\rho)} g^p_{(u-\alpha)_+}\,\mathrm{d}\mu \leq \frac{K}{(R-\rho)^p}\int_{\Omega\cap B(y,R)} (u-\alpha)^p_+\,\mathrm{d}\mu + K\int_{\partial\Omega\cap B(y,R)} |f|(u-\alpha)^p_+\,\mathrm{d}P_\Omega$$

is satisfied.

Theorem 12. *Let $0 < R < \frac{\mathrm{diam}(\Omega)}{4}$ and $\Omega_R = \left\{y \in \Omega : d(y,\partial\Omega) < \frac{R}{2}\right\}$. If $u \in N^{1,p}_*(\Omega)$ is a minimizer of J and $f \in L^\infty(\partial\Omega)$, then $u \in L^\infty(\Omega_R)$ and $Tu \in L^\infty(\partial\Omega_R)$.*

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Induced A_∞-structures

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Abstract: An A_∞-algebra A is a module over a ring equipped with a family of "multiplications" $m_i : A^{\otimes i} \to A$ satisfying the relation

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes s}) = 0$$

for all possible values of n. When $m_i = 0$ for $i \neq 2$, this relation tells us that A is an ordinary associative algebra with multiplication m_2.

We will present how this structure naturally arises in algebra and topology and discuss some examples.

Resumen: Un A_∞-álgebra A es un módulo sobre un anillo equipado con una familia de "multiplicaciones" $m_i : A^{\otimes i} \to A$ satisfaciendo la relación

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes s}) = 0$$

para todos los posibles valores de n. Cuando $m_i = 0$ para todo $i \neq 2$, esta relación nos dice que A es un álgebra asociativa con multiplicación m_2.

Presentaremos cómo esta estructura surge de forma natural en topología y en álgebra, y veremos algunos ejemplos.

Keywords: Hochschild cohomology, A_∞-structure, loop space, associahedra.

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1. A_∞-algebras

In this section we will define A_∞-algebras, explain their origin and how they generalize associative algebras, and provide some examples.

Definition 1. An A_∞-*algebra* A is a graded module over a ring k equipped with a family of multiplication maps $m_i : A^{\otimes i} \to A$ of degree $2-i$ satisfying for all $n \geq 1$ the relation

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes s}) = 0. \tag{1}$$

Let us look at some particular cases of the above relation to recover some well-known algebraic structures.

- The relation implies $m_1 m_1 = 0$, meaning that an A_∞-algebra is in particular a cochain complex with differential m_1. Thus, we can define A_∞-algebras on the category of cochain complexes, and the relations involving m_1 will be a consequence of m_i being a map of complexes.
- The relation also implies the Leibniz rule

  $$m_1 m_2 = m_2(m_1 \otimes 1) + m_2(1 \otimes m_1),$$

  so A_∞-also generalize differential graded algebras (also known as dg algebras).
- When $m_i = 0$ for $i \neq 2$ we obtain the associativity relation

  $$m_2(m_2 \otimes 1) = m_2(1 \otimes m_2).$$

  This means that associative algebras are particular instances of A_∞-algebras.
- In general, for $n = 3$ the relation becomes

  $$m_2(m_2 \otimes 1) - m_2(1 \otimes m_2) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1) + m_3(1 \otimes m_1 \otimes 1) + m_3(1 \otimes 1 \otimes m_1).$$

  This relation implies that m_2 is only associative up to a homotopy given by m_3, i.e., m_2 becomes associative in cohomology with respect to m_1. In this situation we say that m_2 is *homotopy associative*.

If we look at the higher relations we will see a similar pattern in which each m_i is a homotopy that measures the failure of other relations involving lower maps to hold. Therefore, the A_∞ equation (1) is a homotopy coherent extension of the fact that m_2 is homotopy associative.

1.1. Origin of A_∞-algebras

Even though the A_∞ equation (1) seems quite arbitrary, it has some topological roots, so let us see where A_∞-algebras come from.

Let $(X, *)$ be a pointed topological space and let $S^1 = [0,1]/\{0 \sim 1\}$ be the unit circle defined as a quotient of the unit interval with 1 as a base point. Define the *loop space* of X as the space of based loops

$$\Omega X = \{\gamma : S^1 \to X \mid \gamma(1) = *\}.$$

In other words, this is the space of maps from the circle that respect base points. The space ΩX comes equipped with a multiplication map $* : \Omega X \times \Omega X \to \Omega X$ given by concatenation of loops

The operation $\gamma_1 * \gamma_2$ for $\gamma_1, \gamma_2 \in \Omega X$ can be interpreted as running through γ_1 twice as fast on the first half of the circle and then running through γ_2 twice as fast on the second half. We can see that this operation is not associative by looking at Figure 1.

On the top of the picture we see the concatenation $(\gamma_1 * \gamma_2) * \gamma_3$, which is clearly different from $\gamma_1 * (\gamma_2 * \gamma_3)$ below. However, the difference is just about the speed of each loop, so there is a homotopy between these two resulting loops given by a reparametrization. On the right of the picture we see these concatenation represented as trees. In this representation the homotopy is given by sliding one branch through the tree.

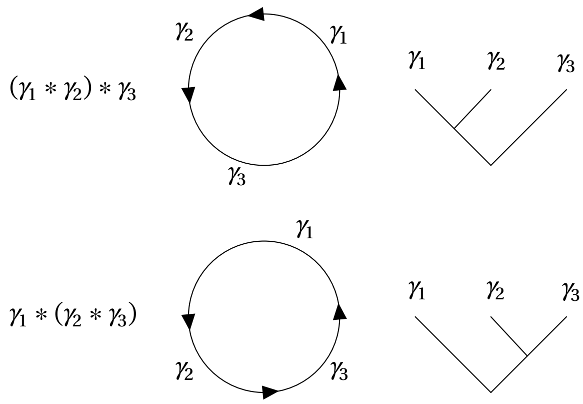

Figure 1: Two ways of concatenating three loops.

If we concatenate four loops we get Figure 2.

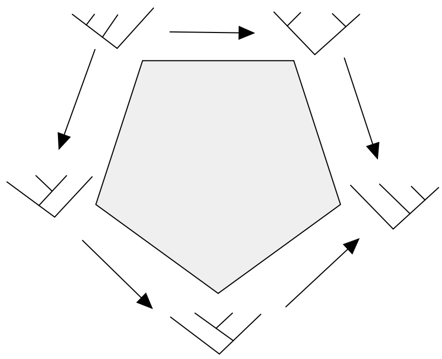

Figure 2: Five ways of concatenating four loops.

In this case we can see that there are two paths of homotopies from one extreme to the other. These paths can be connected by a higher homotopy which allow us to fill the interior of the pentagon. Each point of the filled pentagon corresponds to an intermediate slide of branches. This situation can be extended for any amount of loops. The pictures that we get describe a family of polytopes called *Stasheff associahedra*, since they were first defined by Stasheff in 1963 [6].

In this situation we say that the concatenation map is *homotopy coherent*, since the homotopies are connected by higher homotopies. This shows that ΩX is an example of an A_∞-*space* (see [6] for a precise definition of A_n-space, an A_∞-space is a space satisfying the A_n-space definition for all n.)

The connection between A_∞-spaces and A_∞-algebras is given by the following theorem.

Theorem 2. [4, Proposition 9.2.8] *The cellular chains of an A_∞-space have a structure of A_∞-algebra.*

2. Hochschild complex of an A_∞-algebra

We have seen how a topological A_∞-structure induces an algebraic A_∞-structure. Now, we are going to see how to define further algebraic A_∞-structures from an A_∞-algebra.

Definition 3. The *Hochschild complex* $C^*(A)$ of a k-module A is given by the modules

$$C^m(A) = \hom_k(A^{\otimes m}, A),$$

where $C^0(A) = A$. ◂

For an A_∞-algebra A, we are going to endow $C^*(A)$ with a differential and higher A_∞-maps. But for that we need to define a new algebraic structure on this complex.

Given $f, g_1, \dots, g_n \in C^*(A)$, define the *brace* $f\{g_1, \dots, g_n\}$ as

$$\sum_{k_0+\cdots+k_n=N-n} (-1)^\eta f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes 1^{\otimes k_{n-1}} \otimes g_n \otimes 1^{\otimes k_n}),$$

where N is the arity of f and η comes from iterated shifts as done in the Appendix of [5].

Let A be an A_∞-algebra and let $m = m_1 + m_2 + \cdots$. Define maps $M_i\colon C^*(A;A)^{\otimes n} \to C^*(A;A)$ by

$$M_1(f) := m\{f\} - (-1)^{\deg(f)} f\{m\},$$
$$M_n(f_1, \dots, f_n) := (-1)^{\sum_{i=1}^n (n-i)\deg(f_i)} m\{f_1, \dots, f_n\}, \qquad n > 1.$$

Theorem 4. *The above defined maps M_i define and A_∞-algebra structure on the Hochschild complex $C^*(A)$ of an A_∞-algebra A.*

A proof of the theorem up to signs can be found in Getzler's paper [3]. In particular, when A is an associative algebra, we obtain the classical Hochschild complex with an induced associative multiplication [1].

The original A_∞-structure given by m and the induced A_∞-structure on $C^*(A)$ are related by the following theorem.

Theorem 5. *The map $\Phi\colon A \to C^*(A;A)$ defined by*

$$\Phi(x) = \sum_{n\geq 0} x\{x_1, \dots, x_n\}$$

is a map of A_∞-algebras, i.e., it satisfies $\Phi(M_n) = M_n(\Phi^{\otimes n})$ for all n.

The original statement of the theorem without proof can be found in the paper by Gerstenhaber and Voronov [2].

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Rooted structures in graphs

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Abstract: A *transversal* of a partition is a set which contains exactly one element from each member of the partition and nothing else. A *colouring* of a graph is a partition of its vertex set into anticliques, that is, sets of pairwise nonadjacent vertices. We study the following problem: Given a transversal T of a proper colouring $\mathcal{C}$ of some graph G, is there a partition $\mathfrak{H}$ of a subset of $V(G)$ into connected sets such that T is a transversal of $\mathfrak{H}$ and any two distinct sets of $\mathfrak{H}$ are adjacent?

It has been conjectured by Matthias Kriesell [9] that for any transversal T of a colouring $\mathcal{C}$ of order k of some graph G such that any pair of colour classes induces a connected subgraph, there exists such a partition $\mathfrak{H}$ with pairwise adjacent sets. This would prove Hadwiger's conjecture for the class of uniquely optimally colourable graphs; however it is open for each $k \geq 5$.

This paper will provide an overview about the stated conjecture. It extracts associated results from my PhD thesis and the related papers [2, 10, 11], summarises their relevence to the stated problem, and discusses some unsuccessful attempts.

Resumen: Una *transversal* de una partición es un conjunto que contiene exactamente un elemento de cada miembro de la partición y nada más. Una *coloración* de un grafo es una partición de sus vértices en conjuntos independientes, es decir, conjuntos de vértices no adyacentes entre sí. Nosotros estudiamos el siguiente problema: dada una transversal T de una coloración $\mathcal{C}$ de un grafo G, ¿existe alguna partición $\mathfrak{H}$ de un subconjunto de $V(G)$ en conjuntos conexos tal que T sea una transversal de $\mathfrak{H}$ y cualesquiera dos conjuntos distintos de $\mathfrak{H}$ sean adyacentes?

Matthias Kriesell [9] conjeturó que, para cualquier transversal T de orden k de una k-coloración $\mathcal{C}$ de algún grafo G tal que cualquier par de clases de colores inducen un subgrafo conexo, existe tal partición $\mathfrak{H}$ con conjuntos adyacentes dos a dos. Esto demostraría la conjetura de Hadwiger para la clase de grafos óptimamente coloreables de forma única; sin embargo, el problema sigue abierto para todo $k \geq 5$.

Este artículo presenta una visión general sobre esta conjetura. Expone resultados de mi tesis doctoral y los artículos relacionados [2, 10, 11], resume su relevancia con el problema planteado y discute algunos intentos fallidos.

Keywords: rooted minor, Hadwiger's conjecture.

MSC2010: 05C40, 05C15.

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1. Hadwiger's conjecture

Hadwiger's conjecture states that the order of a largest clique minor in a graph G is at least its chromatic number $\chi(G)$ [8]. It is known to be true for graphs with chromatic number at most 6, with $\chi(G) = 5$ and $\chi(G) = 6$ being merely equivalent to the Four-Colour-Theorem [14]. Even for subclasses of graphs, Hadwiger's conjecture seems to be challenging. Though it is solved for line graphs [13], only partial results exist for claw-free graphs [7]. Paul Erdős [3] stated that it is "one of the deepest unsolved problems in graph theory", thus reinforcing the extreme nature and difficulty of this conjecture.

Instead of restricting to subclasses of graphs, one could also uniformly bound the order of the colour classes. But even when forbidding anticliques of order 3, Hadwiger's conjecture is widely open. This variant is stated in a conjecture of Seymour (see [1]).

Matthias Kriesell suggested in [9] to bound the number of colourings and, in particular, consider uniquely optimally colourable graphs. We will be interested in a rooted version of Hadwiger's conjecture that imposes additional assumptions on the colourings.

2. Kempe colourings

All graphs in the present paper are assumed to be finite, undirected, and simple. For terminology not defined here we refer to contemporary textbooks such as [4] or [6]. By K_S we denote the complete graph on a finite set S. A (minimal) *transversal* of a set $\mathfrak{C}$ of disjoint sets is a set T containing exactly one member of every $A \in \mathfrak{C}$ and nothing else; we also say that $\mathfrak{C}$ is *traversed* by T. A *colouring* of a graph G is a partition $\mathcal{C}$ of its vertex set $V(G)$ into *anticliques*, that is, sets of pairwise non-adjacent vertices. The *order* of a colouring $\mathcal{C}$ is the number of anticliques in $\mathcal{C}$ and an *optimal* colouring is a colouring of smallest order. The *chromatic number* $\chi(G)$ is the order of an optimal colouring of G. A *Kempe chain* is a connected component of $G[A \cup B]$ for some $A \neq B$ from $\mathcal{C}$.

We call a graph G *uniquely* k*-colourable* if $\chi(G) = k$ and for any two optimal colourings $\mathcal{C}$ and $\mathcal{C}'$ of G, we have $\mathcal{C} = \mathcal{C}'$. Such graphs have the property that the union of any two distinct colour classes induces a connected graph [5]. To see this, assume to the contrary that there is a graph G with a unique optimal colouring $\mathcal{C}$ and there are $A, B \in \mathcal{C}$, $A \neq B$, such that $G[A \cup B]$ has at least two components. Let H be one of the components and consider the colouring $\mathcal{C}'$ with

$$\mathcal{C}' = (\mathcal{C} \setminus \{A, B\}) \cup \{(A \setminus V(H)) \cup (B \cap V(H))\} \cup \{(B \setminus V(H)) \cup (A \cap V(H))\}.$$

Then $\mathcal{C}'$ is another optimal colouring of G and distinct from $\mathcal{C}$, a contradiction.

However restricting to uniquely colourable graphs seems excessive provided that in most situations we only make use of the above property that any two colour classes induce a connected graph. This leads us to the following definition.

Definition 1. A colouring $\mathcal{C}$ of a graph G is a *Kempe colouring* if any two vertices from distinct colour classes belong to the same Kempe chain or, in other words, the union of any two colour classes induces a connected subgraph in G. ◂

If G is graph and $\mathcal{C}$ is a Kempe colouring of G for which the vertices $x_1, \ldots, x_k$ are given different colours, then it is easy to see that there exists a system of edge-disjoint x_i,x_j-paths ($i \neq j$ from $\{1, \ldots, k\}$), a so-called *(weak) clique immersion* of order k at $x_1, \ldots, x_k$. The natural question to ask is whether there exists a *clique minor* of the same order such that $x_1, \ldots, x_k$ are in different bags.

A graph H is a *minor* of a graph G if there exists a family $c = (V_t)_{t \in V(H)}$ of pairwise disjoint subsets of $V(G)$, called *bags*, such that V_t is nonempty and $G[V_t]$ is connected for all $t \in V(H)$ and there is an edge connecting V_t and V_s for all $st \in E(H)$. Any such c is called an *H-certificate* in G, and a *rooted H-certificate* if, moreover, $V(H) \subseteq V(G)$ and $t \in V_t$ for all $t \in V(H)$. If there exists a rooted H-certificate, then H is a *rooted minor* of G.

A positive answer to the question above was conjecture by Matthias Kriesell.

Conjecture 2 (Kriesell [9]). *Let G be a graph, $\mathcal{C}$ be a Kempe colouring of size k and T a transversal of $\mathcal{C}$, then G contains a K_k-minor rooted at T.*

Using a result by Fabila-Monroy and Wood [15], a confirmation of Conjecture 2 for $k \leq 4$ follows immediately. In [11], it is proved for line graphs.

Theorem 3 (Kriesell, Mohr [11]). *For every transversal T of every Kempe colouring of the line graph $L(G)$ of any graph G there exists a complete minor in $L(G)$ traversed by T.*

It should be mentioned that a Kempe colouring can have significantly more colours than an optimal colouring. However a positive answer to Conjecture 2 will prove Hadwiger's conjecture for uniquely colourable graphs.

3. Two-coloured paths

In the previous section, we have seen that for each transversal T of any Kempe colouring of order k, there exists a clique immersion of order k at T. It is natural to ask whether the requirement of a Kempe colouring can be weakened to only demanding that two distinct vertices x, y of a transversal T belong to the same connected component of $G[A \cup B]$, where $A, B \in \mathcal{C}$, $\mathcal{C}$ is the colouring of the graph G, and $x \in A, y \in B$.

Conjecture 4. *Let G be a graph and $\mathcal{C}$ be one of its k-colourings (k not necessary optimal). Furthermore, let T be an arbitrary transversal of $\mathcal{C}$.*

Assume that for each pair of distinct vertices $x, y \in T$ there is a Kempe chain containing both vertices x and y. Then G contains a K_k-minor rooted at T.

Conjecture 2 would follow if Conjecture 4 held for all graphs G, colourings $\mathcal{C}$, and transversals T. However Conjecture 4 turned out to be too restrictive to be true: There exists a graph with a 7-colouring that does not contain a rooted K_7-minor.

Theorem 5 (Kriesell, Mohr [10]).

(i) *Let G be a graph and $\mathcal{C}$ be one of its k-colourings (k not necessary optimal). Furthermore, let T be an arbitrary transversal of $\mathcal{C}$. Assume that for each pair of distinct vertices $x, y \in T$ there is a Kempe chain containing both vertices x and y.*

If $k \leq 4$, or $k = 5$ and $G[T]$ is connected, then G contains a K_k-minor rooted at T.

(ii) *There is a graph with a 7-colouring $\mathcal{C}$ and a transversal T of $\mathcal{C}$ such that each pair of distinct vertices $x, y \in T$ belongs to the same Kempe chain, and this graph does not contain a K_7-minor rooted at T.*

We have seen that the setting of Conjecture 4 is insufficient to guarantee a rooted K_7-minor (and any K_k-minor with $k \geq 7$). This troublesome graph K_7 is known to be the smallest 6-connected graph and one may ask whether it is possible to find a 6-connected minor instead.

To address this problem, we move away from colourings and ask the following question [12]:
Given an integer k, does there exist an integer $\ell(k)$ such that for each graph G and $X \subseteq V(G)$ for which there is no separator S in G with $|S| < \ell(k)$ separating vertices of X, G has a k-connected minor (or topological minor) that "contains X"?

This questions demands a local connectedness of the vertices from X. Clearly, $\ell(k)$ must be at least k since X might be equal to $V(G)$. If each separator S in a graph G with $|S| < \ell$ splits the graph into components such that only one contains vertices from X ($|X| \geq \ell + 1$), we say that X is *ℓ-connected in G*. Moreover, the definition of a rooted minor given in Section 2 is not suitable in our setting since the k-connected minor can have significantly more vertices than $|X|$. We adapt the definition and say that a graph H is an *X-minor* of a graph G with $X \subseteq V(G)$ if $X \subseteq V(H)$ and there exists an H-certificate $c = (V_t)_{t \in V(H)}$ in G such that $t \in V_t$ for all $t \in X$. Armed with this refined definition, we prove the following:

Theorem 6 (Böhme, Harant, Kriesell, Mohr, Schmidt [2]). *Let $k \in \{1,2,3,4\}$, G be a graph, and $X \subseteq V(G)$ be a k-connected set in G. Then:*

(i) *G has a k-connected X-minor.*

(ii) *If $1 \leq k \leq 3$, then G has a k-connected topological X-minor.*

Moreover, the theorem is best possible in the sense that there exist graphs G_1 and G_2 with $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_2)$ such that X_1 (X_2) is 5-connected (4-connected) in G_1 (G_2) and neither G_1 nor G_2 contain a 5-connected X_1-minor and a 4-connected topological X_2-minor, respectively.

In summary, Theorems 5 and 6 lead us to the following insights. First, to confirm Conjecture 2 and in turn prove the rooted version of Hadwiger's conjecture for uniquely colourable graphs, it is not possible to restrict to clique immersions. It would seem that a certain connectedness property, which is provided by Kempe colourings, is necessary. Second, lifting the problem away from the colouring doesn't feel like a promising approach either, since a high connectedness gives no guarantee on any highly connected minor.

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Constructing quadratic 2-step nilpotent Lie algebras

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Abstract: A quadratic Lie algebra is a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form. Among all these algebras, we are going to focus on the nilpotent ones whose nilpotency index is two and, particularly, on those which are reduced. There exist different techniques to construct these algebras. Double extension and T^*-extension are recursive methods that allow us to start from smaller dimensions and grow up. Fixing an appropriate basis and using its definition gives us another approach to these algebras. And finally, we have that their classification is equivalent to the alternating trilinear forms one.

Resumen: Un álgebra de Lie cuadrática es un álgebra de Lie dotada de una forma bilineal invariante simétrica no degenerada. Entre todas las álgebras que cumplen estas condiciones, vamos a centrarnos en aquellas que sean nilpotentes y cuyo índice de nilpotencia sea 2, en particular, aquellas reducidas. Existen diferentes técnicas para construir este tipo de álgebras. La doble extensión y T^*-extensión son métodos clásicos recursivos que nos permiten obtenerlas partiendo de dimensiones pequeñas y aumentando progresivamente. Si fijamos una base apropiada y usamos su definición, junto a alguna propiedades, conseguimos una nueva aproximación. Finalmente, tenemos que su clasificación es equivalente a la de formas trilineales alternadas.

Keywords: nilpotent Lie algebra, quadratic algebra, trilinear form.

MSC2010: 15A69, 17B01, 17B05, 17B30.

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1. Keywords

The main concepts we need in this paper are the following:

Definition 1 (Lie algebra). A Lie algebra is a vector space $\mathfrak{n}$ with an alternating bilinear form $[\cdot,\cdot]: \mathfrak{n}\times\mathfrak{n} \to \mathfrak{n}$ called Lie bracket that satisfies the Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$. ◂

Definition 2 (t-step nilpotent). We say a Lie algebra $\mathfrak{n}$ is t-step nilpotent when $\mathfrak{n}^{t+1} = [\mathfrak{n}^t, \mathfrak{n}] = 0$, but $\mathfrak{n}^t \neq 0$, and where $\mathfrak{n}^1 = \mathfrak{n}$, $[A,B] := \operatorname{span}\langle [a,b] : a \in A, b \in B\rangle$. ◂

Definition 3 (quadratic). A quadratic Lie algebra $\mathfrak{n}$ is a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form $f: \mathfrak{n}\times\mathfrak{n} \to \mathbb{K}$, which means that $f([x,y],z) + f(y,[x,z]) = 0$ for every $x,\ y,\ z \in \mathfrak{n}$. ◂

Definition 4 (reduced). An algebra $\mathfrak{n}$ is said to be reduced in case $Z(\mathfrak{n}) \subseteq \mathfrak{n}^2$. ◂

And, as stated in [9, Theorem 6.2]:

Theorem 5. *Any non-reduced and non-abelian quadratic Lie algebra $(\mathfrak{n},\varphi)$ decomposes as an orthogonal direct sum of proper ideals, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{a}$, where $\varphi = \varphi_1 \perp \varphi_2$ and $(\mathfrak{n}_1,\varphi_1)$ is a quadratic reduced Lie algebra and $(\mathfrak{a},\varphi_2)$ is a quadratic abelian algebra.*

Finally, we will note as $\mathfrak{n}_{d,t}$ the free t-step Lie algebra on d generators (see [1] for a formal definition).

2. Constructions

There exist several ways to construct quadratic Lie algebras or equivalent structures. In this section we give an overview of some of them, with focus on the 2-step case.

Unless we specify the contrary, we will work over a generic field $\mathbb{K}$ and (A,f) will be a generic finite-dimensional Lie algebra, while A^* will denote its dual space. Moreover, ad^* will represent the coadjoint representation (i.e., $\operatorname{ad}^*(a)(\alpha)(a') = -\alpha([a,a'])$ for $a, a' \in A$ and $\alpha \in A^*$).

2.1. Double extension

The first way is the classic double extension method (see [7] or [3]). To begin with the extension we need, apart from (A,f) over a field $\mathbb{K}$ of characteristic zero, another finite-dimensional Lie algebra B in the same field and a Lie homomorphism $\phi: B \to \operatorname{Der}_f(A)$ where $\operatorname{Der}_f(A)$ is the space of all f-antisymmetric derivations of A (i.e., $f(d(a),a') + f(a,d(a')) = 0$ for $d \in \operatorname{Der}_f(A)$ and $a, a' \in A$). Let us define $w: A\times A \to B^*$ as $(a,a') \mapsto (b \mapsto f(\phi(b)(a),a'))$ for $b \in B$ and $a, a' \in A$. If we take the vector space $A_B := B \oplus A \oplus B^*$, define the following multiplication:

$$[b + a + \beta,\ b' + a' + \beta'] := [b,b'] + \phi(b)(a') - \phi(b')(a) + [a,a'] + w(a,a') + \operatorname{ad}^*(b)(\beta') - \operatorname{ad}^*(b')(\beta),$$

and the following symmetric bilinear form f_B on A_B:

$$f_B(b + a + \beta, b' + a' + \beta') := \beta(b') + \beta(b) + f(a,a'),$$

for $b, b' \in B$, $a, a' \in A$, $\beta, \beta' \in B^*$. Then, the pair (A_B, f_B) is a metrised Lie algebra over $\mathbb{K}$ and is called the double extension of A by ϕ and B.

And, as we can deduce from [7, Théorème III]:

Corollary 6. *In characteristic zero, every quadratic solvable Lie algebra can be obtained from an abelian Lie algebra extended by successive direct sums and double extensions by one-dimensional algebras.*

In [4, Section 5] we can find examples of indecomposable quadratic t-step nilpotent Lie algebras (arbitary t). The examples include the complete classification up to dimension 7 [4, 5.1. Proposition].

2.2. T^*-extension

The T^*-extension is a one-step method which was introduced in [3]. In contrast to double extension, it can be applied not only to Lie algebras, but to arbitrary nonassociative algebras.

For a Lie algebra B, we consider an arbitrary $w\colon B \times B \to B^*$ bilinear map and define the following multiplication on the vector space $T_w^*B \coloneqq B \oplus B^*$ for $b, b' \in B$ and $\beta, \beta' \in B^*$:

$$[b+\beta,\ b'+\beta'] \coloneqq [b,b'] + w(b,b') + \mathrm{ad}^*(b)(\beta') - \mathrm{ad}^*(b')(\beta).$$

Moreover, we consider the symmetric bilinear form q_B in $B \oplus B^*$ defined as follows:

$$q_B(b+\beta, b'+\beta') \coloneqq \beta(b') + \beta'(b).$$

And, as seen in [3, Lemma 3.1] we know if B, B^*, w and q_B are as above, then the pair $(B \oplus B^*, q_B)$ is a metrised algebra if and only if w is cyclic (i.e., $w(a,b)(c) = w(c,a)(b) = w(b,c)(a)$ for all $a,b,c \in B$).

Finally, we have the following theorem (see [3, Theorem 3.2]), which is really convenient as every quadratic 2-step Lie algebra fulfils every condition.

Theorem 7. *Let (A,f) be a metrised algebra of finite dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then, (A,f) will be isometric to a T^*-extension (T_w^*B, q_B) if and only if n is even and A contains an isotropic ideal I (i.e., $I \subset I^{\perp}$) of dimension $n/2$. In this case: $B \cong A/I$. Note that any isotropic $n/2$-dimensional subspace I of A is an ideal of A if and only if it is abelian, i.e., $I^2 = 0$.*

2.3. Computational approach using Hall Basis

Having a well-defined basis is the first requirement to be able to define algorithmically a construction method. For this purpose, we can use the Hall Basis defined in [6].

The Hall Basis of $\mathfrak{n}_{d,2}$ is $\{x_i : i = d, \dots, 1\} \cup \{[x_i, x_j] : i = 1, \dots, d;\ j = i+1, \dots, d\}$. As we can see, the main advantage of this basis is that the Lie products of every element are already defined, taking into account that every element $[x_i, x_j]$ belongs to the centre as this is a 2-step free nilpotent Lie algebra. And, as stated in [5], any 2-step nilpotent Lie algebra $\mathfrak{n}$ of type d is a homomorphic image of $\mathfrak{n}_{d,2}$ as $\mathfrak{n} \cong \mathfrak{n}_{d,2}/I$, with I an ideal of $\mathfrak{n}_{d,2}$ such that $I \subsetneq \mathfrak{n}_{d,2}^2$.

So we only need to know how the bilinear form works. For this part we can generate a generic symmetric matrix of dimension $\frac{d(d+1)}{2} \times \frac{d(d+1)}{2}$. After that, we just have to reduce the variables in the entries of the matrix by imposing the bilinear form is invariant. The whole process is detailed in [1], where lots of examples are displayed.

Finally, we have to find the kernel of the bilinear form to do the quotient by it, as the bilinear form is non-degenerate. And, every quadratic 2-step nilpotent Lie algebra can be obtained this way as we can see in [1, Proposition 4.1]. This proposition says:

Proposition 8. *Let $(\mathfrak{n}, B)$ be a quadratic 2-step nilpotent Lie algebra of type d and $\varphi\colon \mathfrak{n}_{d,2}/I \to \mathfrak{n}$ be an isomorphism of Lie algebras. If we take the map $\overline{B}\colon \mathfrak{n}_{d,2}/I \times \mathfrak{n}_{d,2}/I \to \mathbb{K}$ defined as $\overline{B}(x+I, y+I) = B(\varphi(x+I), \varphi(y+I))$, then φ is an isometry from $(\mathfrak{n}_{d,2}/I, \overline{B})$ onto $(\mathfrak{n}, B)$.*

It is worthwhile to mention that the kernel of this bilinear form is always of dimension $\frac{d(d+1)}{2} - 2d = \frac{d(d-3)}{2}$ for quadratic 2-step Lie algebras. Indeed, using this property shared by all these algebras, we know that their dimension is always $2d$ and we can simplify the process, as we can see in [2].

2.4. Trivectors

In [8, 3.5 Théorème and 3.6 Corollaire] the relation between quadratic 2-step nilpotent Lie algebras and trivectors appears. They are the following ones:

Theorem 9. *There exists a natural bijection between isomorphism classes of reduced quadratic 2-step nilpotent Lie algebras and dimension $2n$ and the equivalence classes of trilinear forms of rank n.*

Corollary 10. *In an algebraically closed field or $\mathbb{R}$, there exists a finite number of isomorphic classes of reduced quadratic 2-step nilpotent Lie algebras if its dimension is less than 17.*

3. Conclusions

The first clear conclusion we obtain is that having this variety of methods gives us a lot of possibilities. We have several approaches and we can choose the one that fits better for our case.

If we focus on the classic methods (double and T^* extensions), which have been extensively studied, both allow us to incrementally construct all these algebras. The main difference is that:

- Double extension is a more general method but involves several steps.
- T^*-extension is a simpler method, as it is just one step, but it is only valid for some particular Lie algebras. Although it can be used for more general algebras than the Lie ones.

Nevertheless, for the algebras we are interested in, nilpotent 2-step, both methods are perfectly valid for reaching all of them.

On the other hand, the computational approach using Hall Basis is a newer method which can be quite convenient for constructing a lot of examples or checking if some algebra belongs to the class of Lie algebras we are interested in. Moreover, this method can be easily extended to an arbitrary nilpotency index without trouble, and even more, for 2-step Lie algebras we can improve the efficiency using special features of this particular case.

Finally, the fact that trivectors are equivalent allows us to obtain a classification of these algebras, as trivectors have been already classified. Therefore, we can know how many quadratic 2-step Lie algebras are there up to isometrically isomorphisms using less than 9 generators. This data is show in Table 1.

Dimension	6	8	10	12	14	16	≥ 18
Number	1	0	1	2	5	13	∞

Table 1: Non-isometric reduced quadratic 2-step Lie algebras in $\mathbb{C}$ (source [10]).

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On circles enclosing many points

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Abstract: We prove that in every set of n red and n blue points in the plane there are a red and a blue point such that every circle having them in its boundary encloses at least $n(1-1/\sqrt{2})-o(n)$ other points of the set. This is a bichromatic version of a problem introduced by Neumann-Lara and Urrutia. In addition, we show that every set S of n points contains two points such that every circle passing through them encloses at most $\lfloor\frac{2n-1}{3}\rfloor$ other points of S. The results are proved using properties of order-k Voronoi diagrams, in the spirit of the work of Edelsbrunner, Hasan, Seidel and Shen on this problem.

Resumen: Demostramos que en cualquier conjunto de n puntos rojos y n puntos azules en el plano existen un punto rojo y un punto azul tales que cualquier circunferencia que pase por ellos contiene en su interior al menos $n(1-1/\sqrt{2})-o(n)$ puntos del conjunto. Esta es una versión bicromática de un problema propuesto por Neumann-Lara y Urrutia. También probamos que todo conjunto S de n puntos en el plano contiene dos puntos tales que cualquier circunferencia que pase por ellos contiene como mucho $\lfloor\frac{2n-1}{3}\rfloor$ otros puntos de S. Las demostraciones usan propiedades de los diagramas de Voronoi de orden k, al estilo del trabajo de Edelsbrunner, Hasan, Seidel y Shen en este problema.

Keywords: point set, circle containment, Voronoi diagram.

MSC2010: 52C99.

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1. Introduction

Let $\ell(n)$ be the largest number such that every set S of n points in general position in the plane has the following property: There exist $p, q \in S$ such that every circle passing through p and q contains at least $\ell(n)$ other points of S. Neumann-Lara and Urrutia [7] introduced this problem and obtained the bound $\ell(n) \geq \lceil \frac{n-2}{60} \rceil$. This bound was not tight and hence it was improved in a series of papers [1, 4, 5]. The best known bound up-to-date was obtained by Edelsbrunner *et al.* [3], who proved that $\ell(n) \geq n(\frac{1}{2} - \frac{1}{\sqrt{12}}) + O(1) \approx \frac{n}{4.7}$. Later, Ramos and Viaña [9] obtained an independent proof of this lower bound and further proved the following result:

Theorem 1 ([9]). *Every set S of n points in general position in the plane contains two points such that each circle passing through them encloses at least k and at most $n-k-2$ points of S, for $k = (\frac{1}{2} - \frac{1}{\sqrt{12}})n - o(n)$.*

We present an alternative proof of Theorem 1 making use of properties of order-k Voronoi diagrams. The techniques that we use in our proof allow us to obtain two new results: An upper bound condition, Theorem 2, and a bichromatic result, Theorem 3, stated below. The chromatic problem was introduced by Prodromou [8] with d dimensions and $\lfloor \frac{d+3}{2} \rfloor$ colors. In the particular case $d = 2$, it is proved that every set of n red points and m blue points contains a red point and a blue point such that every circle passing through them encloses $\frac{n+m}{36}$ other points of the set. Our result improves this bound.

This is an extended abstract of manuscript [2].

2. Circles and Voronoi diagrams

An order-k Voronoi diagram of a point set S is a subdivision of the plane into regions such that all the points in the same region have the same k closest points of S. The borders between regions are segments of the perpendicular bisectors between pairs of points in S. This is a key concept in our proof of Theorem 1 because the segments of the order-k Voronoi diagram of S are precisely the centers of the circles through two points of S that enclose exactly $k-1$ other points of S [6]. We say that a segment of the perpendicular bisector b_{pq} of p and q has weight k if all the circles through p and q with center in such segment enclose k other points of S. Thus, the segments of the order-k Voronoi diagram have weight $k-1$, see Figure 1.

3. New results

Following the ideas in the previous section, we study an upper bound version of the circle containment problem. Let $u(n)$ be the smallest number such that every set S of n points in general position in the plane has the following property: There exist $p, q \in S$ such that every circle passing through p and q contains at most $u(n)$ other points of S. In Theorem 2 we prove that $u(n) \leq \lfloor \frac{2n-1}{3} \rfloor$.

Theorem 2. *Let S be a set of $n \geq 3$ points in general position in the plane. Then, S contains two points such that every circle passing through them encloses at most $\lfloor \frac{2n-1}{3} \rfloor$ points of S.*

Adapting the proof of Theorem 1 to only consider circles passing through a red point and a blue point, we obtain the following result.

Theorem 3. *Every set S of n red points and $m = \lfloor cn \rfloor$, for $c \in (0, 1]$, blue points in general position in the plane contains a red point p and a blue point q such that any circle passing through them encloses at least $\frac{n+m-\sqrt{n^2+m^2}}{2} - o(n+m)$ points of S.*

For $n = m$, Theorem 3 gives the bound $n(1 - \frac{1}{\sqrt{2}}) - o(n) \approx \frac{n}{3.4}$.

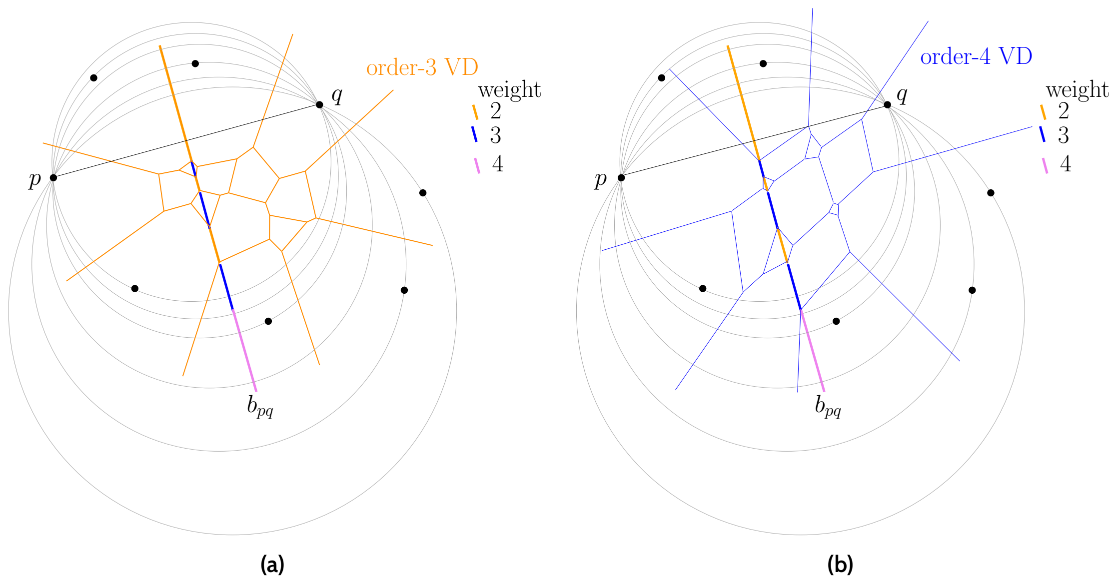

Figure 1: Relation between the order-k Voronoi diagram and the circle containment problem. (a) The segments of weight 2 are edges of the order-3 Voronoi diagram; (b) The segments of weight 3 are edges of the order-4 Voronoi diagram.

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Universal spectral covers and the Hitchin map

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Abstract: We study spectral data for pairs (E, φ), where $E \to X$ is a vector bundle over a smooth projective variety and $\varphi : E \to E \otimes V$ is an endomorphism "twisted" by another vector bundle $V \to X$, satisfying a *commuting condition* $\varphi \wedge \varphi = 0$. When $V = \Omega_X^1$ these pairs are known as *Higgs bundles*, which are intimately related to linear representations of the fundamental group of X.

Studying spectral data for this kind of objects consists on describing the fibres of a certain *Hitchin map*. In order to do this, we review the construction of the *universal spectral cover* and the *spectral correspondence* given in a recent paper by Chen and Ngô [2].

Resumen: Realizamos un estudio de los datos espectrales de pares (E, φ), donde $E \to X$ es un fibrado vectorial sobre una variedad proyectiva lisa y $\varphi : E \to E \otimes V$ es un endomorfismo "torcido" por otro fibrado vectorial $V \to X$, satisfaciendo una *condición de conmutación* $\varphi \wedge \varphi = 0$. Cuando $V = \Omega_X^1$ estos pares se conocen como *fibrados de Higgs*, que están íntimamente relacionados con las representaciones lineales del grupo fundamental de X.

Estudiar los datos espectrales para este tipo de objetos consiste en describir las fibras de una cierta *aplicación de Hitchin*. Para hacer esto, repasamos la construcción de la *cubierta espectral universal* y la *correspondencia espectral* dadas en un artículo reciente de Chen y Ngô [2].

Keywords: spectral data, Higgs bundles, Hitchin morphism.

MSC2010: 14D20, 14H60, 14J60.

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1. Introduction

Let k be an algebraically closed field, X a smooth projective variety over k and $V \to X$ a rank r vector bundle over X. We are interested in studying pairs (E, φ), with $E \to X$ a rank n vector bundle and $\varphi : E \to E \otimes V$ an "endomorphism twisted by V", satisfying a *commuting condition*: $\varphi \wedge \varphi = 0$ as a morphism $E \to E \otimes \wedge^2 V$. Locally, this means that φ can be written as $(\varphi_1, \dots, \varphi_r)$, with $[\varphi_i, \varphi_j] = 0$.

A particularly interesting situation occurs when $V = \Omega^1_X$ is the cotangent bundle of X. In that case we say that such an (E, φ) is a *Higgs bundle*. These objects were introduced by Hitchin in 1987 [5] in the case that X is a Riemann surface, and later generalized by Simpson [6] to Kähler manifolds of higher dimension.

When $k = \mathbb{C}$ is the field of complex numbers, a theorem of Corlette [3] and Simpson [6] identifies the moduli space of *stable* Higgs bundles with the *character variety* parametrizing irreducible representations of the fundamental group $\pi_1(X)$ on $\mathrm{GL}_n(\mathbb{C})$.

We denote by $\mathcal{M}_{n,V}$ the moduli stack of such pairs (E, φ), where E has rank n. The Hitchin morphism is defined as the map

$$h_{n,V} : \mathcal{M}_{n,V} \to \bigoplus_{i=1}^{n} H^0(X, S^i V) \; : \; (E, \varphi) \mapsto (\sigma_1(\varphi), \dots, \sigma_n(\varphi)),$$

where the $\sigma_i(\varphi)$ are the coefficients of the "characteristic polynomial" of φ,

$$p_\varphi(T) = \det(T - \varphi) = T^n + \textstyle\sum_{i=1}^n \sigma_i(\varphi) T^n.$$

We are interested in *studying the fibres of this morphism.*

When $\dim X = 1$ and the twisting bundle is a line bundle $V = L$, if L^n is base point free, then for a generic $b \in \bigoplus_{i=1}^n (X, L^i)$ there exists a smooth *spectral curve* Y_b, with a finite morphism $Y_b \to X$ such that the fibre of the Hitchin map over L is in correspondence with the Picard group of Y_b. This was shown by Hitchin in 1987 [4] in the case of Higgs bundles ($L = \Omega^1_X$) and later generalized by Beauville, Narasimhan and Ramanan [1] for a general line bundle L.

In particular, what Hitchin proved in [4] is that the Hitchin map endows the moduli space of stable Higgs bundles on a Riemann surface of genus $g \geq 2$ with the structure of an *algebraically integrable system.*

We seek similar results to that of Beauville-Narasimhan-Ramanan for general values of $\dim X$ and $\mathrm{rk}\, V$.

A recent paper by Chen and Ngô [2] has shed some light into this problem, by (among other things) giving an interpretation of the Hitchin map in terms of "universal spectral data" and a proving a spectral correspondence for Higgs bundles over projective surfaces.

In this document we review some of the main ideas of Chen and Ngô's interpretation of the Hitchin map and the spectral correspondence.

2. Universal spectral data

Let E be a k-vector space of dimension n. Consider $\varphi_1, \dots, \varphi_r$ a family of r endomorphisms of E that pairwise commute, that is $[\varphi_i, \varphi_j] = 0$. This family generates a k-algebra $A = k[\varphi_1, \dots, \varphi_r]$. The evaluation morphism $k[X_1, \dots, X_r] \to A \subset \mathrm{End}\, E$ endows E with the structure of a $k[X_1, \dots, X_r]$-module. Geometrically, this can be interpreted as a sheaf $\mathcal{F}$ over the r-dimensional affine space $\mathbb{A}^r_k$ such that $p_* \mathcal{F} = E$, where $p : \mathbb{A}^r_k \to \mathrm{Spec}(k)$ is the natural morphism.

In the particular case where $r = 1$, so $A = k[\varphi]$ for some endomorphism $\varphi : E \to E$, since $k[T]$ is a PID, A can be written as $A = k[T]/(m_\varphi)$, where (m_φ) is the ideal generated by a polynomial m_φ which is by definition the *minimal polynomial* of φ. The roots of m_φ are precisely the eigenvalues $x_1, \dots, x_s \in k$ of φ. If $E = \bigoplus_{i=1}^s E_i$ is the spectral decomposition of E associated to φ and $n_i = \dim E_i$, we can consider the characteristic polynomial $p_\varphi(T) = (T - x_1)^{n_1} \cdots (T - x_s)^{n_s}$. The Cayley-Hamilton theorem asserts that $p_\varphi(\varphi) = 0$, and thus the minimal polynomial m_φ divides p_φ. Geometrically, this implies that the sheaf $\mathcal{F}$ is supported in a closed subscheme of $\mathrm{Spec}(k[T]/(p_\varphi))$.

In the general case of $r > 1$, the ring A is of the form $k[X_1, \dots, X_r]/I$, for I some ideal that in principle can be generated by several polynomials. However, we can take the primary decomposition of I, $I = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_s^{\alpha_s}$. This induces a decomposition of E, $E = \bigoplus_{i=1}^s E_i$. If we denote $n_i = \dim E_i$, we can consider the ideal $J = \mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_s^{n_s}$. Using the Nakayama lemma, one can prove a generalized version of the Cayley-Hamilton theorem asserting that $J \subseteq I$. Geometrically, this means that the sheaf $\mathcal{F}$ is supported in a closed subscheme of $\mathrm{Spec}(k[X_1, \dots, X_r]/J)$.

Therefore, we can associate to the algebra $A = k[\varphi_1, \dots, \varphi_r]$ a formal combination of points of $\mathbb{A}_k^r$ $\mathrm{sd}(A) = \sum_{i=1}^s n_i[x_i]$, with $\sum_{i=1}^s n_i = n$, where x_i is the point of $\mathbb{A}_k^r$ associated to the maximal ideal $\mathfrak{m}_i$. This formal combination $\mathrm{sd}(A)$ is known as the *spectral datum* of A.

The set parametrizing formal combinations of points of $\mathbb{A}_k^r$ of length n is the symmetric product $S^n\mathbb{A}_k^r = (\mathbb{A}^r \times \overset{(n)}{\dots} \times \mathbb{A}^r)/\mathfrak{S}^n$. A classical theorem of Weyl shows that there exists a closed embedding

$$\iota_{n,r} : S^n\mathbb{A}_k^r \to \mathbb{A}(k^r \oplus S^2k^r \oplus \cdots \oplus S^nk^r) : \sum_{i=1}^n [x_i] \mapsto (\sigma_1, \dots, \sigma_n),$$

where S^ik^r is the i-th symmetric product vector space of k^r and $\mathbb{A}$ denotes the functor sending a vector space over k to the associated affine space (regarded as a scheme), that is $\mathbb{A}(V) = \mathrm{Spec}(k[V])$. The elements σ_i are defined as

$$\sigma_1 = x_1 + \dots + x_n, \quad \sigma_2 = x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n, \quad \dots \quad \sigma_n = x_1 \cdots x_n.$$

In particular, for $r = 1$ the map $\iota_{n,r}$ is an isomorphism by the Fundamental Theorem on Symmetric Polynomials.

We can now consider an "universal characteristic polynomial" by defining the map

$$\chi_{n,r} : \mathbb{A}_k^r \times S^n\mathbb{A}_k^r \to \mathbb{A}(S^nk^r) : (x, \sum_{i=1}^n [x_i]) \mapsto (x - x_1) \cdots (x - x_n).$$

Indeed, note that $\chi_{n,r}\left(x, \sum_{i=1}^n [x_i]\right) = x^n - \sigma_1 x^{n-1} + \dots + (-1)^n\sigma_n$. We define the *Cayley scheme* as the 0-fibre of this map,

$$\mathrm{Cayley}_n(\mathbb{A}_k^r) = \chi_{n,r}^{-1}(0).$$

The natural projection $p_{n,r} : \mathrm{Cayley}_n(\mathbb{A}_k^r) \to S^n\mathbb{A}_k^r$ is called the *universal spectral cover*.

The following theorem sums up the main properties of this map:

Theorem 1. *The universal spectral cover $p_{n,r}$ is finite of degree n and generically étale over the set $(S^n\mathbb{A}_k^r)'$ consisting of multiplicity-free formal combinations. For $a = \sum_{i=1}^s n_i[x_i]$, we have*

$$p_{n,r}^{-1}(a) = \mathrm{Spec}\left(\frac{k[X_1, \dots, X_r]}{\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_s^{n_s}}\right).$$

Thus, $p_{n,r}$ is flat if and only if $r = 1$ or $n = 1$. Finally, if $A = k[\varphi_1, \dots, \varphi_r]$, the associated sheaf $\mathcal{F}$ is supported in a closed subscheme of $p_{n,r}^{-1}(\mathrm{sd}(A))$.

The last part of this theorem can be interpreted as a "universal version" of the Cayley-Hamilton theorem.

3. The Hitchin map

We come back now to the situation of the introduction, where X is a smooth projective variety over k and $V \to X$ is a rank r vector bundle over X. We can consider now the scheme $S^n(V/X)$, which is the result of twisting the space $S^n\mathbb{A}_k^r$ by the GL_r-torsor attached to V. In other words, $S^n(V/X)$ is a fibre bundle over X with fibre $S^n\mathbb{A}(V_x)$ over $x \in X$. We denote by $\mathcal{B}_{n,V}$ the set of sections of $S^n(V/X)$. The embedding $S^n\mathbb{A}_k^r \hookrightarrow \mathbb{A}(k^r \oplus S^2k^r \oplus \dots \oplus S^nk^r)$ induces an embedding $\iota_{n,V} : \mathcal{B}_{n,V} \hookrightarrow \bigoplus_{i=1}^n H^0(X, S^iV)$.

It is clear now that this gives a fatorization of the Hitchin morphism as $h_{n,V} = \iota_{n,V} \circ \mathrm{sd}$. Therefore, the problem of studying the fibres of $h_{n,V}$ is reduced to the problem of studying the fibres of the *spectral data map* $\mathrm{sd} : \mathcal{M}_{n,V} \to \mathcal{B}_{n,V}$.

Consider now a section $b \in \mathcal{B}_{n,V}$, which is a map $b\colon X \to S^n(V/X)$. If we twist the universal spectral cover by the GL_r-torsor attached to V, we get a map $\mathrm{Cayley}_n(V/X) \to S^n(V/X)$. Taking the fibered product of b and this map we get a degree n finite morphism $\pi\colon Y_b \to X$ factorizing by an embedding $Y_b \hookrightarrow V$. This is the *spectral cover*. Moreover, if b is generically multiplicity free, then π is generically étale. However, π is not a flat morphism in general.

Suppose now that (E, φ) is a pair with spectral data b, then the twisted endomorphism $\varphi\colon E \to E \otimes V$ can be seen as a morphism $V^\vee \to \operatorname{End} E$, which induces a morphism $S^\bullet V^\vee \to \operatorname{End} E$. Since $S^\bullet V^\vee = p_*\mathcal{O}_X$, the previous morphism defines a coherent sheaf $\mathcal{F}$ on V such that $p_*\mathcal{F} = E$. The universal version of the Cayley-Hamilton theorem implies that the support of $\mathcal{F}$ is contained precisely in the spectral cover Y_b. Therefore, we have a correspondence between those pairs (E, φ) with spectral data b and coherent sheaves $\mathcal{L}$ on Y_b with $\pi_*\mathcal{L} = E$. Moreover, we have the following lemma from commutative algebra:

Lemma 2. *Let A be a regular ring and B a finite A-algebra. Any B-module M is maximal Cohen-Macaulay over B if and only if it is locally free as an A-module.*

This implies the following correspondence:

Theorem 3. *The functor π_* gives an equivalence of categories between pairs (E, φ) with spectral data b and maximal Cohen-Macaulay sheaves on Y_b of generic rank 1.*

The main problem with the above result is that, since the map π is not flat in general, the category of maximal Cohen-Macaulay sheaves on Y_b might be empty. A way to solve this problem is by "modifying" π in order to obtain a flat morphism $\tilde{\pi}\colon \tilde{Y}_b \to X$. For example, if $\dim X = 1$, since any coherent sheaf on a curve can be decomposed as a direct sum of a locally free and a torsion sheaf, we can obtain a *flat spectral cover* just by removing the torsion of the structure sheaf. When $\dim X = 2$, a construction by Chen and Ngô [2] yields a *Cohen-Macaulayfication* of the spectral curve.

Moreover, note that if the spectral cover is an integral scheme, the corresponding sheaves over it are torsion-free and, if it is smooth, then they are locally free. In the case where the twisting bundle has rank 1 and its n-th power is base point free, Beauville, Narasimhan and Ramanan found that these good conditions on the spectral cover are in fact satisfied for a generic b. It would be interesting then to find similar conditions for the cases of general values for $\dim X$ and $\operatorname{rk} V$.

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Hyperreflexivity of the space of module homomorphisms between non-commutative L^p-spaces

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Abstract: Let $\mathcal{M}$ be a von Neumann algebra, and let $0 < p, q \leq \infty$. Then, the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all right $\mathcal{M}$-module homomorphisms from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ is a reflexive subspace of the space of all continuous linear maps from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$. Further, the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive in each of the following cases:

(i) $1 \leq q < p \leq \infty$;
(ii) $1 \leq p, q \leq \infty$ and $\mathcal{M}$ is injective, in which case the hyperreflexivity constant is at most 8.

Resumen: Sea $\mathcal{M}$ un álgebra de von Neumann y sean $0 < p, q \leq \infty$. El espacio $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ de los homomorfismos de $\mathcal{M}$-módulos a derechas de $L^p(\mathcal{M})$ en $L^q(\mathcal{M})$ es un subespacio reflexivo del espacio de aplicaciones lineales y continuas de $L^p(\mathcal{M})$ en $L^q(\mathcal{M})$. Además, el espacio $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ es hiperreflexivo en los siguientes casos:

(i) $1 \leq q < p \leq \infty$;
(ii) $1 \leq p, q \leq \infty$ y $\mathcal{M}$ es inyectiva, en cuyo caso la constante de hiperreflexividad es no mayor que 8.

Keywords: non-commutative L^p-spaces, injective von Neumann algebras, reflexive subspaces, hyperreflexive subspaces, module homomorphisms.

MSC2010: Primary 46L52, 46L10, Secondary 47L05.

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1. Introduction

Let $\mathcal{X}, \mathcal{Y}$ be quasi-Banach spaces, and let $\mathcal{S}$ be a closed linear subspace of $B(\mathcal{X}, \mathcal{Y})$. In accordance with Loginov and Šul'man [8], $\mathcal{S}$ is called reflexive if

$$\mathcal{S} = \{T \in B(\mathcal{X}, \mathcal{Y}) : T(x) \in \overline{\{S(x) : S \in \mathcal{S}\}} \,\forall\, x \in \mathcal{X}\}.$$

Following Larson [6, 7], in the case where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $\mathcal{S}$ is called hyperreflexive if there exists C such that

$$\operatorname{dist}(T, \mathcal{S}) \le C \sup_{\|x\| \le 1} \inf\{\|T(x) - S(x)\| : S \in \mathcal{S}\},$$

for all $T \in B(\mathcal{X}, \mathcal{Y})$, and the optimal constant is called the hyperreflexivity constant of $\mathcal{S}$.

If $\mathcal{H}$ is a Hilbert space and $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$, then the double commutant theorem shows that $\mathcal{A}$ is a reflexive subspace of $B(\mathcal{H})$. Christensen [2–4] showed that many von Neumann algebras are hyperreflexive, but the general case is still open.

The non-commutative L^p-spaces that we consider throughout are those introduced by Haagerup (see [5, 9, 10]). Let $\mathcal{M}$ be a von Neumann algebra. For each $0 < p \le \infty$, the space $L^p(\mathcal{M})$ is a contractive Banach $\mathcal{M}$-bimodule or a contractive p-Banach $\mathcal{M}$-bimodule according to $1 \le p$ or $p < 1$, and we will focus on the right $\mathcal{M}$-module structure of $L^p(\mathcal{M})$.

Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be quasi-Banach right $\mathcal{A}$-modules. An operator $T \in B(\mathcal{X}, \mathcal{Y})$ is a right $\mathcal{A}$-module homomorphism if

$$T(xa) = T(x)a \quad (x \in \mathcal{X},\ a \in \mathcal{A}).$$

We write $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ for the space of right $\mathcal{A}$-module homomorphisms from $\mathcal{X}$ to $\mathcal{Y}$.

For $T \in B(\mathcal{X}, \mathcal{Y})$ and $a \in \mathcal{A}$, define linear maps $aT,\ Ta : \mathcal{X} \to \mathcal{Y}$ by

$$(aT)(x) = T(xa), \quad (Ta)(x) = T(x)a \quad (x \in \mathcal{X}).$$

Let $\operatorname{ad}(T) : \mathcal{A} \to B(\mathcal{X}, \mathcal{Y})$ denote the inner derivation implemented by T, so that

$$\operatorname{ad}(T)(a) = aT - Ta \quad (a \in \mathcal{A}).$$

It is clear that T is a right $\mathcal{A}$-module homomorphism if and only if $\operatorname{ad}(T) = 0$, and, in the case where $\mathcal{X}$ and $\mathcal{Y}$ are Banach $\mathcal{A}$-modules, the constant $\|\operatorname{ad}(T)\|$ is intended to estimate the distance from T to the space $\operatorname{Hom}_{\mathcal{A}(\mathcal{X},\mathcal{Y})}$.

In [1], we study the reflexivity and hyperreflexivity of the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$, where $\mathcal{M}$ is a von Neumann algebra and $0 < p, q \le \infty$.

2. Bilinear maps and orthogonality

Our research is based on the analysis of bilinear maps that satisfy a certain orthogonality property.

Proposition. *Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{Z}$ be a topological vector space, and let $\varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{Z}$ be a continuous bilinear map.*

(i) *Suppose that*

$$e \in \operatorname{Proj}(\mathcal{M}) \implies \varphi(e, e^\perp) = 0.$$

Then,

$$\varphi(a, 1_{\mathcal{M}}) - \varphi(1_{\mathcal{M}}, a) = 0 \quad (a \in \mathcal{M}).$$

(ii) *Suppose that $\mathcal{Z}$ is a normed space and let the constant $\varepsilon \ge 0$ be such that*

$$e \in \operatorname{Proj}(\mathcal{M}) \implies \|\varphi(e, e^\perp)\| \le \varepsilon.$$

Then,

$$\|\varphi(a, 1_{\mathcal{M}}) - \varphi(1_{\mathcal{M}}, a)\| \le 8\varepsilon\|a\| \quad (a \in \mathcal{M}).$$

This result remains true even if $\mathcal{M}$ is just a unital C^*-algebra of real rank zero (see [1, Theorem 1.2]).

Proposition. *Let $\mathcal{M}$ be a von Neumann algebra and let $0 < p, q \leq \infty$. Let $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$.*

(i) *If*
$$e \in \operatorname{Proj}(\mathcal{M}) \implies e^{\perp} T e = 0,$$
then $T \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$.

(ii) *If $p, q \geq 1$, then*
$$\|\operatorname{ad}(T)\| \leq 8 \sup_{\|x\| \leq 1} \inf\{\|T(x) - \Phi(x)\| : \Phi \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\}.$$

This proposition is a direct consequence of the previous one, and it can be shown in much more general situations (see [1, Theorems 2.2, 2.3 and 2.4]).

3. Reflexivity and hyperreflexivity

The first part of the second proposition leads us to this result.

Theorem (Alaminos, Godoy and Villena [1, Corollary 2.11]). *Let $\mathcal{M}$ be a von Neumann algebra and let $0 < p, q \leq \infty$. Then, the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is reflexive.*

In [1], we show a slightly stronger result: if $T : L^p(\mathcal{M}) \to L^q(\mathcal{M})$ is a linear map such that
$$T(x) \in \overline{\{\Phi(x) : \Phi \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\}},$$
then $T \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$. Note that the continuity of T is not required.

The hyperreflexivity of these spaces is a substantially more complex problem. In fact, we need additional hypotheses to solve it and the general case is still open.

Theorem (Alaminos, Godoy and Villena [1, Theorems 3.7, 3.8 and 3.9]). *Let $\mathcal{M}$ be a von Neumann algebra and let $1 \leq p, q \leq \infty$.*

(i) *If $p = \infty$ or $q = 1$, then $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive and the hyperreflexivity constant is less or equal than 8.*

(ii) *If $\mathcal{M}$ is injective and $p, q \geq 1$, then $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive and the hyperreflexivity constant is less or equal than 8.*

(iii) *If $q < p$, then $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive and the hyperreflexivity constant is less or equal than a constant $C_{p,q}$ that does not depend on $\mathcal{M}$.*

The proof of the first and second parts of the theorem consists of showing that, given $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$, there is a homomorphism Φ such that
$$\|T - \Phi\| \leq \|\operatorname{ad}(T)\|,$$
and then the second part of the second proposition concludes the demonstration.

The third part is shown by assuming towards a contradiction that for each $n \in \mathbb{N}$ there is a von Neumann algebra $\mathcal{M}_n$ and an operator $T_n \in B(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))$ such that
$$\operatorname{dist}(T_n, \operatorname{Hom}_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) > n\|\operatorname{ad}(T_n)\| \quad (n \in \mathbb{N}).$$

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3rd *BYMAT Conference* (2020)

On discrete isoperimetric type inequalities

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Abstract: The isoperimetric inequality is one of the oldest and most outstanding results in mathematics, and can be summarized by saying that the Euclidean balls minimize the surface area measure $\mathrm{S}(\cdot)$ (Minkowski content) among those compact convex sets with prescribed positive volume $\mathrm{vol}(\cdot)$ (Lebesgue measure). It admits the following "*neighbourhood* form": for any compact convex set $K \subset \mathbb{R}^n$, and all $t \geq 0$,

$$\mathrm{vol}(K + tB_n) \geq \mathrm{vol}(rB_n + tB_n), \tag{1}$$

where $r > 0$ is such that $\mathrm{vol}(rB_n) = \mathrm{vol}(K)$ and B_n denotes the (closed) Euclidean unit ball.

In this talk we discuss and show a discrete analogue of the isoperimetric inequality in its form (1) for the *lattice point enumerator* $\mathrm{G}_n(K) = \#(K \cap \mathbb{Z}^n)$ of a bounded subset $K \subset \mathbb{R}^n$: we determine sets minimizing the functional $\mathrm{G}_n\big(K + t[-1, 1]^n\big)$, for any $t \geq 0$, among those bounded sets K with given positive lattice point enumerator. We also show that this new discrete inequality implies the classical result for compact sets. The results of this talk will appear in [5].

Resumen: La desigualdad isoperimétrica es uno de los resultados más antiguos de las matemáticas, y puede ser sintetizada en el hecho de que las bolas euclídeas minimizan la medida de área de superficie $\mathrm{S}(\cdot)$ (contenido de Minkowski) entre todos los conjuntos compactos y convexos con volumen positivo prescrito $\mathrm{vol}(\cdot)$ (medida de Lebesgue). Admite la siguiente "versión local": para todo conjunto compacto y convexo $K \subset \mathbb{R}^n$, y todo $t \geq 0$,

$$\mathrm{vol}(K + tB_n) \geq \mathrm{vol}(rB_n + tB_n), \tag{2}$$

donde $r > 0$ es tal que $\mathrm{vol}(rB_n) = \mathrm{vol}(K)$ y B_n denota la bola unidad euclídea (cerrada).

En esta charla discutimos y probamos un análogo discreto de la desigualdad isoperimétrica en su forma (2) para el *enumerador de puntos de retículo* $\mathrm{G}_n(K) = \#(K \cap \mathbb{Z}^n)$ de un conjunto acotado $K \subset \mathbb{R}^n$: determinamos los conjuntos que minimizan el funcional $\mathrm{G}_n\big(K + t[-1, 1]^n\big)$ para cualquier $t \geq 0$, entre todos los conjuntos acotados K con un enumerador de puntos de retículo positivo dado. También mostramos que esta nueva desigualdad discreta implica el correspondiente resultado clásico para conjuntos compactos. Los resultados de esta charla aparecerán en [5].

Keywords: isoperimetric inequality, lattice point enumerator, integer lattice.

MSC2010: Primary: 52C07, 11H06; Secondary: 52A40.

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1. Introduction

The classical *isoperimetric inequality* in its form for convex bodies (compact and convex sets) in $\mathbb{R}^n$ states that the volume $\mathrm{vol}(\cdot)$ (Lebesgue measure) and surface area $\mathrm{S}(\cdot)$ (Minkowski content) of any n-dimensional convex body K satisfy

$$\left(\frac{\mathrm{S}(K)}{\mathrm{S}(B_n)}\right)^n \geq \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_n)}\right)^{n-1}, \tag{3}$$

where B_n denotes the Euclidean (closed) unit ball. In other words, Euclidean balls minimize the surface area among those convex bodies with prescribed positive volume.

There exist various facets of the isoperimetric inequality, due to its different versions and extensions. Among other analogues of it we emphasize its equivalent analytic version, the *Sobolev inequality* (see e.g. [4, Section 5]), and its form for *mixed volumes*, the so-called *Minkowski first inequality* (see e.g. [10, Theorem 7.2.1]). *Diskant's inequality*, which can be regarded as an improvement of the latter, and the Bonnesen-type inequalities in the plane also deserve special attention (see e.g. [10, Section 7.2] and the references therein). The isoperimetric inequality also has various ramifications into other settings, such as its versions in the spherical and hyperbolic spaces (see e.g. [2]); it has been the starting point for new engaging related results, such as a reverse isoperimetric inequality (see [1]), and it has led to various remarkable consequences not only in geometry but also in analysis (see e.g. [3]). For an extensive survey article on this inequality we refer the reader to [7].

Let us denote by $+$ the Minkowski sum of sets, i.e., $A + B = \{a + b \,:\, a \in A,\ b \in B\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$. Also denote by rA the set $\{ra \,:\, a \in A\}$, for any $r \geq 0$. The isoperimetric inequality (3) admits the following "*neighbourhood* form" (see e.g. [6, Proposition 14.2.1]): for any n-dimensional convex body $K \subset \mathbb{R}^n$, and all $t \geq 0$, we have

$$\mathrm{vol}(K + tB_n) \geq \mathrm{vol}(rB_n + tB_n), \tag{4}$$

where rB_n, $r > 0$, is a ball of the same volume as K. In fact, by subtracting $\mathrm{vol}(K) = \mathrm{vol}(rB_n)$ and dividing by t in both sides of (4), and taking limits as $t \to 0^+$, one immediately gets (3) from (4).

The neighbourhood $K + tB_n$, $t \geq 0$, of the n-dimensional convex body K coincides with the set of all points of $\mathbb{R}^n$ having (Euclidean) distance from K at most t. Exchanging the role of the unit ball B_n in (4) by another (n-dimensional) convex body $E \subset \mathbb{R}^n$, i.e., changing the involved "distance", one is naturally led to the fact

$$\mathrm{vol}(K + tE) \geq \mathrm{vol}(rE + tE) \tag{5}$$

for all $t \geq 0$, where again $r > 0$ is such that rE has the same volume as K. Thus, the advantage of using the volume of a neighbourhood of K, instead of its surface area, is that it can be extended to other spaces in which the latter notion makes no sense; it just suffices to consider a metric and a measure on the given space. Relevant examples of spaces in which isoperimetric inequalities in this form hold are the unit sphere, the *Gauss space* or the n-dimensional discrete cube $\{0,1\}^n$ (see e.g. [6, Section 14.2]).

2. Main results

Let us start by defining a family of sets which will be shown to be optimal under the hypothesis of the discrete isoperimetric inequality. Given a vector $u = (u_1 \dots, u_n) \in \mathbb{Z}^n$ and fixing $i_u \in \{1, \dots, n\}$, we will write

$$u' = (u_1 \dots, u_{i_u-1}, u_{i_u+1}, \dots, u_n) \in \mathbb{Z}^{n-1}.$$

With this notation, in [8] the following well-order $\prec$ in $\mathbb{Z}^n$ is defined:

Definition 1. If $n = 1$ we define the order $\prec$ given by

$$0 \prec 1 \prec -1 \prec 2 \prec -2 \prec \cdots \prec m \prec -m \prec \dots$$

For $n \geq 2$ we set, for $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$,

$$M(w) = \max_{\prec}\{w_i \,:\, i = 1, \dots, n\} \quad \text{and} \quad i_w = \min\{i \,:\, w_i = M(w)\},$$

and we define $\prec$ recursively as follows: for any $u, v \in \mathbb{Z}^n$ with $u \neq v$,

(i) if $M(u) \prec M(v)$, then $u \prec v$;

(ii) if $M(u) = M(v)$, then $u \prec v$ if either $i_v < i_u$ or ($i_v = i_u$ and) $u' \prec v'$.

Moreover, we write $u \preceq v$ if either $u \prec v$ or $u = v$. ◂

This order will allow us to define the extended lattice cube $\mathcal{I}_r$ of r points as the initial segment in $\mathbb{Z}^n$ with respect to $\prec$. To define the sets $\mathcal{C}_r$, which will be referred to as *extended cubes*, first we need the following definition, which can be seen as a particular case of the family of *weakly unconditional sets*, first introduced in [9] (we refer the reader to this work for further properties and relations of them with certain Brunn-Minkowski type inequalities): for any non-empty finite set $A \subset \mathbb{R}^n$, we write

$$\mathcal{C}_A = \{(\lambda_1 x_1, \dots, \lambda_n x_n) \in \mathbb{R}^n \,:\, (x_1, \dots, x_n) \in A, \lambda_i \in [0, 1] \text{ for } i = 1, \dots, n\}.$$

(see Figure 1).

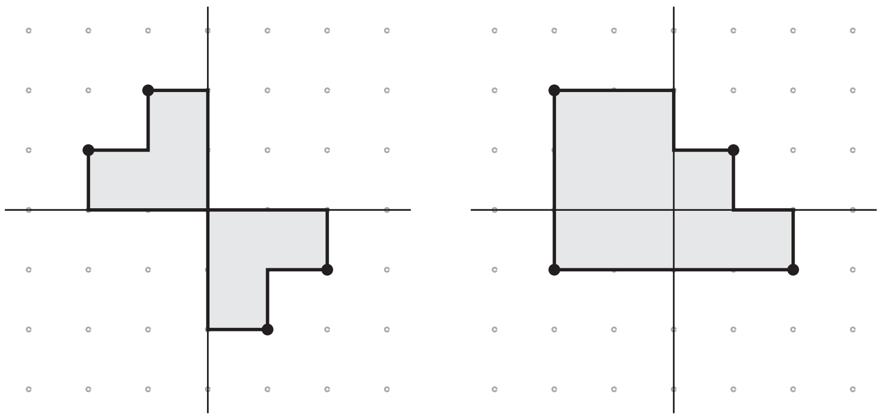

Figure 1: Sets $\mathcal{C}_A \subset \mathbb{R}^2$ for different finite sets $A \subset \mathbb{Z}^2$.

Definition 2. Let $r \in \mathbb{N}$. By $\mathcal{I}_r$ we denote the initial segment in $(\mathbb{Z}^n, \prec)$ of length r, i.e., the set of the first r points with respect to the order $\prec$ in $\mathbb{Z}^n$ (see Figure 2, left). Moreover, by $\mathcal{C}_r$ we denote the set given by $\mathcal{C}_r := \mathcal{C}_{\mathcal{I}_r}$ (see Figure 2, right). ◂

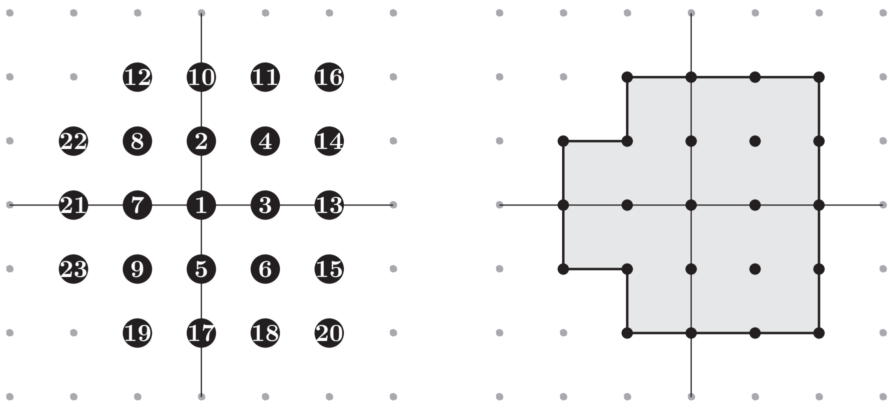

Figure 2: The extended lattice cube $\mathcal{I}_{23}$ in $\mathbb{Z}^2$ (left) and the corresponding extended cube $\mathcal{C}_{23}$ in $\mathbb{R}^2$ (right).

We note that if $r = m^n$ for some $m \in \mathbb{N}$ then $\mathcal{I}_r$ is indeed a lattice cube. More precisely, $\mathcal{I}_r = \{-m/2 + 1, -m/2+2, \dots, m/2-1, m/2\}^n$ if m is even and $\mathcal{I}_r = \big\{-(m-1)/2, -(m-1)/2+1, \dots, (m-1)/2, (m-1)/2\big\}^n$ if m is odd (cf. Figure 2, left). This further implies that $\mathcal{C}_r$ is a cube whenever $r = m^n$ for some $m \in \mathbb{N}$.

We are now ready to present the two main results. First, we obtain a discrete analogue of the classical isoperimetric inequality.

Theorem 3. *Let $K \subset \mathbb{R}^n$ be a bounded set with $\mathrm{G}_n(K) > 0$ and let $r \in \mathbb{N}$ be such that $\mathrm{G}_n(\mathcal{C}_r) = \mathrm{G}_n(K)$. Then,*

$$\mathrm{G}_n\big(K + t[-1,1]^n\big) \geq \mathrm{G}_n\big(\mathcal{C}_r + t[-1,1]^n\big) \tag{6}$$

for all $t \geq 0$.

And finally, we show that this result is, in a sense, stronger than the classical one.

Theorem 4. *The discrete isoperimetric inequality (6) implies the classical isoperimetric inequality (5), with $E = [-1,1]^n$, for non-empty compact sets.*

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Opportunistic maintenance under periodic inspections in heterogeneous complex systems

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Abstract: A complex system consisting of monitored and non-monitored components is studied. Monitored components are subject to a degradation process, following a homogeneous gamma process. They are subject to a condition-based maintenance: the system is periodically inspected, and if the degradation level of a monitored component reaches a preventive threshold, the component is replaced by a new one. Furthermore, non-monitored components can fail between inspections. Time between these sudden failures follows an exponential distribution. Failures are self-announcing and the repair of the failed component is performed after a fixed delay time. In turn, these repair times are opportunities for preventive maintenance of the monitored components. Assuming a cost for each maintenance action, the expected cost rate of this system is analytically obtained. Numerical examples are given considering identical and non-identical components. Preventive thresholds and time between inspections that minimize the expected cost rate are evaluated.

Resumen: Se estudia un sistema complejo formado por componentes monitorizadas y no monitorizadas. Las componentes monitorizadas están sujetas a un proceso de degradación gamma homogéneo. Están sujetas a un mantenimiento basado en la condición del sistema: el sistema es inspeccionado periódicamente, y si el nivel de degradación de una componente alcanza el umbral preventivo, dicha componente es reemplazada por una nueva. Además, las componentes no monitorizadas pueden fallar entre inspecciones. El tiempo entre estos fallos sigue una distribución exponencial. Los fallos son *self-announcing*, y la reparación de las componentes estropeadas se realizan después de un tiempo de retraso fijado. De hecho, estos tiempos de reparación son oportunidades para realizar un mantenimiento preventivo de las componentes monitorizadas. Asumiendo un coste para cada acción de mantenimiento, se obtiene analíticamente el coste esperado de este sistema. Se muestran ejemplos numéricos considerando componentes idéndicas o no. Se evalúan los umbrales preventivos y el tiempo entre inspecciones que minimizan el coste esperado.

Keywords: maintenance, gamma process, monitored component, optimization.

MSC2010: 90B25, 60K10.

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1. Introduction

A system is a set of components with the aim of carrying out a certain function. Nevertheless, systems are affected by external and internal degradation.

One example of this internal degradation is the pitting corrosion, which consists in the appearance of pits simultaneously on a system. On the other hand, we also can find systems subject to external deterioration, such as the changes in some material due to the temperature or humidity. These external factors are considered as shocks, which result in traumatic failures.

Maintenance plays an important role in areas such as engineering, where the failure of the system leads to high costs and production downtime. Rausand and Hoyland [2] classified the maintenance tasks into two groups:

(i) **Corrective maintenance:** it is performed when a system is not working. The purpose of this maintenance policy is to return the system to a good condition in which it can perform its function properly.

(ii) **Preventive maintenance:** it is a planned maintenance performed when the system is working, in order to avoid downs of the system and prevent total failures. This maintenance policy can be divided in other classes, for instance,

- *Age-based maintenance:* maintenance actions are performed when the system exceeds a certain fixed age.
- *Condition-based maintenance:* this is also called *preventive maintenance*. With this policy, the system is maintained when its deterioration level exceeds a certain threshold.

In our model, condition-based maintenance is implemented through two different thresholds that control the state of the system: a preventive threshold (denoted by **M**) and a corrective threshold (denoted by **L**), lower than the previous one.

2. System description

The general assumptions of the model are the following:

(i) Monitored components of the system are subject to a continuous degradation, which follows a gamma process with shape and scale parameters α_i and β_i. Let $X_i(t)$ be the degradation of the monitored component i at time t. Its density function is given by:

$$f_{\alpha_i(t),\beta_i} = \frac{\beta_i^{\alpha_i t}}{\Gamma(\alpha_i t)} x^{\alpha_i t-1} \exp\{-\beta_i x\}, \quad x \geq 0,$$

where $\Gamma(\cdot)$ is the well-known gamma function.

(ii) Non-monitored components represents the sudden shocks to which the system is subject. Failures arrivals are exponentially distributed, that is, they follow a Poisson arrival process. Let Y be the time between these failures, then the survival function of Y is given by:

$$\bar{F}_Y = \exp\{-\lambda t\},$$

where λ is the parameter of the underlying Poisson process. Notice that non-monitored components can only be maintained upon failure, and we cannot predict when the failure will occur.

(iii) Failures of both monitored and non-monitored components are independent. When a component fails, a signal is immediately sent to the repair time, and it arrives with a delay of τ time units to start the reparation.

(iv) The system is subject to periodic inspections, that is, the deterioration level of the monitored components is checked each T time units, which is the inspection period.

(v) An opportunistic maintenance policy is implemented on the system: maintenance and inspection times of the system are seen as opportunities to check the state of the rest of the monitored components and perform a preventive maintenance of them if necessary.

3. Mathematical modelling

A *system renewal* is the maintenance time in which all the monitored components are replaced and the time to the next inspection is T time units. However, describing the state using renewal theory is complicated, since between renewals many preventive replacements can occur. To deal with it, semi-regenerative processes are used instead of renewal processes. A *semi-regenerative cycle* is defined as the time between two successive maintenance actions (which are the semi-regeneration points).

With that, we are able to study the evolution of the system with a Markov chain.

Let O_k be the time between the $(k-1)$-th and the k-th maintenance actions. The multiple process

$$(X_1(O_k), X_2(O_k), \dots, X_m(O_k))$$

is a Markov chain with state space $[0, M) \times \overset{m)}{\cdots} \times [0, M)$.

If the previous chain comes back to the initial state $(\mathbf{0}, \dots, \mathbf{0})$ almost surely (that is, is a regeneration point), then there exists a stationary measure π solution of the equation

(1)
$$\pi(\cdot) = \int_0^M \int_0^M \dots \int_0^M \mathbb{Q}(\cdot|\mathbf{x})\pi(\mathrm{d}\mathbf{x}),$$

where $\mathbb{Q}(\cdot|\mathbf{x})$ denotes the *kernel* of the process.

A result that assures a finite expected time to the system renewal is given. The proof can be seen in Proposition 4.1 of [1].

Lemma 1. *If $\mu < 1$, where μ is*

$$\mu = 1 - F_Y(T-\tau)\prod_{i=1}^{m}(F_{\alpha_i\tau,\beta_i}(M))F_{\alpha_i(T-\tau),\beta_i}(L-M),$$

then the stationary distribution π in (1) exists.

With the existence of the stationary measure π, the state of the system can be described at any time, so we can study now the objective function of the model.

4. Optimization of the objective function

Theorem 2. *For any realisation of the process, the long-run average reward per time unit is equal to the expected reward earned during one cycle divided by the expected length of one cycle. That is,*

$$P\left[\lim_{t\longrightarrow\infty}\frac{\mathbb{E}[C(t)]}{t} = \frac{\mathbb{E}[C(O_1)]}{\mathbb{E}[O_1]}\right] = 1,$$

where O_1 stands for the time to the next maintenance (that is, the length of a maintenance cycle) and $C(t)$ is the total cost at time t.

Each maintenance task implies a certain cost. Let C_∞ be the asymptotic cost rate. With the renewal-reward theorem, the cost can be developed as

$$C_\infty(T, M) = \frac{\mathbb{E}[C^c(O_1)]}{\mathbb{E}[O_1]} + \frac{\mathbb{E}[C^p(O_1)]}{\mathbb{E}[O_1]} + \frac{\mathbb{E}[C^{nm}(O_1)]}{\mathbb{E}[O_1]} + \frac{\mathbb{E}[C(I(O_1))]}{\mathbb{E}[O_1]} + \frac{\mathbb{E}[C(D(O_1))]}{\mathbb{E}[O_1]} - \frac{\mathbb{E}[R(O_1)]}{\mathbb{E}[O_1]},$$

where $\mathbb{E}[C^c(O_1)]$ and $\mathbb{E}[C^p(O_1)]$ are the expected costs due to preventive and corrective maintenance of monitored components in a cycle, respectively; $\mathbb{E}[C^{nm}(O_1)]$ denotes the expected cost due to the corrective replacement of non-monitored components; $\mathbb{E}[C(I(O_1))]$ corresponds to the expected cost due to inspections; $\mathbb{E}[C(D(O_1))]$ is the expected cost due to downtime, and $\mathbb{E}[R(O_1)]$ stands for the expected reward obtained in a semi-regenerative cycle.

$$C(T_{opt}, M_{opt}) = \inf\{C_\infty(T, M), T < 2\tau, M < L\}.$$

The following sequence of costs is used to study our maintenance strategy:

- Corrective replacement cost of monitored component i: $C_i^c = 80$ monetary units, for all $i \in I$.
- Preventive replacement cost of monitored component i: $C_i^p = 30$ monetary units, for all $i \in I$.
- Corrective replacement cost of non-monitored components: $C^f = 80$ monetary units.
- Downtime cost of monitored component i: $c_i = 5$ monetary units per time unit, for all $i \in I$.
- Downtime cost of non-monitored components: $c^{nm} = 5$ monetary units per time unit.

Furthermore, a reward provided by the monitored components of the system is considered. It depends on the deterioration level of the monitored components, and it decreases as the deterioration level of a component increases, so a classical exponential function is used to model it. Given the deterioration level x of the monitored component i, the reward function r_i is

$$r_i(x) = \theta_0 + g \exp\{-\gamma_i x\}, \quad 0 \leq x \leq L, \quad \gamma_i > 0, \quad \forall\, i,$$

where θ_0 and g are constants greater than 0.

To deal with the optimization of the objective cost function, we propose the following method. Firstly, typical Monte-Carlo simulation is used to search for potential solutions of the optimal values of the time between inspections (or inspection period) T and the preventive threshold, denoted by M.

After that, some meta-heuristic algorithms, such as *Pattern Search* and the *Genetic Algorithm*, are employed to optimize the previous parameters T and M. Nowadays, meta-heuristics are widely employed in stochastic problems, to provide a sufficiently good solution to an optimization problem, despite the fact that they don't guarantee a globally optimal solution in some problems.

The results obtained with this method are shown in Table 1:

m	T_0	M_0
2	4.39	2.95
3	4.28	3.17
4	3.13	3.38
5	2.47	3.78
6	2.29	3.93
7	1.89	4.18
8	1.64	4.37
9	1.53	4.65
10	1.48	4.70

Table 1: (a) Starting points with MC.

m	(T_{opt}, M_{opt})	$C_\infty(T_{opt}, M_{opt})$
2	(5.20, 2.02)	8.94
3	(3.86, 2.40)	11.02
4	(3.19, 2.88)	13.05
5	(3.20, 2.38)	13.10
6	(2.71, 3.51)	16.44
7	(2.30, 3.01)	16.90
8	(1.73, 3.22)	18.84
9	(1.46, 3.36)	20.11
10	(1.19, 3.45)	21.45

(b) T_{opt}, M_{opt} and C_∞ using the *Pattern Search*.

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3rd *BYMAT Conference* (2020)

Yano's extrapolation theorem

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Abstract: Yano's extrapolation theory provides a tool to obtain estimates on L^1 spaces starting from information on estimates for L^p spaces for every $1 < p < p_0$, for some $p_0 > 1$. This document provides an introduction to this theory by sketching the proof of Yano's extrapolation theorem [3]. The main tool developed in the proof of this theorem is the technique known as *layer cake*, which is nowadays used in many other proofs in Fourier analysis.

Resumen: La teoría de extrapolación de Yano da una herramienta para obtener estimaciones en espacios L^1 partiendo de información sobre estimaciones para espacios L^p para todo $1 < p < p_0$, para algún $p_0 > 1$. Este documento da una introducción a esta teoría esbozando la demostración del teorema de extrapolación de Yano [3]. La herramienta principal desarrollada en la prueba de este teorema es la técnica conocida como *layer cake*, que se usa a día de hoy en muchas demostraciones en análisis de Fourier.

Keywords: extrapolation theory, Yano's theorem, endpoint estimates, layer cake method.

MSC2010: 42B99, 46E30.

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1. Introduction

If you are familiar with functional analysis, or more specially with the theory of bounding operators, you have probably heard about operator interpolation techniques, where we use operator bounding at the "ends" of a family of spaces, to get bounds on the rest of the spaces of the family.

However, when it comes to extrapolation techniques, the intention is precisely the opposite. That is, use what we know about the bounds of a certain operator in the "interior" of a family of spaces to obtain bounds of this operator at the endpoints of the family. Beyond concrete points, what characterizes this type of theorems is the use of a specific type of techniques. In this sense, two great schools stand out: Yano's and Rubio de Francia's.

In these pages, we intend to present in a simple way the extrapolation technique of Yano, by explaining the theorem that he published in 1951 [3]. The various applications of this method to the study of the bounding of several operators is a unique knowledge paradigm in the field of Fourier analysis.

2. Contextualization of the problem

Definition 1 ($L\log L$ space). For a measurable function f defined in (a,b), we will say that $f \in L^{*k}[a,b]$ if

$$\int_a^b |f(x)| \log^k\left(1+f^2(x)\right) \mathrm{d}x, \qquad k>0.$$

Note that this is not a norm, even though it allows us to characterize the functions in that space.

Definition 2. As usual, for two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, for an operator

$$T\colon X \to Y,$$

we are going to define the norm of T as

$$\|T\| = \sup_{\|f\|_X \leq 1} \frac{\|Tf\|_Y}{\|f\|_X}.$$

In Fourier analysis, we are often concerned with operators T which transform a measurable function f defined in $[0,2\pi]$ into another measurable function also defined in $[0,2\pi]$ such that

(i) for every $p>1$ we have
$$\|T\|_{L^p} \leq A_p,$$

(ii) for every $f \in L^{*k}[0,2\pi]$ we have
$$\|Tf\|_{L^1[0,2\pi]} \leq A_k \int_0^{2\pi} |f(x)| \log^k\left(1+f^2(x)\right) \mathrm{d}x + B_k,$$

where A_p, A_k and B_k are constants depending only on p, k and k, respectively.

Usually, given an operator T, it is checked that T satisfies each one of the above conditions separately. But, what if we could deduce that T satisfied the last condition based on T satisfying the first one? This is what Yano's theorem allows us to do.

Theorem 3. *Let T be a sublinear operator which transforms every integrable function to a measurable function, both being defined in a finite interval (a,b) such that*

(i) $|Tf| = |T(-f)|$,

(ii) *the inequality*

$$\|T\|_{L^p[a,b]} \leq \frac{A}{(p-1)^k},$$

holds for $1 < p \leq 2$*, for some* $k > 0$ *and the constant* A *depending only on the length of the interval* (a, b).

Then, we have that for every $f \in L^{*k}[a, b]$

$$\|Tf\|_{L^1[a,b]} \leq A_k \int_0^{2\pi} |f(x)| \log^k \left(1 + f^2(x)\right) \mathrm{d}x + B_k,$$

where A_k *and* B_k *are constants depending only on* k *and the lenght of the interval* $b - a$.

For the proof, we only need to check the theorem in the case where $f \geq 1$ because for any other function, we can decompose it into the difference of two functions greater than one and apply the condition (i) in order to obtain the desired result. Indeed,

$$f = \left(f\chi_{f\geq 0} + 1\right) - \left(1 - f\chi_{f<0}\right) = g_1 - g_2,$$

where $g_1, g_2 \geq 1$.

So, given an arbitrary function $f \geq 1$, we decompose it in the following way:

$$f = \sum_{n\geq 0} 2^n f_n, \qquad \text{where} \quad f_n = 2^{-n} f \chi_{\{2^n \leq f < 2^{n+1}\}}.$$

The sublinearity of the operator T allow us to work with these special functions f_n which have the particularity that $1 \leq f_n(x) < 2$ for every $n \geq 0$ and any $x \in [a, b]$. Moreover, the definition of these functions makes it possible for us to return to the initial function f if desired.

In fact, from the above decomposition and applying the Hölder inequality, it is easy to see that for any sequence $\{p_n\}$ of exponents such that $1 < p_n \leq 2$ it is satisfied that

$$\|Tf\|_{L^1[a,b]} \leq C \sum_n 2^n \|Tf_n\|_{L^{p_n}[a,b]} \leq C \sum_n \frac{2^n}{(p_n - 1)^k} \|f_n\|_{L^{p_n}[a,b]},$$

with the constant C only depending on the length of the interval and on the constant A which appears on the second hypothesis of the theorem.

At this point, it only remains to choose the exponent p_n, not fixed yet, to conclude the desired theorem. For instance, we choose

$$p_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 + \dfrac{1}{n} & \text{if } n \neq 0. \end{cases}$$

For the end of the theorem we only need to use Young's inequality in the right way and the fact that

$$2^n f_n(x) n^k \leq C|f(x)| \log^k \left(1 + f^2(x)\right),$$

for every $x \in [a, b]$.

This way of treating a function, by decomposing it into simpler and control-bounded functions, is known as the *layer cake* method. This technique has been used in many other proofs in order to obtain extrapolation theorems, but also in other areas of Fourier analysis. See, for example [1] or the proof of Lemma 1.4.20 in [2, page 56], where the partition made is a little bit different since it considers the sets

$$A_n = \{x \in X : f^*(2^{n+1}) < |f(x)| \leq f^*(2^n)\},$$

where f^* denotes the rearrangement invariant of f.

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Local Bollobás type properties and diagonal operators

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Abstract: We study properties related to the density of norm-attaining operators. To do so, we introduce the set $\mathcal{A}_{\|\cdot\|}(X, Y)$ of norm-one norm-attaining operators from X to Y such that, given some $\varepsilon > 0$, there exists $\eta(\varepsilon, T)$ such that, if $\|T(x)\| > 1 - \eta$, then there is x_0 with $\|x_0 - x\| < \varepsilon$ and T attains its norm at x_0. These are operators such that, whenever they almost attain their norm at a point, they do attain it at a nearby point. The analogous set $\mathcal{A}_{\mathrm{nu}}$ for the numerical radius is also introduced and studied. We give examples of operators that belong to these sets and, in particular, we give a characterisation of what diagonal operators belong to these sets for the classical Banach sequence spaces. The contents are from [5].

Resumen: Estudiamos propiedades relacionadas con la densidad de operadores que alcanzan su norma. Para ello, introducimos el conjunto $\mathcal{A}_{\|\cdot\|}(X, Y)$ de operadores de norma uno de X en Y tales que, dado un $\varepsilon > 0$, existe $\eta(\varepsilon, T)$ de forma que, si $\|T(x)\| > 1 - \eta$, entonces existe un x_0 con $\|x_0 - x\| < \varepsilon$ y T alcanza su norma en x_0. Estos son operadores tales que, si casi alcanzan su norma en un punto, la alcanzan en un punto cercano. El conjunto análogo $\mathcal{A}_{\mathrm{nu}}$ para el radio numérico también es introducido y estudiado. Vamos a dar ejemplos de operadores que pertenecen a estas clases y, en particular, daremos una caracterización de qué operadores diagonales pertenecen a estos conjuntos para los espacios de Banach de sucesiones clásicos. Los contenidos forman parte de [5].

Keywords: Bishop-Phelps-Bollobás, norm attaining operators, diagonal operators.

MSC2010: 46B20, 46B04, 46B07.

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1. Introduction

In this contribution, we will summarize some of the results from the recent work [5] that were presented in a talk in the 3rd BYMAT Conference in 2020.

1.1. Notation and terminology

Let X and Y be Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by B_X, S_X and X^* the closed unit ball, the unit sphere and the topological dual of X, respectively. $\mathcal{L}(X, Y)$ represents the space of bounded and linear operators from X to Y, and we shall write $\mathcal{L}(X) = \mathcal{L}(X, X)$. We will use the notation c_0, ℓ_1, ℓ_∞ and ℓ_p $(1 < p < \infty)$ for the classical Banach sequence spaces, and the notation $x = (x(1), x(2), x(3), \ldots)$ will be used for any $x \in X$ where X is a Banach sequence space. If $x \in X$ and $x^* \in X^*$, the dual action may be written as $\langle x^*, x\rangle$ or as $x^*(x)$ indistinctly.

We say that an operator $T\colon X \to Y$ *attains its norm* (or is *norm attaining*) if there exists some $x \in S_X$ such that $T(x) = \|T\| = \sup_{x\in B_X} |T(x)|$ (that is, when the supremum is actually attained), and the set of norm attaining operators is denoted by $\mathrm{NA}(X, Y)$. The set of states of X is defined as $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. The *numerical radius* of an operator $T\colon X \to X$ is defined as $\nu(T) := \sup_{(x,x^*)\in\Pi(X)} |x^*(x)|$. It is easy to see that the numerical radius is a seminorm which satisfies $0 \le \nu(T) \le \|T\|$ for every operator $T\colon X \to X$. We say that an operator $T\colon X \to X$ *attains its numerical radius* (or is *numerical radius attaining*) if there exist some $(x, x^*) \in \Pi(X)$ such that $|x^*(T(x))| = \nu(T)$, and the set of such operators is denoted by $\mathrm{NRA}(X)$.

If X and Y are Banach sequence spaces, an operator $T\colon X \to Y$ is said to be *diagonal* if it is defined as $T(x) = (\alpha_1 x(1), \alpha_2 x(2), \alpha_3 x(3), \ldots)$ for all $x \in X$, for some bounded sequence of scalars $\{\alpha_k\}_{k=1}^{+\infty} \in \mathbb{K}$.

1.2. Brief historical background

In 1961, Bishop and Phelps [3] proved that, for any Banach space X, the set $\mathrm{NA}(X, \mathbb{K})$ is always dense in X^*. Bollobás [4] gave a numerical refinement of that result in 1970, stating that you can always approximate a functional $x^* \in S_{X^*}$ and a point $x \in S_X$ at which it almost attains its norm by a pair $(y, y^*) \in \Pi(X)$. It is natural to wonder if any of these results also hold in the case of operators instead of functionals, however Lindenstrauss [8] proved that there exist spaces X and Y such that $\mathrm{NA}(X, Y)$ is not dense in $\mathcal{L}(X, Y)$.

In order to study quantitatively when a pair of spaces satisfies that $\mathrm{NA}(X, Y)$ is dense in $\mathcal{L}(X, Y)$, Acosta, Aron, García and Maestre [2] introduced in 2008 the Bishop-Phelps-Bollobás property, abbreviated as BPBp (see [2, Definition 1.1]). Roughly speaking, a pair of Banach spaces (X, Y) has the BPBp if, whenever we have an operator $T \in S_{\mathcal{L}(X,Y)}$ and a point $x \in S_X$ at which it almost attains its norm, we can always approximate them by an operator $S \in S_{\mathcal{L}(X,Y)}$ and a point $y \in S_X$ at which it attains its norm. We refer the interested reader to the survey [1] and references therein for more information and background on the BPBp.

Motivated by that work, Guirao and Kozhushkina [7] introduced in 2013 the *Bishop-Phelps-Bollobás property for numerical radius* (BPBp-nu for short), which is a natural adaptation of the BPBp to the case of numerical radius instead of norms. These properties, as well as several variations, have been profusely studied in the recent years. One of those properties is particularly relevant to this work, the $\mathbf{L}_{o,o}$, which is a local version of the BPBp where $S = T$ and η depends on the previously fixed T (see [6, Definition 2.1]).

1.3. Introducing the problem

Most of the works studying BPBp-like properties focus in finding what spaces can, or can not, satisfy properties of that kind. In the very recent work [5], the authors tackled the study from a different point of view: finding what *operators* can satisfy properties of that kind. We are going to present here briefly some of the findings from that work. We start by introducing two necessary concepts for this study, which are classes of operators such that whenever they almost attain their norm (or numerical radius) at some point (or state), they do attain it at a nearby point (or state). Note that this concept is closely related to the $\mathbf{L}_{o,o}$.

Definition 1. Let X, Y be Banach spaces.

(i) $\mathcal{A}_{\|\cdot\|}(X, Y)$ stands for the set of all norm-attaining operators $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon, T)$, there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

(ii) $\mathcal{A}_{\text{nu}}(X)$ stands for the set of all numerical radius attaining operators $T \in \mathcal{L}(X, X)$ with $\nu(T) = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $(x, x^*) \in \Pi(X)$ satisfies $|x^*(T(x))| > 1 - \eta(\varepsilon, T)$, there is $(x_0, x_0^*) \in \Pi(X)$ such that $|x_0^*(T(x_0))| = 1$, $\|x_0 - x\| < \varepsilon$, and $\|x_0^* - x^*\| < \varepsilon$. ◂

In Section 2 we will list some of the results from [5], focusing mainly in those where the involved operators are diagonal operators.

2. Results

2.1. First results

In order to determine what operators satisfy certain properties, it is natural to begin wondering what happens with operators between finite dimensional spaces and functionals. [5, Theorem 2.1] claims that, if X is a finite dimensional Banach space and Y is any Banach space, then every operator from $S_{\mathcal{L}(X,Y)}$ is in $\mathcal{A}_{\|\cdot\|}(X, Y)$, and every operator $T \in \mathcal{L}(X)$ with $\nu(T) = 1$ is in $\mathcal{A}_{\text{nu}}(X)$. As for functionals that may or may not belong to $\mathcal{A}_{\|\cdot\|}$, [5, Theorems 2.1 and 2.2] study the cases when $X = c_0$, ℓ_1, ℓ_∞ and the case where X is uniformly convex.

What can be said about other operators? Is there any relation between the sets $\mathcal{A}_{\|\cdot\|}(X, X)$ and $\mathcal{A}_{\text{nu}}(X)$ in general? The following examples should make it clear that this is, in fact, not trivial, even in the particular case of Hilbert spaces.

Example 2. Consider the following operators $T\colon \ell_2 \to \ell_2$:

(i) If $T(x) := x$, for all $x \in \ell_2$, then $T \in \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ and $T \in \mathcal{A}_{\text{nu}}(\ell_2)$.

(ii) If $T(x) := (0, x(1), x(2), x(3), x(4), \dots)$, for all $x \in \ell_2$, then $T \in \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ but $T \notin \mathcal{A}_{\text{nu}}(\ell_2)$.

(iii) If $T(x) := (2x(2), -2x(1), x(3), 0, 0, \dots)$, for all $x \in \ell_2$, then $T \notin \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ but $T \in \mathcal{A}_{\text{nu}}(\ell_2)$.

(iv) If $T(x) := \left(x(1), \frac{1}{2}x(2), \frac{2}{3}x(3), \frac{3}{4}x(4), \frac{4}{5}x(5), \dots\right)$, for all $x \in \ell_2$, then $T \notin \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ and $T \notin \mathcal{A}_{\text{nu}}(\ell_2)$, even though T attains both its norm ant its numerical radius, and $\|T\| = \nu(T) = 1$. ◂

We refer the reader to [5] for a collection of results and examples in the matter involving, for example, compact operators, adjoint operators, canonical projections, Hilbert spaces, and direct sums of spaces.

2.2. Diagonal operators

Let us examine items (i) and (iv) from Example 2. In both cases, T is a diagonal operator satisfying $\|T\| = \nu(T) = 1$ and $T \in \text{NA}(\ell_2, \ell_2) \cap \text{NRA}(\ell_2)$; however, their situations are completely different regarding the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$. In [5], a characterisation is made to determine what are the diagonal operators that belong to the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$ when the involved spaces are the classical Banach sequence spaces c_0 and ℓ_p $(1 \leq p \leq +\infty)$. We will summarize here some of the intuitions behind as well as the main results.

Example 3. Consider the diagonal operators $T\colon c_0 \to c_0$ defined as $T(x) = (\alpha_1 x(1), \alpha_2 x(2), \dots)$ for all $x \in c_0$, all of which satisfying $\|T\| = \nu(T) = 1$:

(i) If $\alpha_n := \frac{n}{n+1}$, then T can not be in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$, since T does not attain its norm or numerical radius. So at least one of the α_n needs to have absolute value 1 to be in our sets.

(ii) If $\alpha_1 = 1$ and $\alpha_n = 1 - \frac{1}{n}$ for $n > 1$, then T is also not in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$, since the only points $x \in S_{c_0}$ where it attains its norm are of the form $s \cdot e_1$ with $|s| = 1$, but the sequence $\{|T(e_{n+1})|\}_{n=1}^{+\infty}$ is strictly increasing and converges to 1, and similar with the numerical radius. So to be in our sets, not only one of the α_n needs to have $|\alpha_n| = 1$, but also, those that are not 1 have to be far from 1 (that is, 1 can not be an accumulation point of $\{|\alpha_n|\}_{n=1}^{+\infty}$).

(iii) Finally, if $\alpha_1 = 1$ and $\alpha_n = \frac{1}{n}$ for $n > 1$, then T is in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cap \mathcal{A}_{\mathrm{nu}}(c_0)$, since the only points where the norm is almost attained are close to some point $x \in S_{c_0}$ with $|x(1)| = 1$ (similar for $\mathcal{A}_{\mathrm{nu}}$). ◂

The intuitions presented above addresses us to necessary and sufficient conditions for a diagonal operator to be in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\mathrm{nu}}(c_0)$, and they can be generalized to other spaces and to the complex case. We summarize the main results on the matter from [5, Theorems 2.13, 2.15 and 2.17, and Corollary 2.16].

Theorem 4. *Let (X, Y) be (c_0, c_0), (ℓ_p, ℓ_p) $(1 \le p \le +\infty)$ or (ℓ_p, c_0) $(1 \le p < +\infty)$. Let $T : X \to Y$ be the norm one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ if and only if both of these conditions are satisfied:*

(i) *There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.*

(ii) *If $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, then either $J = \mathbb{N}$ or $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$.*

Theorem 5. *Given $1 \le p < +\infty$, let $T : c_0 \to \ell_p$ be the norm one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ if and only if there is some $N \in \mathbb{N}$ such that $\alpha_n = 0$ for all $n > N$.*

Theorem 6. *Let $X = c_0$ or ℓ_p, $1 \le p < \infty$. Let $T : X \to X$ be the numerical radius one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{nu}(X)$ if and only if the following two conditions hold:*

(i) *There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.*

(ii) *If $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, then the cardinality of the set $\{\alpha_n : n \in J\}$ is finite and $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$ when $J \neq \mathbb{N}$.*

In particular, if $\{\alpha_n\}_{n=1}^{+\infty} \subset \mathbb{R}$, $T \in \mathcal{A}_{nu}(X)$ if and only if $T \in \mathcal{A}_{\|\cdot\|}(X, X)$.

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Axial algebras of Monster type $(2\eta, \eta)$ for orthogonal groups over $\mathbb{F}_2$

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Abstract: Axial algebras are commutative non-associative algebras generated by special elements called axes satisfying a prescribed fusion law. They were introduced by Hall, Rehren and Shpectorov. This class has applications in physics, group theory and also elsewhere in mathematics. Here, we introduce the axial algebras of Monster type $(2\eta, \eta)$ and give an overview of the flip construction for such algebras. We also apply it to the non-degenerate orthogonal groups $O^\varepsilon(2k, 2)$. We describe the classes of involutions (flips) of $O^\varepsilon(2k, 2)$ and for each flip we investigate the corresponding flip subalgebra. In this way, we build a new rich family of examples of algebras of Monster type $(2\eta, \eta)$.

Resumen: Las álgebras axiales son álgebras conmutativas no asociativas generadas por elementos especiales, llamados ejes, que satisfacen una ley de fusión prescrita. Fueron introducidas por Hall, Rehren y Shpectorov. Tienen aplicaciones en física, teoría de grupos y también en otras áreas de las matemáticas. Aquí introducimos las álgebras axiales de tipo Monster $(2\eta, \eta)$ y damos una visión general de la construcción por involución para tales álgebras. También la aplicamos a los grupos ortogonales no degenerados $O^\varepsilon(2k, 2)$. Describimos las clases de involuciones (*flips*) de $O^\varepsilon(2k, 2)$ y para cada *flip* investigamos la correspondiente subálgebra. De este modo, construimos una nueva e interesante familia de ejemplos de álgebras de tipo Monster $(2\eta, \eta)$.

Keywords: Axial algebras, Monster type, orthogonal groups.

MSC2010: 17D99.

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1. Introduction

In 2009, Alexander Ivanov [5] turned the key properties used by Masahiko Miyamoto and Shinya Sakuma in their calculations of vertex operator algebras generated by two Ising vectors into the axioms of a new class of algebras called Majorana algebras. A Majorana algebra is a commutative non-associative algebra A over the field of real numbers generated by special idempotents with the fusion law $\mathcal{M}\left(\frac{1}{4}, \frac{1}{32}\right)$ and satisfying some additional properties. In 2015, Jon Hall, Felix Rehren and Sergey Shpectorov [3, 4] refined and generalised the axioms of Majorana algebras and these new axioms became the axioms of axial algebras.

In this text, we start by providing the background on axial algebras. Then we introduce the notion of 3-transposition groups, from which we derive the Matsuo algebras. After that, we turn to the flips of Matsuo algebras and we explain how a flip leads to a flip subalgebra that is an algebra of Monster type $(2\eta, \eta)$. In the case of orthogonal groups over the field with two elements, we obtain two types of flips and further split them into conjugacy classes. For each class we determine the dimension of the flip subalgebra. Our approach is similar to that of Vijay Joshi [6] who completed the case of symplectic groups over $\mathbb{F}_2$.

2. Background

2.1. Axial algebras

Let $\mathbb{F}$ be a field. All algebras in this text are over $\mathbb{F}$ and they are non-associative, that is, not necessarily associative. For a set $\mathcal{F}$, the set of all subsets of $\mathcal{F}$ is denoted by $2^{\mathcal{F}}$.

Definition 1. Let $\mathcal{F}$ be a finite subset of $\mathbb{F}$ and $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ be a symmetric binary operation. The pair $(\mathcal{F}, *)$ is a *fusion law* over $\mathbb{F}$. ◂

Examples of fusion laws can be seen in Tables 1 and 2.

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

Table 1: Fusion law $\mathcal{J}(\eta)$

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 2: Fusion law $\mathcal{M}(\alpha, \beta)$

Definition 2. Let A be a commutative algebra. For $a \in A$, the adjoint endomorphism $\mathrm{ad}_a : A \to A$ is defined by $b \mapsto ab$ for all $b \in A$. ◂

For $\lambda \in \mathbb{F}$, let $A_\lambda(a) = \{b \in A : ab = \lambda b\}$ be the λ-eigenspace of ad_a.

Definition 3. Let $\mathcal{F}$ be a *fusion law* over $\mathbb{F}$. Then, $a \in A$ is an $\mathcal{F}$-axis if

(i) a is an idempotent, i.e., $a^2 = a$;

(ii) ad_a is semisimple and every eigenvalue of ad_a is in $\mathcal{F}$, i.e., $A = A_{\mathcal{F}}(a) = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$;

(iii) $A_\lambda A_\mu \subseteq \bigoplus_{\nu \in \lambda * \mu} A_\nu(a)$ for all $\lambda, \mu \in \mathcal{F}$, where $\lambda * \mu$ is the product in $\mathcal{F}$, hence a subset of $\mathcal{F}$. ◂

Definition 4. Let A be a commutative algebra. We call A an *$\mathcal{F}$-axial algebra* if it is generated by a set of $\mathcal{F}$-axes. ◂

Definition 5. An $\mathcal{F}$-axis a is *primitive* if $A_1(a) = \langle a \rangle$, that is, $A_1(a)$ is 1-dimensional. An $\mathcal{F}$-axial algebra is *primitive* if it is generated by a set of primitive $\mathcal{F}$-axes. ◂

Jordan algebras are examples of axial algebras.

Definition 6 ([7]). A *Jordan algebra* is a commutative algebra A satisfying the following condition:

(J) *(Jordan Identity)* $x^2(yx) = (x^2y)x$ for all $x, y \in A$. ◂

Every idempotent in a Jordan algebra satisfies the Peirce decomposition that amounts to the fusion law $\mathcal{J}(\frac{1}{2})$.

Definition 7. An axial algebra of *Jordan type* η is a primitive axial algebra generated by a set of axes satisfying the fusion law $\mathcal{J}(\eta)$. ◂

Definition 8. An axial algebra of *Monster type* (α, β) is a primitive axial algebra generated by a set of axes satisfying the fusion law $\mathcal{M}(\alpha, \beta)$. ◂

The Griess algebra is a 196, 844-dimensional algebra over the field of real numbers. This algebra is an axial algebra of Monster type $\left(\frac{1}{4}, \frac{1}{32}\right)$.

2.2. 3-Transposition groups

Definition 9 ([1]). Suppose that G is a finite group and C is a normal subset of involutions (elements of order 2) of G. If C generates G and for all $c, d \in C$, $o(cd)$ is at most 3, then the pair (G, C) is a 3*-transposition group.* ◂

Let V be a vector space over $\mathbb{F}_2$, $q : V \to \mathbb{F}_2$ a non-degenerate quadratic form and $(\cdot, \cdot)$ the associated symplectic form. Let $G = O^\varepsilon(2k, 2)$ be the orthogonal group associated with V and q, where $\dim V = n = 2k$ and q is of type $\varepsilon \in \{+, -\}$. Take $w \in V$. The map $r_w : u \mapsto u + (u, w)w$ is called a *transvection* and it lies in G if $q(w) = 1$. Take $C = \{r_w : w \in V, q(w) = 1\}$ to be the class of transvections. Then, (G, C) is a 3-transposition group.

2.3. Matsuo algebras

Definition 10. Let (G, C) be a 3-transposition group and $\mathbb{F}$ be a field with $\operatorname{char} \mathbb{F} \neq 2$. Take $\eta \in \mathbb{F}$, $\eta \neq 0, 1$. Let $A = M_\eta(G, C)$ be the algebra with the basis C and the product $\circ$ defined by

$$c \circ d = \begin{cases} c & \text{if } c = d, \\ 0 & \text{if } o(cd) = 2, \\ \frac{\eta}{2}(c + d - e) & \text{if } o(cd) = 3, \end{cases}$$

where $c, d \in C$ and $e = c^d$. ◂

This algebra A is the *Matsuo algebra* corresponding to (G, C) and it is of Jordan type η.

2.4. Flip subalgebras

Consider a Matsuo algebra $A = M_\eta(G, C)$.

Definition 11. A *flip* is an involutive automorphism of A. ◂

Involutive automorphisms of G preserving C act on A, and hence they are flips.

Definition 12. Let $a, b \in C$ be such that $ab = 0$, i.e., a and b are orthogonal. Then, $a + b$ is called a *double axis.* ◂

Definition 13. Let σ be a flip of A. The *flip subalgebra* is generated by all single and double axes contained in the fixed subalgebra A_σ. ◂

Theorem 14 ([2]). *Every flip subalgebra is a primitive axial algebra of Monster type* $(2\eta, \eta)$.

3. Flip subalgebras in the orthogonal case

The following theorems give us the information about the flip subalgebras for all possible flips in the orthogonal case.

Theorem 15 ([8]). *Let U be a maximal totally isotropic subspace of V with a basis $\{u_1, \dots, u_k\}$, where each u_i is non-singular. Let $\sigma = \tau_i = r_{u_1} r_{u_2} \cdots r_{u_i}$ for all $1 \leq i \leq k$. If i is odd, then the dimension of the flip subalgebra is $2^{n-3} + 2^{n-i-2}$. If i is even, then the dimension is $2^{n-3} + 2^{n-i-2} - \delta 2^{k-2}$, where $\delta = 1$ for plus type and $\delta = -1$ for minus type.*

Theorem 16 ([8]). *Let U be a maximal totally singular subspace of V with a basis $\{u_1, \dots, u_{k-\beta}\}$, where $\beta = 0$ for plus type and $\beta = 1$ for minus type. Let $\sigma = \sigma_s = \sigma_{U_1} \sigma_{U_2} \cdots \sigma_{U_s}$, $1 \leq s \leq \left\lfloor \frac{k-\beta}{2} \right\rfloor$, where $U_j = \langle u_{2j-1}, u_{2j} \rangle$, for all $1 \leq j \leq \left\lfloor \frac{k-\beta}{2} \right\rfloor$, and $\sigma_{U_j} = \prod_{0 \neq u \in U_j} r_u$. Then, the dimension of the flip subalgebra is $2^{n-2} + 2^{n-2s-2} - \delta 2^{k-1}$.*

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Best approximations and greedy algorithms

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Abstract: We have obtained the exact order estimates for approximations by greedy algorithms of the classes $L^{\psi}_{\beta,p}$ of periodic functios in the space L_q for some relations between parameters p and q.

Resumen: Se han obtenido las estimaciones de orden exacto para las aproximaciones por algoritmos *greedy* de las clases $L^{\psi}_{\beta,p}$ de funciones periódicas en el espacio L_q, para algunas relaciones entre los parámetros p y q.

Keywords: greedy approximation, greedy algorithms, best approximations.

MSC2010: 42A10, 41A30.

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1. Introduction

Let L_q be a space of functions f which are 2π-periodic and summable to a power q, $1 \le q < \infty$ (resp., essentially bounded for $q = \infty$), on the segment $[-\pi, \pi]$. The norm in this space is defined as follows:

$$\|f\|_{L_q} = \|f\|_q = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} |f(x)|^q \, \mathrm{d}x \right)^{1/q}, & 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{x \in [-\pi,\pi]} |f(x)|, & q = \infty. \end{cases}$$

For a function $f \in L_1$, we consider its Fourier series

$$\sum_{k \in \mathbb{Z}} \hat{f}(k) \mathrm{e}^{\mathrm{i}kx},$$

where $\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x$ are the Fourier coefficients of the function f. In what follows, we always assume that the function $f \in L_1$ satisfies the condition

$$\int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = 0.$$

Further, let $\psi \ne 0$ be an arbitrary function of natural argument and let β be an arbitrary fixed real number. If a series

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{\psi(|k|)} \mathrm{e}^{\mathrm{i}(kx + \beta \frac{\pi}{2} \operatorname{sign} k)}$$

is the Fourier series of a summable function, then, following Stepanets [3], we can introduce the (ψ, β)-derivative of the function f and denote it by f_β^ψ. By L_β^ψ we denote the set of functions f satisfying this condition. In what follows, we assume that the function f belongs to the class $L_{\beta,p}^\psi$ if $f \in L_{\beta,p}^\psi$ and

$$f_\beta^\psi \in U_p = \{\varphi : \varphi \in L_p, \|\varphi\|_p \le 1\}, \quad 1 \le p \le \infty.$$

If $\psi(|k|) = |k|^{-r}$, $r > 0$, and $k \in \mathbb{Z} \setminus \{0\}$, then the (ψ, β)-derivative of the function f coincides with its (r, β)-derivative (denoted by f_β^r) in the Weyl–Nagy sense.

We give the definition of the greedy approximation under investigation. Let $\{\hat{f}(k(l))\}_{l=1}^\infty$ be the Fourier coefficients $\{\hat{f}(k)\}_{k \in \mathbb{Z}}$ of the function $f \in L_1$, arranged in non-increasing order of their absolute value, i.e.,

$$|\hat{f}(k(1))| \ge |\hat{f}(k(2))| \ge \dots$$

Denote for $f \in L_q$

$$G_m(f, x) = \sum_{l=1}^{m} \hat{f}(k(l)) \mathrm{e}^{\mathrm{i}k(l)x}$$

and, if $F \subset L_q$ is a certain function class, then we set

$$G_m(F)_q := \sup_{f \in F} \|f(\cdot) - G_m(f, \cdot)\|_q. \tag{1}$$

At present, there are many works devoted to the investigation of quantity (1) for important classes of functions. For details and the corresponding references, see, e.g., [7].

By B we denote the set of functions ψ satisfying the following conditions:

(i) ψ is positive and nonincreasing;

(ii) there exists a constant $C > 0$ such that $\frac{\psi(\tau)}{\psi(2\tau)} \le C$, $\tau \in \mathbb{N}$.

Thus, the functions $1/\tau^r$, $r > 0$; $\ln^\gamma(\tau + 1)/\tau^r$, $\gamma \in \mathbb{R}$, $r > 0$, $\tau \in \mathbb{N}$, and some other functions belong to the set B.

For the quantities A and B, the notation $A \asymp B$ means that there exist positive constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$. If $B \le C_2 A$ ($B \ge C_1 A$), then we can write $B \ll A$ ($B \gg A$). All C_i, $i = 1, 2, \dots$, encountered in our paper may depend only on the parameters appearing in the definitions of the class and metric in which we determine the error of approximation.

2. Main results

The following assertion is true:

Theorem. *Let $1 < p < q \leq 2$, $\psi \in B$, $\beta \in \mathbb{R}$ and let, in addition, there exist $\varepsilon > 0$ such that the sequence $\psi(t)t^{\frac{1}{p}-\frac{1}{q}+\varepsilon}$, $t \in \mathbb{N}$, does not increase. Then, the following order estimate is true:*

$$G_m(L^{\psi}_{\beta,p})_q \asymp \psi(m)m^{\frac{1}{p}-\frac{1}{2}}.$$

Proof. The upper bounds follow from the estimate for the approximation of functions from the classes $L^{\psi}_{\beta,p}$ by their Fourier sums [3]:

$$\mathcal{E}_m(L^{\psi}_{\beta,p})_2 = \sup_{f \in L^{\psi}_{\beta,p}} \left\| f(x) - \sum_{k=-m}^{m} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx} \right\|_2 \asymp \psi(m)m^{\frac{1}{p}-\frac{1}{2}}.$$

We now determine the lower bounds. We will use the Rudin-Shapiro polinomials $\mathcal{R}_l(x)$:

$$\mathcal{R}_l(x) = \sum_{j=2^{l-1}}^{2^l-1} \varepsilon_j \mathrm{e}^{\mathrm{i}jx}, \quad \varepsilon_j = \pm 1,\ x \in \mathbb{R},$$

satisfying the order estimate (see, e.g., [1]) $\|\mathcal{R}_l\|_\infty \ll 2^{l/2}$.

We also need the well-known de la Vallee-Poussin kernels

$$V_m(x) = \frac{1}{m} \sum_{l=m}^{2m-1} D_l(x), \quad x \in \mathbb{R},\ m \in \mathbb{N},$$

where $D_l(x) = \sum_{|k|\leq l} \mathrm{e}^{\mathrm{i}kx}$ is the Dirichlet kernel.

Further, for $\varepsilon = \pm 1$ we set $\Lambda_{\pm 1} := \{k : \widehat{\mathcal{R}}_l(k) = \pm 1\}$, and let $\varepsilon = \pm 1$ be such that $|\Lambda_\varepsilon| > |\Lambda_{-\varepsilon}|$. Then, for given m, we take $l \in \mathbb{N}$ from the relation $2^{l-2} \leq m < 2^{l-1}$, take a small positive parameter δ and consider a function

$$f(x) = C_3 \psi(2^l) 2^{l(\frac{1}{p}-1)} f_1(x), \quad C_3 > 0,$$

where $f_1(x) = V_m(x) + \varepsilon\delta\mathcal{R}_m(x)$ and $0 < \delta \leq m^{\frac{1}{2}-\frac{1}{p}}$.

We now show that, for a certain choice of the constant $C_3 > 0$, the function f belongs to the class $L^{\psi}_{\beta,p}$. To this end, it suffices to verify that $\|f^{\psi}_{\beta}\|_p \ll 1$.

For this purpose, we use the estimate [2] $\|t^{\psi}_{\beta}\|_p \ll \psi^{-1}(n)\|t\|_p$ (for any polynomial $t \in T_n$, $1 < p < \infty$), and the well-known relation (see, e.g., [4]) $\|V_{2^l}\|_p \asymp 2^{l(1-\frac{1}{p})}$, $1 \leq p \leq \infty$.

Hence, we can write

$$\begin{aligned}
\|f^{\psi}_{\beta}\|_p &\ll \psi^{-1}(m)\|f\|_p \leq \psi^{-1}(m)\psi(2^l)2^{l(\frac{1}{p}-1)}(\|V_m\|_p + \delta\|\mathcal{R}_m\|_p) \\
&\leq \psi^{-1}(m)\psi(2^l)2^{l(\frac{1}{p}-1)}(\|V_m\|_p + \delta\|\mathcal{R}_m\|_\infty) \\
&\ll \psi^{-1}(m)\psi(2^l)2^{l(\frac{1}{p}-1)}(2^{l(1-\frac{1}{p})} + 2^{l(\frac{1}{2}-\frac{1}{p})}2^{\frac{l}{2}}) \ll 1.
\end{aligned}$$

This implies that, for a proper choice of the constant $C_3 > 0$, function $f \in L^{\psi}_{\beta,p}$.

By using the estimate (see, e.g., [5, p. 581]) that, for $1 \leq q \leq 2$ and $1 < p \leq 2$,

$$\|f_1 - G_m(f_1)\|_q \gg m^{\frac{1}{2}},$$

we obtain

$$\sup_{f\in L^{\psi}_{\beta,p}} \|f-G_m(f)\|_q \gg \psi(2^l)2^{l(\frac{1}{p}-1)}\|f_1-G_m(f_1)\|_q \gg \psi(m)m^{\frac{1}{p}-1}m^{\frac{1}{2}} = \psi(m)m^{\frac{1}{p}-\frac{1}{2}}.$$

The required lower bound is established, which proves the theorem. ■

Remark. The assertion of the theorem for a special case of the classes $W^r_{p,\beta}$ was established by Temlyakov [6]. ◂

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Singular and fractal properties of functions associated with three-symbolic system of real numbers coding

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Abstract: We consider continuous singular and piecewise singular functions defined in terms of given polybasic three-symbolic representation of real numbers, which depend on three parameters and are a generalization of classic ternary representations of real numbers. Their local and global properties (structural, variational, extreme, differential, integral and fractal) are studied. We investigate a family of continuous functions that store a central digit in a given polybasic three-symbolic representation of real numbers, that depends on three parameters and is a generalization of classic ternary representation of real numbers. It is proved that the set of such functions is continuous. A special role in this family has unique strictly decreasing function, called the inversor of digits. We also thoroughly study the properties of several model representatives of countable subclasses of functions with one and two infinite levels, respectively. They are piecewise singular. We found equivalent definitions for them.

Resumen: En esta contribución, se consideran funciones continuas singulares y singulares a trozos, definidas en términos de una determinada representación polibásica trisimbólica de los números reales, que depende de tres parámetros y es una generalización de la representación ternaria clásica de los números reales. Se estudian sus propiedades locales y globales (estructurales, variacionales, extremas, diferenciales, integrales y fractales). Se investiga una familia de funciones continuas que almacenan un dígito central en una determinada representación polibásica trisimbólica de los números reales, que depende de tres parámetros y es una generalización de la representación ternaria clásica de los números reales. Se demuestra que el conjunto formado por dichas funciones es continuo. Una función especial en esta familia tiene una única función estrictamente decreciente, llamada inversor de los dígitos. También se estudian a fondo las propiedades de varias representaciones de subclases contables de funciones con uno y dos niveles infinitos respectivament. Estas son singulares a trozos y encontramos una definición equivalente para ellas.

Keywords: singular function, Q_3-representation digits of real number, self-affine set, Hausdorff-Besicovitch dimension, the set of level of function.

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1. Introduction

Continuous functions often differ fundamentally in their local properties. We are not talking about smooth functions, but about functions with a complex local structure, which have "features" in each arbitrarily small interval. This "category" of functions includes tortuous and singular functions. The first ones do not have intervals of monotonicity, but in each arbitrarily small segment they have the largest and the smallest value. The theorems of Banach-Mazurkiewicz and Zamfirescu show that families of such functions are "numerous" and, therefore, deserve attention. Recently, interest in such functions has been growing, and many works have been devoted to them, in particular [1–4]. There is a common problem for them: the problem of having an effective "apparatus" of tasks and research. Following the principle "from simple to complex", functions that have relatively simple local properties have been studied so far, namely: they have the properties of self-similarity and self-affinity. For this class of functions, we are looking for alternative ways of setting and studying, seeing some potential in the theory of functional equations.

The well-known Lebesgue theorem states that every monotonic function f can be decomposed into a linear combination $\alpha_1 f_d + \alpha_2 f_{ac} + \alpha_3 f_s$ ($\alpha_i > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$) of three monotonic functions: discrete f_d, absolutely continuous f_{ac} and singular f_s. Moreover, in 1981 T. Zamfirescu proved that the "majority" of continuous monotonic functions are singular, since the latter in the metric space of all continuous monotonic functions with supremum-metric form a set of Be of the second category. For more than 100 years of development, the theory of singular functions has been enriched mainly due to individual theories (individual functions or finitely parametric families of functions have been studied), but the general theory is still poorly developed, it contains little, but it is small. At the same time, the study of singular functions has recently been intensified due to their connection with the theory of fractals.

On the basis of the general interest in singular functions, a natural interest in nonmonotonic singular functions and nontrivial mixtures of singular and absolutely continuous functions arises. Examples of nonmonotonic singular Kantor-type functions (functions whose constancy intervals form a set of full measure) are easily constructed. The first examples of nowhere monotonic singular functions were constructed in the 1950s by Indian mathematicians (Shukla U. K., Gard K. M.). Simple examples of singular monotonic functions nowhere appear in the works of M. V. Pratsiovytyi and A. N. Agadjanov. There are only a few works dedicated to such functions. Mixtures of singular and absolutely continuous functions have not yet been the subject of serious study. The logical question is: in which "relatively simple" classes are such functions dominant? And where do they "appear" naturally?

There are a number of problems associated with singular functions, one of which is the problem of effective ways to set and research them. Recently, various systems of representation of real numbers with both finite and infinite alphabets have been used for this purpose, one of which is Q-representation of numbers, first introduced in 1986 by M. V. Pratsiovytyi [3]. It was used to study singular distribution functions. We use Q-representation of numbers to study nonmonotonic piecewise singular functions. We are interested, in particular, in the fractal aspect of the study.

2. Object of research

The investigated functions are defined in terms of Q_3-representations of real numbers from the segment $[0,1]$, namely: $x = \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left[\beta_{\alpha_k(x)} \prod_{j=1}^{k-1} q_{\alpha_j(x)}\right] \equiv \Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, where (q_0, q_1, q_2) is a predetermined set of positive numbers such that $q_0 + q_1 + q_2 = 1$, $\beta_0 = 0$, $\beta_1 = q_0$, $\beta_2 = q_0 + q_1$, $\alpha_n(x) \in \{0, 1, 2\}$.

The main object of research is a continuous function f satisfying the conditions

$$f(\Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{Q_3}_{\gamma_1\gamma_2\ldots\gamma_n\ldots} \quad \text{and} \quad \gamma_n = \begin{cases} \gamma_n(y) = \gamma_n(\alpha_1, \alpha_2, \ldots, \alpha_n), \\ \gamma_n = 1 \Leftrightarrow \alpha_n = 1. \end{cases} \tag{1}$$

That is, if $\gamma_n = 1$ if and only if $\gamma_n = 1$, then the function f stores the digit 1 (without magnification). The set of all such functions is denoted by P_c.

3. Results

Theorem 1. *The set of P_c continuous functions on $[0;1]$, which store the number 1 in the Q_3-representation of numbers, is continuum.*

Proof. To prove this fact, it suffices to show that the function f, which for a predetermined $y_0 \in C \equiv C[Q_3,\{0,2\}] = \{x : x = \Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},\ \alpha_1 \in \{0,2\}\}$ gives $f(1) = y_0$. From this, the equivalence of subset P_c and continuum set C is obvious, which is equivalent to continuum P_c. ∎

A trivial example of a function that satisfies this definition is the function $f(x) = x$. Another example of such a function is the inversor, which is a continuous function I with inhomogeneous differential properties [4], which is denoted by the equality $I(\Delta^{Q_3}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{Q_3}_{[2-\alpha_1][2-\alpha_2]\ldots[2-\alpha_n]\ldots}$. The function I can be defined as the only solution to the system of functional equations $f(\beta_i + q_i x) = \beta_{[2-i]} + q_{[2-i]} f(x)$, $i = 0,1,2$, in the class of continuous functions.

Theorem 2 ([4]). *The inversor I has the following properties:*

(i) *it is a correctly defined continuous monotone function and singular if $q_0 \neq q_2$;*

(ii) *its graph $\Gamma_I = \{(x, I(x)) : x \in [0,1]\}$ is a self-affine set, namely: $\Gamma_I = \bigcup_{i=0}^{2} \phi_i(\Gamma_I) \equiv \phi(\Gamma_I)$, where ϕ_i is an affine transformation such that* $\phi_i : \begin{cases} x' = q_i x + \beta_i, \\ y' = -q_{[2-i]} y + \beta_{[3-i]}, \quad i \in A_3; \end{cases}$

(iii) *there is equality:* $\displaystyle\int_0^1 I(x)\,\mathrm{d}x = \frac{2q_0q_1 + q_0^2}{1 - 2q_0q_1 - q_1^2}.$

Using the definition of the inversor, you can effectively specify functions with some symmetries of the graph, which belong to the set P_c. For example, a function f that is the only solution in the set P_c of the functional equation $f(x) = f(I(x))$, $x \in [0,1]$ under condition $f(\Delta^{Q_3}_{(0)}) = \Delta^{Q_3}_{(02)}$.

Consider another example, the function $g(x)$.

Define the digits γ_n of the function $g(x)$ as follows. Let $\{0,2\} \ni i$ be a fixed parameter. We put

(i) $\gamma_1 = \alpha_1$; moreover, if $\alpha_1 = 1$, then $\gamma_{1+r} = \alpha_{1+r}$, $r \in N$, $g(\Delta^{Q_3}_{(0)}) = \Delta^{Q_3}_{(02)}$ and $g(\Delta^{Q_3}_{(2)}) = \Delta^{Q_3}_{(20)}$;

(ii) if $\alpha_1 = \ldots = \alpha_m = i$ and $\alpha_{m+1} = 2 - i$, then $\gamma_{m+1+r} = \begin{cases} \alpha_{m+1+r}, & m \text{ is odd number}, \\ 2 - \alpha_{m+1+r}, & m \text{ is even number}; \end{cases}$

(iii) if $\alpha_1 = \ldots = \alpha_m = i$, $\alpha_{m+1} = \ldots = \alpha_{m+k} = 1$ and $\alpha_{m+k+1} = 2-i$, then $\gamma_{m+k+1+r} = \begin{cases} \alpha_{m+k+1+r}, & m = 2t+1, \\ 2 - \alpha_{m+k+1+r}, & m = 2t. \end{cases}$

If $\alpha_{m+k+1} = i$, then $\gamma_{m+k+1+r} = \begin{cases} 2 - \alpha_{m+k+1+r}, & m = 2t+1, \\ \alpha_{m+k+1+r}, & m = 2t, \end{cases}$ where $m, k, r, t \in N$.

It is obvious that the function $g(x)$ is correctly defined and satisfies (1), i.e., it belongs to P_c.

Theorem 3. *The function $g(x)$ thus defined is the only solution of the functional equation with the conditions*

$$\begin{cases} g(I(x)) = I(g(x)), \quad x \in [0,1], \\ g(\Delta^{Q_3}_{(0)}) = \Delta^{Q_3}_{(02)}, \\ g(\Delta^{Q_3}_{(2)}) = \Delta^{Q_3}_{(20)}. \end{cases}$$

Proof. From the definition g for $i, j \in \{0,2\}$ of the function, the relations follow:

$$g(\Delta^{Q_3}_{1\alpha_1\ldots\alpha_n\ldots}) = \Delta^{Q_3}_{1\alpha_1\ldots\alpha_n\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m+1}[2-i]\alpha_{2m+3}\ldots\alpha_{2m+3+n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots i}_{2m+1}[2-i]\alpha_{2m+3}\ldots\alpha_{2m+3+n}\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m}[2-i]\alpha_{2m+2}\ldots\alpha_{2m+2+n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots i[2-i]}_{2m}i[2-\alpha_{2m+2}]\ldots[2-\alpha_{2m+2+n}]\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m+1}\underbrace{1\ldots 1}_{k}j\alpha_{r_1}\ldots\alpha_{r_n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots i}_{2m+1}\underbrace{1\ldots 1}_{k}[2-i]\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_1}\right]\ldots\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_n}\right]\ldots},$$

$$g(\Delta^{Q_3}_{\underbrace{i\ldots i}_{2m}\underbrace{1\ldots 1}_{k}j\alpha_{r_1}\ldots\alpha_{r_n}\ldots}) = \Delta^{Q_3}_{\underbrace{i[2-i]\ldots[2-i]}_{2m}\underbrace{1\ldots 1}_{k}i\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_1}\right]\ldots\left[j+(-1)^{\left[\frac{j}{2}\right]}\alpha_{r_n}\right]\ldots}.$$

It is easy to see that for all the above relations, equality $g(I(x)) = I(g(x))$ holds. Therefore, equality $g(I(x)) = I(g(x))$ holds for the function g.

We show that g is the only solution. Suppose that in the set P_c there exists a function ψ different from g which satisfies the conditions of the theorem. That is, there are points $x_0 = \Delta^{Q_3}_{c_1c_2\ldots c_n\ldots}$ that $y_0 = \psi(x_0) \neq g(x_0) = y_1$. Then for each such number x_0 there exists the smallest natural k such that $h = \alpha_k(y_0) \neq \alpha_k(y_1) = l$, $h \neq 1 \neq l$, $h, l \in \{0, 2\}$ and $\alpha_j(y_0) = \alpha_j(y_1)$ for $j < k$. We choose among them x_0 for which k is the smallest. If there are more than one x_0, then we take the one for which the number h is the smallest.

From the fact that $\alpha_j(y_0) = \alpha_j(y_1)$ for $j < k$ it follows that y_0 and y_1 belong some segment $\Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}}$.

Let $h < l$. From the above it follows that $h = 0$, $l = 2$. Then, there exists a point $x_1 < x_0$, i.e., $x_1 \in \Delta^{Q_3}_{c_1c_2\ldots c_{k-1}[c_k-1]}$. Given the previous considerations, consider different Q_3-rational values: $x_0 \equiv \Delta^{Q_3}_{c_1c_2\ldots c_{k-1}c_k(0)} = \Delta^{Q_3}_{c_1c_2\ldots c_{k-1}[c_k-1](2)} \equiv x_1$. The function ψ at the points x_0 and x_1 takes values

$$y_0 = \psi(x_0) = \Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}h\tau_1\tau_2\ldots\tau_n\ldots} = \Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}0\tau_1\tau_2\ldots\tau_n\ldots}, \quad y_0^* = \psi(x_1) = \Delta^{Q_3}_{c'_1c'_2\ldots c'_{k-1}ds_1s_2\ldots s_n\ldots},$$

where $d \in A_3$, $\tau_i, s_i \in \{0, 2\}$, $i \in N$. If the conditions $\tau_i = 2$, $s_i = 0$ and $d = 1$ are satisfied for all i, then the function ψ coincides with g, which contradicts the assumption. If the sequences (τ_n) and (s_n) are arbitrarily different from the previous case or $d \neq 1$, then the function ψ is discontinuous. Therefore, $\psi \notin P_c$ which contradicts the assumption.

If $h > l$, i.e., $h = 2$ and $l = 0$, then the numbers y_0 and y_0^* are equal if and only if the conditions $\tau_i = 0$, $s_i = 2$ and $d = 1$ are satisfied for all i. In this case ψ coincides with g, which contradicts the assumption. For the remaining values of d, (τ_n), (s_n) we obtain a discontinuous function, i.e., $\psi \notin P_c$.

Therefore, only function g satisfies the conditions of the theorem. The theorem is proved. ∎

Theorem 4. *The function g has the following properties: for $q_0 = q_2$, it is a piecewise linear, and for $q_0 \neq q_2$ it is a mixture of singular and piecewise linear; the graph of the function g is "symmetrically similar" with respect to the point $(\Delta^{Q_3}_{(1)}; \Delta^{Q_3}_{(1)})$; and it has two infinite levels $y_0 = \Delta^{Q_3}_{(02)}$ and $y'_0 = \Delta^{Q_3}_{(20)}$.*

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On unique-extension renormings

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Abstract: E. Oja, T. Viil, and D. Werner proved that every weakly compactly generated Banach space X having a norm with the property that every linear functional on X has a unique Hahn-Banach extension to its bidual X^{**} (which R. R. Phelps referred to as "X having *property U* in X^{**}") can be renormed to have the stronger property that every linear continuous functional defined on any linear subspace of X has a unique Hahn–Banach extension to X^{**} (the so-called *total smoothness* property of X). We proved that, thanks to a deep theorem of M. Raja, the above result can be obtained even in a stronger form and without any extra conditions on the space X (i.e., omitting the "weakly compactly generated" on the statement). Here we recall this result and present some extensions in the direction of what is called "weak Hahn-Banach smoothnes". This is partially based on a joint work with A. J. Guirao and V. Montesinos.

Resumen: E. Oja, T. Viil, and D. Werner probaron que todo espacio de Banach débilmente compactamente generado que tenga una norma con la propiedad de que todo funcional lineal y continuo en X tenga una única extensión de Hahn-Banach a su bidual X^{**} (es decir, "X tiene la *propiedad U* en X^{**}", en la terminología de R. R. Phelps) puede ser renormado para tener la propiedad más fuerte de que todo funcional lineal y continuo definido en cualquier subespacio lineal de X tiene una única extensión de Hahn-Banach a X^{**} (lo que se conoce como *total suavidad* de X). Probamos que, gracias a un profundo teorema de M. Raja, se puede obtener una versión incluso más fuerte del resultado anterior sin ninguna condición adicional en el espacio X (es decir, omitiendo "débilmente compactamente generado" en el enunciado). Reproducimos el resultado y proponemos algunas extensiones usando el concepto de suavidad Hahn-Banach débil del espacio. Esto está parcialmente basado en un trabajo conjunto con A. J. Guirao y V. Montesinos.

Keywords: renorming, Hahn-Banach extensions, Hahn-Banach smoothness, total smoothness, LUR norm, Kadets-Klee property.

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1. Introduction

The present contribution —based on the joint work [2]— is motivated by a recent paper [4], where it is proved that *every weakly compactly generated Banach space whose norm has the Hahn-Banach smooth property has an equivalent norm with the (stronger) totally smooth property*. This improved an earlier result of Sullivan [8] proving this statement under the assumption of separability. Our main contribution, that solves an open problem in [4], is that no extra requirement —besides the Hahn-Banach property of the norm— on the space is needed. This is a consequence of an important theorem due to M. Raja [7], where the so-called Kadets-Klee property for the w and w^* topologies in the dual of a Banach space allows for a renorming of the space with the local uniformly rotund property of its dual norm. By "renorming" a Banach space we mean defining an equivalent norm on it —naturally seeking for better geometric or analytic properties. Details and definitions needed are given below.

The basic Hahn-Banach theorem does not ensure uniqueness of the existing norm-preserving extension from a subspace to the whole space. This issue was considered by Phelps, who introduced the following definition:

Definition 1 (Phelps). Let $(X, \|\cdot\|)$ be a Banach space, and M a linear (not necessarily closed) subspace of X. We will say that M has *property U* in X if each continuous linear functional on M has a unique norm-preserving extension to X. ◂

We shall consider every Banach space X canonically embedded in its bidual space X^{**}.

Definition 2 (Sullivan). The norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ is said to be *Hahn–Banach smooth* (*HBS*, for short) if $(X, \|\cdot\|)$ has property U in X^{**} (i.e., every $x^* \in X^*$ has a unique norm-preserving extension to X^{**}). ◂

Definition 3. The norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ is said to be *totally smooth* (*TS*, for short) if every linear subspace M of X has property U in X^{**} (i.e., for every linear subspace M of X, every $f \in M^*$ has a unique norm-preserving extension to X^{**}). ◂

Obviously, if a norm has the HBS property, then it has the TS property. Notice that properties U, HBS, and TS, are of isometric nature. Indeed (some needed definitions will appear in Section 2 below),

(i) the Hilbertian norm on a Hilbert space H has property U. However, as happens in every Banach space, H has an equivalent norm $\|\cdot\|$ that fails to be Gâteaux differentiable at some $x_0 \in S_H$. Thus, two different norm-preserving extensions of $x_0^*|_M$ exist, where $M := \operatorname{span}\{x_0\}$ and x_0^* belongs to the subdifferential of $\|\cdot\|$ at x_0. This shows that U is not invariant under renormings.

(ii) On the other hand, it is not hard to prove that a Banach space $(X, \|\cdot\|)$ is reflexive if, and only if, every equivalent norm on X is HBS. If $(X, \|\cdot\|)$ is a Banach space with a separable dual, it is well known (see [3]) that X^* admits an equivalent dual LUR norm $|||\cdot|||^*$, so the topologies w and w^* coincide on S_{X^*}. Proposition 6 below shows then that $|||\cdot|||$ on X is HBS. By a previous observation, if moreover the space X is not reflexive, then it has an equivalent norm $|\cdot|$ which is not HBS. Thus, HBS is not invariant under renormings.

(iii) Finally, Theorem 9 below shows that the property TS of a norm is equivalent to the HBS property plus the strict convexity of its dual norm. Thus, the Hilbertian norm on a Hilbert space H is obviously TS, although H admits a non-rotund equivalent norm (this is a dualization of the argument in (i) above). This shows that TS is also non-invariant under renormings.

The statement of our main result follows. As we mentioned, this improves results in [8] and [4], and solves a problem in [4].

Theorem 4. *Let $(X, \|\cdot\|)$ be a Banach space. Then, the following statements are equivalent:*

(i) *X has an equivalent norm with property HBS.*

(ii) *X^* has an equivalent w^*-w-Kadets-Klee norm.*

(iii) *X has an equivalent norm whose dual norm is LUR.*

(iv) *X has an equivalent norm with property TS.*

The following classes of Banach spaces satisfy one —and then all— of the conditions in Theorem 4: (i) Asplund spaces that are weakly compactly generated, and (ii) spaces whose dual is a subspace of a weakly compactly generated Banach space. In particular, separable Banach spaces satisfy one of the conditions in Theorem 4 if, and only if, they are Asplund.

2. The walkthrough

We shall provide a sketch of the proof of Theorem 4. We shall need some extra definitions. Recall that a norm $\|\cdot\|$ on a Banach space is called *strictly convex* or *rotund* if the unit sphere does not contain non-trivial line segments. It is called *locally uniformly rotund* (*LUR*, for short) if for every $x, x_n \in S_X$, $n \in \mathbb{N}$, such that $\|x + x_n\| \to 2$, then $x_n \to x$.

Definition 5. Let $\|\cdot\|$ be a norm in a Banach space X and $\tau_1 \subset \tau_2 \subset \|\cdot\|$ two vector topologies on X. We say that $\|\cdot\|$ has the ***τ_1-τ_2**-Kadets-Klee property* if the topologies τ_1 and τ_2 coincide on its unit sphere. ◂

Proof of (i)$\Longleftrightarrow$(ii) in Theorem 4: it will follow from Propositions 6 and 7 below.

Proposition 6 (Godefroy). *Let $(X, \|\cdot\|)$ be a Banach space. Then, $x^* \in S_{X^*}$ has a unique norm-preserving extension to X^{**} if, and only if, the w^* and w topologies on S_{X^*} coincide on x^*. In particular, $\|\cdot\|$ is HBS if, and only if, its dual norm has the w^*-w-Kadets-Klee property.*

This last proposition suggests that in order to find an equivalent HBS norm on a Banach space X —if possible—, we should try to renorm the dual space X^* with a w^*-w-Kadets-Klee norm. There is an extra requirement that in some cases is hard to achieve: the equivalent norm on the dual space must be a dual norm. In our situation, this is obtained for free: as a particular case of the following result, any w^*-w-Kadets-Klee norm on X^* is already a dual norm.

Proposition 7. *Let $\|\cdot\|$ be a τ_1-τ_2-Kadets-Klee norm which is τ_2-lower semicontinuous. Then, it is also τ_1-lower semicontinuous.*

(ii)$\Longleftrightarrow$(iii): this is a consequence of the following deep result proved by Raja [6, 7], a landmark in renorming theory.

Theorem 8 (Raja). *Let X be a Banach space. If X admits an equivalent norm whose dual is w^*-w-Kadets-Klee, then it admits an equivalent norm whose dual is LUR.*

That (iv)$\Longrightarrow$(i) was already mentioned above.

To finalize the proof of Theorem 4, the only remaining thing is to prove that (iii)$\Longrightarrow$(iv). Notice that the norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ has the TS property if, and only if, every linear subspace M of X has property U in X and $\|\cdot\|$ has the HBS property. We need the following result.

Theorem 9 (Taylor-Foguel). *Let $(X, \|\cdot\|)$ be a Banach space. Then, every linear subspace M of X has property U on X if, and only if, the dual norm $\|\cdot\|^*$ is rotund.*

This last theorem and the paragraph above allow us to decompose the TS property in the following way: The norm $\|\cdot\|$ of a Banach space is TS if, and only if, it has HBS property and its dual norm is strictly convex. It is easy to see that the property of having a dual LUR norm is stronger than having those two properties simultaneously. In fact, any LUR norm is already a strictly convex norm, and also, as an easy application of the Riesz lemma, we have that if $\|\cdot\|^*$ is a dual LUR norm then the w^* and the norm topologies (and so, any topology in between them) coincide on its unit sphere. Proposition 6 shows that this implies the HBS property on its predual norm $\|\cdot\|$, and the proof of Theorem 4 is over.

3. Some further topics

We present here some remarks on, and some extensions of the previous results. Most of this can be found in detail in [1].

Remark 10. First of all, it is of key importance to prove that Theorem 4 is a real extension of the result of Oja, Viil and Werner: Indeed, it could happen that all four conditions stated in Theorem 4 would imply the original Banach space X being WCG. However, this is not the case. To see this, it is enough to take any Hausdorff non-Eberlein compact space K such that $K^{\omega_1} = \varnothing$ (for example, $K = [0, \omega_1]$), and consider the corresponding $C(K)$ space. It is proved in [3] that $C(K)^*$ admits a dual LUR norm (in particular, $C(K)$ admits an HBS norm), but it is not a WCG space (as K is not Eberlein). ◂

Some of the work done in Section 2 can be extended to more general cases. For this purpose, we may introduce some extra definitions related to the uniqueness of extensions (if X is a Banach space, then the subset of X^* consisting of all norm-attaining functionals on X will be denoted by $\mathrm{NA}(X)$): (i) If M is a linear (not necessarily closed) subspace of X, we will say that M has *property wU* in X if each element in $\mathrm{NA}(M)$ has a unique norm-preserving extension to X. (ii) The space $(X, \|\cdot\|)$ is said to be *weak Hahn–Banach Smooth* (*wHBS* for short) if $(X, \|\cdot\|)$ has property wU in X^{**} (i.e., every $x^* \in \mathrm{NA}(X)$ has a unique norm-preserving extension to X^{**}). The last definition is due to Sullivan. (iii) Let $\|\cdot\|$ be a norm in a Banach space X, $A \subset X$ be a cone, and $\tau_1 \subset \tau_2 \subset \|\cdot\|$ two vector topologies on X. We say that $\|\cdot\|$ has the ***τ_1-τ_2**-Kadets-Klee property with respect to **A*** when both topologies τ_1 and τ_2 coincide when restricted to $A \cap S_{(X,\|\cdot\|)}$

These definitions allow us to generalize Proposition 7 and the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 4.

Proposition 11. *Let $\|\cdot\|$ be a norm in the Banach space X that is τ_1-τ_2-Kadets-Klee with respect to a cone $A \subset X$ that satisfies $\overline{A \cap B_{(X,\|\cdot\|)}}^{\|\cdot\|}$. Then, if the norm is τ_2-lower semicontinuous, it is also τ_1-lower semicontinuous.*

Proposition 12. *Let $(X, \|\cdot\|)$ be a Banach space. Then, X admits a wHBS norm if, and only if, X^* admits a norm which is w^*-w-Kadets-Klee with respect to $NA(X)$.*

There are further similarities between the two properties HBS and wHBS. For example, it can be proved that a norm $\|\cdot\|$ is very smooth if, and only if, $\|\cdot\|$ its simultaneously Gâteaux smooth and wHBS, and this scheme is the analogous version of the TS decomposition above, but for the unique extension of the norm-attaining elements. It is natural to ask if wHBS on X also implies the existence of dual norm on X^* with good convexity properties, just as HBS implies the dual LUR norm on X^*. However, this is far from being true, since Talagrand proved that there are some spaces ($C([0, \mu])$ with uncountable μ that admit a Fréchet smooth equivalent norm (a much stronger property than being wHBS and even very smooth) but its dual spaces do not admit a dual strictly convex norm (see [3]).

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Best approximations and entropy numbers of the classes of periodic functions of many variables

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Abstract: We obtain order estimates for several approximative characteristics of the classes of periodic multivariate functions, that are connected to nonlinear approximation. Namely, we investigate a behaviour of the best orthogonal and M-term trigonometric approximations of the classes of functions with bounded generalized derivative (the classes of Weyl-Nagy type). We indicate cases, when there are advantages of nonlinear methods over the approximation of corresponding functional classes by step hyperbolic Fourier sums and by trigonometric polynomials with "numbers" of harmonics from step hyperbolic crosses.

Further we get the estimates of entropy numbers for the classes of functions with certain restrictions on their modulus of continuity (the classes of Nikol'skyi-Besov type). All the error approximations are measured in a metric of the Lebesgue space.

Resumen: Obtenemos estimaciones de orden para varias características aproximativas de las clases de funciones periódicas multivariantes, que están relacionadas con la aproximación no lineal. Concretamente, investigamos el comportamiento de las mejores aproximaciones ortogonales y trigonométricas de término M de las clases de funciones con derivada generalizada acotada (las clases de tipo Weyl-Nagy). Indicamos los casos, cuando hay ventajas de los métodos no lineales sobre la aproximación de las clases funcionales correspondientes por sumas de Fourier hiperbólicas escalonadas y por polinomios trigonométricos con "números" de armónicos de cruces hiperbólicos escalonados.

Además, obtenemos las estimaciones de los números de entropía para las clases de funciones con ciertas restricciones en su módulo de continuidad (las clases de tipo Nikol'skyi-Besov). Todas las aproximaciones de error se miden en una métrica del espacio de Lebesgue.

Keywords: entropy numbers, best approximation, step hyperbolic cross.

MSC2010: 42A10.

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1. Introduction

Let $\mathbb{R}^d$, $d \geq 1$, be the Euclidean space with elements $\boldsymbol{x} = (x_1, \dots, x_d)$ and $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + \dots + x_d y_d$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. By $L_q := L_q(\pi_d)$, $\pi_d = \prod_{j=1}^{d} [0, 2\pi]$, $1 \leq q \leq \infty$, we denote the space of functions $f(\boldsymbol{x})$, that are 2π-periodic by each variable, equipped with the usual norm. Suppose further, that for the functions $f \in L_1$ the condition $\int_0^{2\pi} f(\boldsymbol{x})\, \mathrm{d}x_j = 0$, $j = \overline{1, d}$, holds.

Let us consider the Fourier series for $f \in L_1$, i.e.,

$$\sum_{k \in \mathbb{Z}^d} \hat{f}(k) \mathrm{e}^{\mathrm{i}(k,x)},$$

where $\hat{f}(\boldsymbol{k}) = (2\pi)^{-d} \int_{\pi_d} f(\boldsymbol{t}) \mathrm{e}^{-\mathrm{i}(\boldsymbol{k},\boldsymbol{t})} \mathrm{d}\boldsymbol{t}$ are the Fourier coefficients of f. Further, let $\boldsymbol{\psi} = (\psi_1, \dots, \psi_d)$, $\psi_j \neq 0$, $j = \overline{1, d}$, be arbitrary sequences of natural argument, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$, $\mathring{\mathbb{Z}}^d = (\mathbb{Z} \setminus \{0\})^d$. Assume that the series

$$\sum_{\boldsymbol{k} \in \mathring{\mathbb{Z}}^d} \prod_{j=1}^{d} \frac{\mathrm{e}^{\mathrm{i}\frac{\pi \beta_j}{2} \operatorname{sgn} k_j}}{\psi_j(|k_j|)} \hat{f}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}(\boldsymbol{k},\boldsymbol{x})},$$

where $\mathring{\mathbb{Z}}^d = (\mathbb{Z} \setminus \{0\})^d$, are the Fourier series of some function $f_{\boldsymbol{\beta}}^{\boldsymbol{\psi}}$ summable on π_d.

Let us denote by $L_{\boldsymbol{\beta},p}^{\boldsymbol{\psi}}$, $1 \leq p \leq \infty$, a set of functions f, for which $(\boldsymbol{\psi}, \boldsymbol{\beta})$-derivatives exist, and the condition $\|f_{\boldsymbol{\beta}}^{\boldsymbol{\psi}}\|_p \leq 1$ is satisfied. The univariate classes $L_{\beta,p}^{\psi}$ were introduced in 1983 by A. I. Stepanets [10]. The study of different approximative characteristics on the respective classes of multivariate functions was initiated by A. S. Romanyuk and further continued by his students. Note also that these classes generalize the well-known Weyl-Nagy classes $W_{\boldsymbol{\beta},p}^{\boldsymbol{r}}$, namely, $L_{\boldsymbol{\beta},p}^{\boldsymbol{\psi}} \equiv W_{\boldsymbol{\beta},p}^{\boldsymbol{r}}$ in the case $\psi_j(|\tau|) \equiv |\tau|^{-r_j}$, $\tau \in \mathbb{Z} \setminus \{0\}$, $r_j > 0$, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$.

When investigating best trigonometric approximations on the classes $L_{\boldsymbol{\beta},p}^{\boldsymbol{\psi}}$, we impose some additional conditions on the sequences ψ_i, $i = 1, \dots, d$. So, let D be a set of functions ψ of natural argument that satisfy the conditions

- ψ are positive and non increasing;
- there exists $M > 0$ such that for all $l \in \mathbb{N}$ we have $\frac{\psi(l)}{\psi(2l)} \leq M$.

Note that to the indicated set belong, in particular, the functions $\phi(|\tau|) = |\tau|^{-r}$; $\phi(|\tau|) = \ln^{\alpha}(|\tau| + 1)$, $\alpha < 0$; $\phi(|\tau|) = \ln^{\alpha}(|\tau| + 1)|\tau|^{-r}$, where $\tau \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathbb{R}$, $r > 0$.

The results are presented in terms of functions

$$\Phi(n) = \min_{(\boldsymbol{s},\boldsymbol{1})=n} \prod_{j=1}^{d} \psi_j(2^{s_j}), \qquad \Psi(n) = \max_{(\boldsymbol{s},\boldsymbol{1})=n} \prod_{j=1}^{d} \psi_j(2^{s_j}),$$

where the vectors $\boldsymbol{s}, \boldsymbol{1} \in \mathbb{N}^d$. In the case $\psi_j(|\tau|) = |\tau|^{-r}$, $j = \overline{1, d}$, $r > 0$, we have $\Phi(n) = \Psi(n) = 2^{-nr}$ and, besides, for $d = 1$ the functions $\Phi(n)$ and $\Psi(n)$ coincide and take the form $\psi_1(2^n)$.

In what follows, we also establish estimates for the entropy numbers of the classes $B_{p,\theta}^{\Omega}$ in the metric of the space L_q, $1 \leq q \leq \infty$, under certain conditions imposed on the function Ω and the parameters p, θ. For the first time, the indicated classes with $\theta = \infty$ were considered by N. N. Pustovoitov [6]. In the paper by S. Yongsheng and W. Heping [11], these classes were extended to the case $1 \leq \theta < \infty$. They can be regarded as a generalization of the classes $B_{p,\theta}^{\boldsymbol{r}}$ with respect to a smooth parameter. The classes $B_{p,\theta}^{\Omega}$ are defined with the help of a majorant function $\Omega(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}_+^d$, for the mixed modulus of continuity $\Omega_l(f, \boldsymbol{t})_p$ of order l, $l \in \mathbb{N}$, of the function $f \in L_p(\pi_d)$, $1 \leq p \leq \infty$, and a numerical parameter θ, $1 \leq \theta \leq \infty$.

The results are given in terms of order relations. So, for two nonnegative sequences $\{a(n)\}_{n=1}^{\infty}$ and $\{b(n)\}_{n=1}^{\infty}$, the relation (order inequality) $a(n) \ll b(n)$ means that there exists a constant $C > 0$, independent of n and such that $a(n) \leq Cb(n)$. The relation $a(n) \asymp b(n)$ is equivalent to $a(n) \ll b(n)$ and $b(n) \ll a(n)$.

2. Estimates of the best trigonometric approximation

We now define the approximate characteristics. So, for $f \in L_q$, $1 \le q \le \infty$, the quantity

$$e_M(f)_q = \inf_{\boldsymbol{k}^j, c_j} \left\| f - \sum_{j=1}^{M} c_j \mathrm{e}^{\mathrm{i}(\boldsymbol{k}^j, \boldsymbol{x})} \right\|_q \tag{1}$$

is called the best M-term trigonometric approximation of function f, where $\{\boldsymbol{k}^j\}_{j=1}^M$ is the set of vectors $\boldsymbol{k}^j = (k_1^j, \dots, k_d^j) \in \mathbb{Z}^d$, $c_j \in \mathbb{C}$, $j = \overline{1, M}$. For $F \subset L_q$, we put $e_M(F)_q = \sup_{f \in F} e_M(f)_q$ and call it the best M-term trigonometric approximation of the functional class F.

We consider also a close to (1) characteristic $e_M^{\perp}(f)_q$ (respectively, $e_M^{\perp}(F)_q$), where the coefficients c_j of corresponding polynomials are the Fourier coefficients of the function f with respect to the system $\{\boldsymbol{k}^j\}_{j=1}^M$. The detailed history and further references could be found in the monograph by D. Dũng, V. N. Temlyakov and T. Ullrich [2].

Let us formulate some results for the best M-term approximation of the classes $L_{\beta,p}^{\psi}$, that we proved in the papers [7–9]. We considered first the limit case $p = 1$.

Theorem 1. *Let $1 < q \le 2$, $\psi_j \in D$, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$, and $\varepsilon > 0$ be such that $\psi_j(|\tau|)\,|\tau|^{1-1/q+\varepsilon}$, $j = \overline{1, d}$, do not increase. Then, for any natural M and n that satisfy the condition $M = M(n) \asymp 2^n n^{d-1}$, the following relations hold:*

$$\Phi(n) M^{1-1/q} (\log M)^{2(d-1)(1/q-1/2)} \ll e_M\big(L_{\beta,1}^{\psi}\big)_q \ll \Psi(n) M^{1-1/q} (\log M)^{2(d-1)(1/q-1/2)}.$$

Note, that in the univariate case we get an exact-order estimate $e_M\big(L_{\beta,1}^{\psi_1}\big)_q \asymp \psi_1(M) M^{1-1/q}$, $1 < q \le 2$. Analogous estimates (with the same left and right bounds) hold also for the quantity $e_M^{\perp}\left(L_{\beta,1}^{\psi_1}\right)_q$, $1 < q < \infty$.

Theorem 2. *Let $2 < q < \infty$, $\psi_j \in D$, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$, and $\varepsilon > 0$ be such that $\psi_j(|\tau|)\,|\tau|^{1+\varepsilon}$, $j = \overline{1, d}$, do not increase. Then, for any natural M and n that satisfy the condition $M = M(n) \asymp 2^n n^{d-1}$, the following relations hold:*

$$\Phi(n) M^{1/2} \ll e_M\big(L_{\beta,1}^{\psi}\big)_q \ll \Psi(n) M^{1/2}.$$

We see that, in the case $2 < q < \infty$, the best M-term approximation (that uses arbitrary coefficients of approximation polynomials) gives better bounds than the corresponding best orthogonal approximation, while for $1 < q \le 2$ they coincide in order.

In the paper [3], we get the estimates for $e_M\left(L_{\beta,p}^{\psi}\right)_q$ in the case of "small smoothness" of respective functions. We showed, that the best M-term approximation in this case gives better bounds, than the corresponding best orthogonal approximation and the approximation of functions from the class $L_{\beta,p}^{\psi}$ by trigonometric polynomials with numbers of harmonics from the so-called step hyperbolic crosses.

3. Estimates of entropy numbers

Let X be a Banach space, $B_X(\boldsymbol{y}, r)$ be a ball of radius r and center at point $\boldsymbol{y} \in \mathbb{R}^d$. For a compact set $A \subset X$ and $\varepsilon > 0$ by $\varepsilon_k(A, X)$ we denote entropy numbers (see., e.g., [1]) of this set:

$$\varepsilon_k(A, X) = \inf\left\{\varepsilon \,:\, \exists \boldsymbol{y}^1, \dots, \boldsymbol{y}^{2^k} \in X \,:\, A \subseteq \bigcup_{j=1}^{2^k} B_X(\boldsymbol{y}^j, \varepsilon)\right\}.$$

In [4, 5], we obtained the estimates for entropy numbers of the classes of periodic multivariate functions $B_{p,\theta}^{\Omega}$, under the so-called Bari-Stechkin (see [1]) conditions (S^{α}) and (S_l) on the function Ω, in the space L_q, $1 \le q \le \infty$. In particular, the following statement holds in the case $d = 2$.

Theorem 3. *Let $d = 2$, $2 \le p \le \infty$, $1 \le \theta \le \infty$, and $\Omega(\boldsymbol{t}) = \omega\Big(\prod_{j=1}^{d} t_j\Big)$, where the function ω satisfies condition (S^α) with some $\alpha > 1/2$ and condition (S_l). Then, for any natural M and n such that $M = M(n) \asymp 2^n n$, the following relation holds:*

$$\varepsilon_M \left(B^{\Omega}_{p,\theta}, L_\infty\right) \asymp \omega\left(2^{-n}\right)(\log M)^{1-1/\theta}.$$

Note that we got estimates of the quantity $\varepsilon_M \left(B^{\Omega}_{p,\theta}, L_q\right)$ for diferent relations between p and q.

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Numerical index of absolute symmetric norms on the plane

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Abstract: We give a lower bound for the numerical index of two-dimensional real spaces with absolute and symmetric norm. This allows us to compute the numerical index of the two-dimensional real L_p-space for $3/2 \leq p \leq 3$.

Resumen: Damos una cota inferior del índice numérico para espacios reales dos-dimensionales con normas absolutas y simétricas. Esto nos permite calcular el índice numérico del espacio real dos-dimensional L_p para $3/2 \leq p \leq 3$.

Keywords: numerical range, numerical radius, numerical index, absolute symmetric norm, L_p-spaces..

MSC2010: 46B20, 47A12.

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1. Introduction

The numerical index of a Banach space is a constant relating the norm and the numerical range of bounded linear operators on the space. Given a Banach space X, we will write X^* for its topological dual and $\mathcal{L}(X)$ for the Banach algebra of all (bounded linear) operators on X. For an operator $T \in \mathcal{L}(X)$, its *numerical range* is defined as

$$V(T) := \{x^*(Tx) : x^* \in X^*, x \in X, \|x^*\| = \|x\| = x^*(x) = 1\},$$

and its *numerical radius* is

$$v(T) := \sup\{|\lambda| : \lambda \in V(T)\}.$$

Clearly, v is a seminorm on $\mathcal{L}(X)$ satisfying $v(T) \leq \|T\|$ for every $T \in \mathcal{L}(X)$. The *numerical index* of X is the constant given by

$$n(X) := \inf\{v(T) : T \in \mathcal{L}(X), \|T\| = 1\},$$

or, equivalently, $n(X)$ is the greatest constant $k \geq 0$ satisfying $k\,\|T\| \leq v(T)$ for every $T \in \mathcal{L}(X)$. There has been a deep development of this field of study with the contribution of several authors. The state of the art on the subject can be found in the survey paper [5] and references therein.

It is clear that $0 \leq n(X) \leq 1$ for every Banach space X. There are some classical Banach spaces for which the numerical index has been calculated. If H is a Hilbert space of dimension greater than one, then $n(H) = 0$ in the real case and $n(H) = 1/2$ in the complex case. Besides, $n(L_1(\mu)) = 1$ and the same happens to all its isometric preduals. In particular, it follows that $n(C(K)) = 1$ for every compact K.

2. Numerical index of absolute symmetric norms and L_p-spaces

The problem of computing the numerical index of the L_p-spaces has been latent since the beginning of the theory [4]. In order to present the known results on this matter we need to fix some notation. For $1 < p < \infty$, we write ℓ_p^m for the m-dimensional L_p-space, $q = p/(p-1)$ for the conjugate exponent to p, and

$$M_p := \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = \max_{t\geq 1} \frac{|t^{p-1} - t|}{1 + t^p},$$

which is the numerical radius of the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defined on the real space ℓ_p^2. This can be found in [7, Lemma 2]. However, some results have been obtained on the numerical index of the L_p-spaces [1–3, 7, 8], we summarize them in the following list.

(i) The sequence $(n(\ell_p^m))_{m\in\mathbb{N}}$ is decreasing.

(ii) $n(L_p(\mu)) = \inf\{n(\ell_p^m) : m \in \mathbb{N}\}$ for every measure μ such that $\dim(L_p(\mu)) = \infty$.

(iii) In the real case, $n(L_p[0,1]) \geq M_p/12$.

(iv) In the real case, $\max\left\{\frac{1}{2^{1/p}}, \frac{1}{2^{1/q}}\right\} M_p \leq n(\ell_p^2) \leq M_p$.

The presence of the numerical radius of the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the value of the numerical index of L_p-spaces is not a coincidence. For those two-dimensional real spaces with absolute and symmetric norm whose numerical index is known, it coincides with the numerical radius of the mentioned operator. This happens, for instance, to a family of octagonal norms and to the spaces whose unit ball is a regular polygon, see [6, Theorem 2 and Theorem 5]. In the paper [9], it is shown that the same happens for many absolute and symmetric norms on $\mathbb{R}^2$, this is the content of Theorem 1. We say that a norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ is *absolute* if $\|(1,0)\| = \|(0,1)\| = 1$ and

$$\|(a,b)\| = \|(|a|,|b|)\|,$$

for every $a, b \in \mathbb{R}$, and that the norm is *symmetric* if $\|(b,a)\| = \|(a,b)\|$ for every $a, b \in \mathbb{R}$. Some of the most important examples of absolute and symmetric norms are ℓ_p-norms on $\mathbb{R}^2$.

If X is $\mathbb{R}^2$ endowed with an absolute and symmetric norm, the existence of a basis of the space of operators $\mathcal{L}(X)$ formed by onto isometries is particularly useful:

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad I_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first main result of the paper [9] is the following.

Theorem 1. [9, Theorem 2.2] *Let X be $\mathbb{R}^2$ endowed with an absolute and symmetric norm. Let $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ be such that $|x_0^*(I_4 x_0)| = v(I_4)$ and write $c_j = |x_0^*(I_j x_0)|$ for every $j = 1, \dots, 4$. If $c_4 = 0$, then $n(X) = 0$. If, otherwise, $c_4 > 0$, then*

$$n(X) \geq \min \left\{ c_4, \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}} \right\}.$$

Moreover, if the inequality $c_4 \left(1 + \frac{1}{c_2} + \frac{1}{c_3} \right) \leq 1$ holds, then

$$n(X) = v(I_4).$$

As a major consequence, the numerical index of ℓ_p^2 for $3/2 \leq p \leq 3$ is calculated, which improves partially [7, Theorem 1] and throws some light to the long standing problem of computing the numerical index of L_p-spaces.

Theorem 2. [9, Theorem 2.3] *Let $p \in \left[\frac{3}{2}, 3\right]$. Then,*

$$n(\ell_p^2) = M_p = \sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

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Evolution of mathematical modelling as a method of scientific cognition and its didactic functions in educational process

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Abstract: We investigate historical aspects of developing mathematical modelling method and analyze the functions of mathematical modelling in the process of cognition. It is justified that modelling in the educational process acts simultaneously as a method of scientific knowledge, a part of the content of educational material and an effective mean of its study.

We show that the development of students' ideas about the role of mathematical modelling in scientific knowledge and practice, the development of their ability to build mathematical models of life phenomena, are an important task of modern school. A possibility of improvement pupils' literacy in mathematics is highlighted by developing their correct conceptions about the method of mathematical modelling.

Resumen: Investigamos los aspectos históricos del desarrollo del método de modelización matemática y analizamos las funciones de la modelización matemática en el proceso de cognición. Se justifica que la modelización en el proceso educativo actúa simultáneamente como un método de conocimiento científico, una parte del contenido del material educativo y un medio eficaz de su estudio.

Mostramos que el desarrollo de las ideas de los alumnos sobre el papel de la modelización matemática en el conocimiento y la práctica científica, el desarrollo de su capacidad para construir modelos matemáticos de los fenómenos de la vida, son tareas importantes de la escuela moderna. La posibilidad de mejorar la alfabetización matemática de los alumnos se pone de manifiesto en el desarrollo de sus concepciones correctas sobre el método de modelización matemática.

Keywords: model, mathematical modelling, cognition.

MSC2010: 00A35, 97M10.

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1. Introduction

Mathematical modelling is one of the main modern methods of reality cognition. It is widely used in all areas of research.

The purpose of our research is to consider historical aspects of the formation of the modelling method and to highlight its epistemological functions.

There are various interpretations of a model and modelling process concepts in modern scientific literature. A term "model" is understood as a mental or materially realized system, which is able to replace an object of study by reflecting or reproducing it, so that the study of model provides new information about this object (see V. A. Shtoff [5]). Accordingly, modelling is a scientific method of studying different systems by building models of these systems, which preserve some of the main features of the subject of study, and a study of functioning of models with the transfer of data to the subject of research.

The mathematical model of the system is understood as a set of relations (formulas, equations, inequalities, etc.), which determine characteristics of states of the system depending on its parameters, external conditions, initial conditions and time. By the definition of V. M. Glushkov, a mathematical model is a set of symbolic mathematical objects and the relations between them. For M. M. Amosov [1], a mathematical model is a system that reflects another system.

The term "model" covers an extremely wide range of material and ideal objects. Determining the epistemological role of modelling theory, that is, its significance in the process of cognition, it is necessary, first of all, to start from the historical aspects of the formation of the modelling method.

2. Historical aspects of mathematical modelling

An investigation of historical aspect of developing mathematical modelling method shows that the progress is closely linked to the development of a science.

Mathematical modelling originated in ancient times. Ancient Greek philosophers, for example, Archimedes (287 – 212 B.C.), Democritus (460 – 370 B.C.) or Epicurus (341–270 B.C.), have already explained physical properties of objects creating some analogies on intuitive base. The appearance of experimental modelling was connected with the names of Leonardo da Vinci (1452 – 1519), Johannes Kepler (1571 – 1630), Galileo Galilei (1564 – 1642), Nicolaus Copernicus (1473 – 1543). The scientists applied analog models, created graphic constructions from real objects and obtained results in further research. It was Galileo Galilei who proved the inability to bring a similarity of mechanical systems to their geometric similarity. He also claimed that conclusions obtained from only geometrically similar to its prototype models often lead to mistakes.

A powerful force for the mathematical modelling development was the appearance of mathematical notation systems. The historically first comprehensive mathematical model was the classical mechanics of Isaac Newton (1642 –1727). The scientist has initiated modelling method as those of theoretical research.

The term "model" was introduced to mathematics in the 19th century because of an emergence of the hyperbolic geometry of Nikolai Lobachevsky (1792 – 1856) and the spherical geometry of Georg Riemann (1835 – 1900). The term was later used by the German mathematician Felix Klein (1849 – 1925). Nevertheless, an application of mathematical modelling in the 19-20th centuries was accompanied by certain difficulties. It happened so because of the lack of researcher's mathematical education to describe mathematically new phenomena of science. Therefore, mathematical modelling was applied only in those branches of knowledge that have gained a high level of development. The first mathematized science is physics. The example of physics mathematization shows a parallel development of both sciences. However, in some cases mathematics has left physics behind preparing a necessary apparatus for it.

In the latter half of the 20th century there appeared a great value of works investigating mathematical modelling in epistemological and didactic aspects. A fundamental research in this field was conducted by the following scientists: N. H. Alekseev, B. M. Kedrov, V. A. Shtoff, A. I. Uyemov, L. M. Fridman, L. R. Kalapusha, etc.

There are various classifications of models in modern scientific literature. Scientists highlight functions of modelling, analyze the connections of modelling with other experimental and theoretical methods of cognition (see V. E. Bakhrushin [1], L. R. Kalapusha [3], B. A. Glinskii, B. S. Gryaznov, B. S. Dynin and E. P. Nikitin [2]).

It is shown that a mathematical model can be created in three ways:

(i) as a result of direct study of the real process (phenomenological model);

(ii) as a result of the deduction process, when the new model is a special case of a certain general model (asymptotic model);

(iii) as a result of the induction process, when the new model is a generalization of other models (ensemble models).

The model, as a special epistemological form, can be understood only when considering the set of its various functions. The analysis of scientific and methodical literature showed the diversity of views of scientists on the definition of epistemological functions of the modelling method.

V. A. Shtoff [5] states: "In the theoretical thinking one see a domination of the one side, in sensory perceptions and observations — the other, whereas in the model they are linked together, and in this regard we have a specifics of the model and one of its most important epistemological functions."

I. Novik [4] distinguishes five main functions of modelling:

(i) illustrative,

(ii) translational,

(iii) substitution-heuristic,

(iv) approximation,

(v) extrapolation-prognostic.

The scientist notes that these functions are not alternative, they coexist in models, but their presence in each model is optional; moreover, some other functions may be found in certain models.

A number of scientists (B. A. Glinskii, E. P. Nikitin and others, see [2]) distinguish such modelling functions as:

(i) interpretive (explanations based on logic and formalized language of presentation),

(ii) explanatory (shows that this object is a subject to a particular law or set of laws),

(iii) predictive (operation, the task of which is to obtain data on objects and processes or non-existent, or existing, but not known), and

(iv) criterion (verifying the truth of knowledge about the original).

We can undoubtedly state that mathematical modelling as a method of reality cognition is used not only because it can replace an experiment. It has a great independent significance because

(i) with the help of mathematical modelling, it is possible to develop different mathematical models on the basis of the same data, and these models would interpret the studied phenomenon differently;

(ii) in the process of model building, one can make various additions to the hypothesis under study and get simplification;

(iii) in the case of complex mathematical models, one can use computers;

(iv) it is possible to conduct model experiments.

In the latter half of the 20th century there appeared a great value of works investigating mathematical modelling in epistemological and didactic aspects. A fundamental research in this field was conducted by the following scientists: N. H. Alekseev, B. M. Kedrov, V. A. Shtoff, A. I. Uyemov, L. M. Fridman, L. R. Kalapusha, etc.

It was established that an acquaintance of pupils (students) with the method of modelling, including mathematical, helps them to understand logic of scientific knowledge and to learn its methodology (see the book by L. R. Kalapusha [3]).

Modelling in the educational process acts simultaneously as:

(i) a method of scientific knowledge,

(ii) a part of the content of educational material, and

(iii) an effective mean of its study.

The development of students' ideas about the role of mathematical modelling in scientific knowledge and practice, the development of their ability to build mathematical models of life phenomena, are an important task of modern school.

In particular, we need to pay special attention to developing students' skills to reformulate an applied problem into the language of mathematics and to create adequate mathematical models. It is important, that students concentrate correctly, highlighting the essential and non-essential properties of objects; abstract from insignificant properties; correctly interpret the relationships between objects of the problem. The teachers should form a particular attitude of students to the acquired knowledge through the disclosure of the essence of mathematical modelling.

Despite the widespread use of the method of mathematical modelling, the development of relevant students' skills is not systematic and is mostly done during mathematics lessons. This significantly reduces didactic effectiveness of the use of this method in learning process, in particular, in increasing mathematical literacy of students. To overcome this limitation, in our opinion, it is possible to use interdisciplinary connections more effectively.

For example, when generalizing the basic properties of directly proportional and linear functions in mathematics lessons, it is advisable to use the knowledge of students that were already obtained in physics lessons when studying thermal phenomena. We offer them, as a homework, to build graphs of the amount of heat obtained during the combustion of this type of fuel, its mass, for example, for dry firewood, anthracite, gasoline. We analyze these graphs in math lessons. It is possible not only to repeat the basic properties of direct proportionality, but also to form in students the concept of function as a mathematical model.

The development of correct ideas of students about the nature of the reflection of mathematical phenomena and processes of the real world, the role of mathematical modelling in scientific knowledge and in practice, is of great importance for the formation of their mathematical literacy.

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Towards a database of isogeny graphs

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Abstract: In number theory, it is often productive to gather arithmetic data in order to conjecture new results and discover unknown behaviour. The most notable modern case of this is the LMFDB, which contains lots of information on arithmetically interesting objects such as fields, algebraic curves and modular functions. Despite such a large collection of data, the isogeny-based cryptography community still lacks a range of examples of supersingular isogeny graphs. This work is a first attempt at generating these examples for genus 1, and it involves exploring elliptic curve isogenies and computing some of their graph invariants.

Resumen: En teoría de números, suele ser productivo recabar datos aritméticos para poder conjeturar nuevos resultados y descubrir comportamientos desconocidos. El caso moderno más notable es el de la base de datos LMFDB, que contiene información sobre objetos de interés aritmético tales como cuerpos, curvas algebraicas o funciones modulares. A pesar de existir tal colección de datos, la comunidad de criptografía basada en isogenias todavía carece de un repositorio de ejemplos de grafos de isogenias supersingulares. Este trabajo es un primer intento de generar estos ejemplos para género 1, e involucra explorar isogenias de curvas elípticas y computar algunos invariantes de dichos grafos.

Keywords: elliptic curves, isogeny graphs, distributed computing.

MSC2010: 11-04, 14K02.

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1. Introduction

This paper is the starting point of a project to systematically produce data on isogeny graphs. Supersingular isogeny graphs of elliptic curves are used in proposed postquantum protocols, and having examples of them can help to further experiment with them. Our end goal is to investigate higher-dimensional abelian varieties over finite fields, and this is the first step to produce a framework for the task.

Not only we produce adjacency matrices of isogeny graphs, but we also want to list their graph invariants. Some quantitative work has already been done in [1], and we follow them to compute several of our metrics. We have used SageMath 9 over Python 3 [6] for our purposes.

2. Elliptic curves and isogeny graphs

An *elliptic curve* over a finite field $\mathbb{F}_q$ of characteristic $p \neq 2, 3$ is given by an equation

$$E\colon\ y^2 = x^3 + Ax + B,\ \ A, B \in \mathbb{F}_q$$

satisfying $4A^3 + 27B^2 \neq 0$. Such a curve has a group structure, displaying the simplest examples of abelian varieties. An *isogeny* between two elliptic curves is an algebraic map $E \to E'$ which is compatible with the group structures. Isogenies are characterised by the properties of being surjective and having finite kernel. The *degree* of a separable isogeny is the size of its kernel. If $\deg(\phi) = \ell$, we say ϕ is an ℓ-isogeny. An isomorphism is the case of an isogeny with trivial kernel.

Two elliptic curves are isomorphic over $\bar{\mathbb{F}}_q$ if and only if they have the same j*-invariant*, defined as

$$j(E) = 1728 \frac{4A^3}{4A^3 + 27B^2}.$$

For each $j \in \bar{\mathbb{F}}_q$, there is an elliptic curve with that invariant, which we denote by E_j.

An elliptic curve is said to be *supersingular* if it has no p-torsion points. Hence, the *supersingular isogeny graph* $\Gamma_1(\ell; p)$, with $\ell \neq p$ two different primes, is defined as follows:

(i) Its vertices are the j-invariants of supersingular elliptic curves over $\bar{\mathbb{F}}_p$. These j-invariants are all in $\mathbb{F}_{p^2}$, and so they can be represented by two integers modulo p.

(ii) Given two vertices j and j', ϕ is an edge from j to j' if there is an ℓ-isogeny $\phi\colon\ E_j \to E_{j'}$. Multiple edges are allowed, although they are fairly rare in the genus 1 case.

For each ℓ-isogeny $\phi\colon\ E_j \to E_{j'}$ there is always a dual ℓ-isogeny $\hat{\phi}\colon\ E_{j'} \to E_j$, so we can regard $\Gamma_1(\ell; p)$ as being an undirected graph.

We can find a supersingular j-invariant in $\mathbb{F}_{p^2}$ in $\tilde{O}((\log p)^3)$ using Bröker's algorithm [3]. The graph $\Gamma_1(\ell; p)$ is always connected, so we can easily list all of its vertices with an exploration algorithm.

There are at least two known methods to compute an isogeny [2]. However, to compute the number of edges in $\Gamma_1(\ell; p)$ from j to j' it is sufficient to factor a modular polynomial. This allows us to work without equations for the curves E_j and $E_{j'}$, which would potentially require working over a larger finite field. Fix a prime p, and let N be any non-zero integer coprime with p. The Nth *modular polynomial* $\Phi_N(X, Y)$ is the equation that defines the planar model of the modular curve $X_0(N)$ classifying elliptic curves over $\mathbb{C}$ with a cyclic group of order N. The function field of this curve is $\mathbb{C}(j(\tau), j(N\tau))$, and so two curves E_j and $E_{j'}$ over $\mathbb{C}$ have an N-isogeny between them whenever $\Phi_N(j, j') = 0$. In fact, one can prove that $\Phi_N(X, Y) \in \mathbb{Z}[X, Y]$. Reducing the polynomial modulo p, we get the main result for our purposes: E and E' over $\bar{\mathbb{F}}_q$ are N-isogenous via a cyclic isogenies if, and only if, $\Phi_N(j(E), j(E')) = 0$.

Therefore, given $j = j(E)$, the N-neighbors of E are given by the roots of $\Phi_N(j, Y) \in \mathbb{F}_{p^2}[Y]$. In the case $N = \ell$ prime, this polynomial has at most $\ell + 1$ distinct roots (in general, the number of roots is given by Dedekind's ψ function).

2.1. Graph properties

Once we have the list of nodes of $\Gamma_1(\ell; p)$ and its adjacency matrix, we want to compute several of its properties, which we now explain.

(i) *Diameter and largest eigenvalues.* The graph $\Gamma_1(\ell; p)$ is $(\ell + 1)$-regular and almost-undirected (i.e., it is undirected at every vertex except for a bounded number of them). Therefore, its diameter can be controlled by the eigenvalues of the adjacency matrix. More precisely, we know that the eigenvalues can be ordered as $\ell + 1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n > -(\ell + 1)$.

If we fix the prime ℓ and let $\lambda_\star(p) = \max\{|\lambda_2|, |\lambda_n|\}$, the diameter of the family $\{\Gamma_1(\ell; p)\}_p$ grows like $O(\log p)$ as long as there exists some fixed constant $\Lambda_\ell < \ell + 1$ with $\lambda_\star(p) \leq \Lambda_\ell$ for all p. This is indeed the case for supersingular graphs of elliptic curves (they have the Ramanujan property), but the result for higher-dimensional varieties is still conjectural [5].

(ii) *Size of the spine.* The spine of $\Gamma_1(\ell; p)$ is the induced subgraph of vertices that are defined over $\mathbb{F}_p$. Knowing the structure of the spine is useful since it tends to be a very small subgraph where finding paths is simpler. If we are able to solve that particular problem, then finding a path between any two vertices reduces to finding paths to the spine.

(iii) *Number of isogenous conjugate pairs.* Each j-invariant outside of the spine, $j \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, has a Frobenius conjugate j^p. This corresponds to the Frobenius (inseparable) isogeny $E \to E^{(p)}$, given by $(x, y) \mapsto (x^p, y^p)$. An ℓ-isogenous conjugate pair is a pair of vertices (j, j^p) connected by an ℓ-degree isogeny.

3. Distributed computation

Due to the great computational cost of calculating the data, the task has to be performed in parallel. Parallel programming on a single computer has been useful for graphs with relatively small p, where they could be calculated using a typical 4-core personal computer in a reasonable time. In order to calculate the graphs with $p \approx 30\,000$, we needed to scale the computational power beyond a single computer and use distributed computing. Therefore, the calculations have been carried out in a parallel and distributed way, having more than 500 threads of parallel execution in a distributed way[1].

3.1. Graph computation procedure

The computation of each graph and its properties has been divided in five sequential stages (i.e., a stage can only be run after the previous ones). We now describe them.

The first step to start working with graphs is *computing the nodes* of the ℓ-isogeny graph, so for a given p we want to compute a list with all the nodes from $\Gamma_1(\ell; p)$. To discover the nodes we use a slight modification of the breadth-first search algorithm (BFS) starting from an initial node. Given a node j, we obtain all its neighbours by factoring $\Phi_\ell(j, Y)$. Because the node list depends exclusively on p, and for efficiency reasons, we explore the graph $\Gamma_1(\ell; p)$ with $\ell = 2$. The BFS algorithm is not well suited to be run in parallel, so parallelization has been achieved by just exploring multiple graphs simultaneously.

Checks over the node list. On the one hand, we know we must have $\left[\frac{p-1}{12}\right] + \varepsilon$ nodes in $\Gamma_1(\ell; p)$ (with $\varepsilon = 0, 1, 1, 2$ according to $p \equiv 1, 5, 7, 11 \mod 12$). On the other hand, we can test any given node for supersingularity with SageMath's function `E.is_supersingular`. Using these two facts, we can guarantee that the computed node list is complete and correct.

Computation of the adjacency matrix of $\Gamma_1(\ell; p)$. Given p and ℓ, we compute all the neighbours of each node in $\Gamma_1(\ell; p)$. This task is highly parallelizable, since it is enough to split the list of nodes into batches of similar size and assign one batch to every thread of execution. Once we have the list of neighbours for every node it can be easily converted to the adjacency matrix.

[1] Code can be found at `https://github.com/gfinol/IsogenyGraph`.

Checks on adjacency matrix. We check that the matrix is square, has correct dimensions and all nodes have out-degree $\ell + 1$. It is important to notice that these are sanity checks to discard possible errors on the computation rather than checks to prove the correctness of the whole matrix.

Finally, we *compute the graph metrics* using the adjacency matrices. Similarly to our other tasks, we compute them for several graphs simultaneously.

3.2. Lithops

To scale computational power beyond one machine we have used the Lithops[2] framework, which provides an API mimicking the Python multiprocessing library and allows us to execute our code transparently [7] in a distributed serverless environment without having a physical computer cluster nor having to manage one. Thanks to the similar APIs, the code can be executed in parallel on a single machine or distributed using FaaS by just changing the module import from multiprocessing to Lithops. This also allows us to use SageMath in a distributed environment.

4. Results and future work

We have computed all graphs $\Gamma_1(\ell; p)$ for primes $13 \leq p < 30\,000$ and degrees $\ell \in \{2, 3, 5, 7, 11\}$, along with the graph properties specified in Section 2.1. The data has been uploaded to Zenodo [4].

We have built a framework to compute examples for larger p and ℓ in the future, and that will also allow us to explore isogeny graphs of higher-dimensional abelian varieties. This will provide us with data to further confirm existing conjectures [5] on such graphs.

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A multi-objective sampling design for spatial prediction problem

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Abstract: In many fields such as image classification, geostatistical surveys, and air pollution, regarding the limitations of resources, time and technology, spatial sampling plays a crucial role. In spatial sampling, a set of sample locations are chosen such that the spatial prediction at unobserved locations be optimal. While many studies are focused on only one objective function as the predictor variance, and the predictor entropy, in applied problems we are interested in more than one objective. In this study, the optimization problem of spatial sampling with minimum cost is investigated from both the perspective of covariogram estimation and kriging variance. For this purpose, a bi-objective optimization problem of soil sampling has been considered. The first objective function is the mean total error and the second objective function is the cost of the distance travelled by the sampler. The mean total error is the sum of the ordinary kriging variance and uncertainties of the estimated covariogram parameters. The non-dominated sorting genetic algorithm-II and Taguchi method is applied to this problem. The results show the proper performance of this algorithm in multi-objective spatial sampling for spatial predictions.

Resumen: El muestreo espacial juega un papel crucial en muchos campos, como la clasificación de imágenes, los estudios geoestadísticos y la contaminación atmosférica, con respecto a las limitaciones de recursos, tiempo y tecnología. En el muestreo espacial, se elige un conjunto de ubicaciones de la muestra de forma que la predicción espacial en las ubicaciones no observadas sea óptima. Mientras que muchos estudios se centran en una sola función objetivo, como la varianza y la entropía del predictor, en los problemas aplicados interesa más de un objetivo. En este estudio, el problema de optimización del muestreo espacial con coste mínimo se investiga tanto desde la perspectiva de la estimación del covariograma como de la varianza de kriging. Para ello, se ha considerado un problema de optimización bi-objetivo de muestreo de suelos. La primera función objetivo es el error total medio y la segunda función objetivo es el coste de la distancia recorrida por el muestreador. El error total medio es la suma de la varianza ordinaria de kriging y las incertidumbres de los parámetros estimados del covariograma. Se aplica a este problema el algoritmo genético de ordenación no dominante-II y el método de Taguchi. Los resultados muestran el buen funcionamiento de este algoritmo en el muestreo espacial multiobjetivo para las predicciones espaciales.

Keywords: spatial sampling, spatial prediction, multi-objective optimization, uncertainty.

MSC2010: 62H11, 58E17.

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1. Introduction

Optimization of spatial sampling is a critical issue applied in many areas, including geostatistics, air pollution, and epidemiology. In spatial sampling, a set of sample locations is chosen. The spatial prediction at unobserved locations is optimal for the predictor variance and the predictor entropy. While there are several types of research on the single-objective spatial sampling, in many applied problems we are interested in more than one objective in sampling [4]. This paper aims to use Non-dominated Sorting Genetic Algorithm-II (NSGA-II) in spatial sampling subject to the spatial correlation of data. The rest of this paper is organized as follows. Section 2 is devoted to the problem statement. Materials and methods are introduced in Section 3. This section recalls some preliminaries on multi-objective optimization theory and customizes the NSGA-II for the introduced spatial sampling optimization model. Numerical results are provided in Section 4.

2. Problem statement

In this paper, the soil sampling of a field in Silsoe, Bedfordshire, UK, shown in Figure 2, is investigated [5]. It is assumed that the sampler enters and leaves the field at the corner $\{100, 100\}$ and collects 50 sample points across this domain. The first objective function is the spatial mean total error defined by [5, 7]

$$\bar{\sigma}_P^2 = \frac{1}{\mathcal{A}} \int_{s\in\mathcal{A}} (\sigma_{OK}^2(\mathbf{s}) + E[\tau^2(\mathbf{s})])\,\mathrm{d}s, \tag{1}$$

where $\sigma_{OK}^2(\mathbf{s})$ and $E[\tau^2(\mathbf{s})]$ are the squared prediction error (ordinary kriging variance) and the uncertainty in the estimated spatial model (covariogram) parameters, respectively, and

$$\sigma_{OK}^2(s_0) = Var(Z(s_0) - \tilde{Z}(s_0|\boldsymbol{\theta})) = \boldsymbol{C}(s_0 - s_0|\boldsymbol{\theta}) - \boldsymbol{\lambda}^T\boldsymbol{d},$$

$$E[\tau^2(s_0)] = \sum_{i=1}^{q}\sum_{j=1}^{q} Cov(\theta_i, \theta_j)\frac{\partial\boldsymbol{\lambda}^T}{\partial\theta_i}\boldsymbol{C}\frac{\partial\boldsymbol{\lambda}}{\partial\theta_j},$$

where $\boldsymbol{\lambda}^T$, $\frac{\partial\lambda}{\partial\theta_i}$, and $Cov(\theta_i, \theta_j)$ are the vector of kriging weights, the n-vector of partial derivatives of the kriging weights with respect to the ith variance parameter, and the covariance between the ith and jth parameters, respectively. Furthermore,

$$\begin{pmatrix} (\lambda_i)_{i=1}^n \\ \psi \end{pmatrix} = \underbrace{\begin{bmatrix} \left(C(s_i - s_j|\boldsymbol{\theta})\right)_{i,j=1}^n & \vec{\mathbf{1}} \\ \vec{\mathbf{1}} & 0 \end{bmatrix}^{-1}}_{A} \times \underbrace{\begin{pmatrix} (C(s_0 - s_i|\boldsymbol{\theta}))_{i=1}^n \\ 1 \end{pmatrix}}_{d},$$

$$\frac{\partial\boldsymbol{\lambda}}{\partial\theta_i} = \boldsymbol{A}^{-1}\left(\frac{\partial\boldsymbol{d}}{\partial\theta_i} - \frac{\partial\boldsymbol{A}}{\partial\theta_i}\boldsymbol{A}^{-1}\boldsymbol{d}\right),$$

$$Cov(\theta_i, \theta_j) \approx \boldsymbol{F}^{-1}(\theta_i, \theta_j) = \left(\frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{C}^{-1}\frac{\partial\boldsymbol{C}}{\partial\theta_i}\boldsymbol{C}^{-1}\frac{\partial\boldsymbol{C}}{\partial\theta_j}\right]\right)^{-1}.$$

Note that C is the spherical covariogram [1]. Marchant and Lark [5] and Wadoux *et al.* [7] investigated some single-objective optimization problems with the same objective function (1). We consider the cost of the sampling defined by $Cost = d(\mathbf{s}) \times C_m$ as the other objective function where C_m and $d(\mathbf{s})$ are a fixed cost per each meter travelled by the sampler and the total distance walked to visit all points, respectively. We set, $C_m = 1$ and $d(\mathbf{s}) = \sum_{i=1}^{n-1} \|s_{i+1} - s_i\|$, $\mathbf{s} = (s_1, \ldots, s_n)$.

3. Materials and methods

In this section, we provide some preliminaries on multi-objective optimization theory and Spatial NSGA-II Algorithm.

3.1. Multi-objective optimization

A general multi-objective optimization problem can be formulated as follows:

$$\min f(x) = (f_1(x), \dots, f_m(x)), \qquad g_j(x) \leq 0, \quad j \in J_\ell, \quad x \in X \subseteq \mathbb{R}^n, \tag{2}$$

where $f_i, g_j : \mathbb{R}^n \to \mathbb{R}$ ($i \in I_m := \{1, \dots, m\}$, $j \in J_\ell := \{1, \dots, \ell\}$), and $m > 1$. The nonempty set $S := \{x \in X \subseteq \mathbb{R}^n : g_j(x) \leq 0, j \in J_\ell\}$ is called the feasible set.

Definition 1 ([3]). A feasible solution $\hat{x} \in S$ is called efficient or Pareto optimal solution of Problem (2), if there is no $x \in S$ such that $f_k(x) \leq f_k(\hat{x})$ for each $k \in I_m$ and $f_i(x) < f_i(\hat{x})$ for some $i \in I_m$. The set of all Pareto optimal solutions of Problem (2) is called Pareto frontier for this problem. ◂

Now, the bi-objective spatial optimization problem can be represented as

$$\min_{\mathbf{s} \in \mathcal{A}} f(\mathbf{s}) = (\bar{\sigma}_P^2, d(\mathbf{s})),$$

$$\text{s.t. } \bar{\sigma}_P^2 - 1 \leq 0, \quad d(\mathbf{s}) - 5000 \leq 0, \|s_i - sj\| \geq 20 \quad \forall s_i, s_j \in \mathcal{A}, i \neq j,$$

where $\mathcal{A}$ is the interested area of study.

3.2. Spatial NSGA-II Algorithm

The NSGA-II algorithm is suggested by [2]. In the Figure 1, the diagram of the customized version of the NSGA-II for spatial sampling is shown. Note that the parameters of the algorithm are set by the Taguchi method [6].

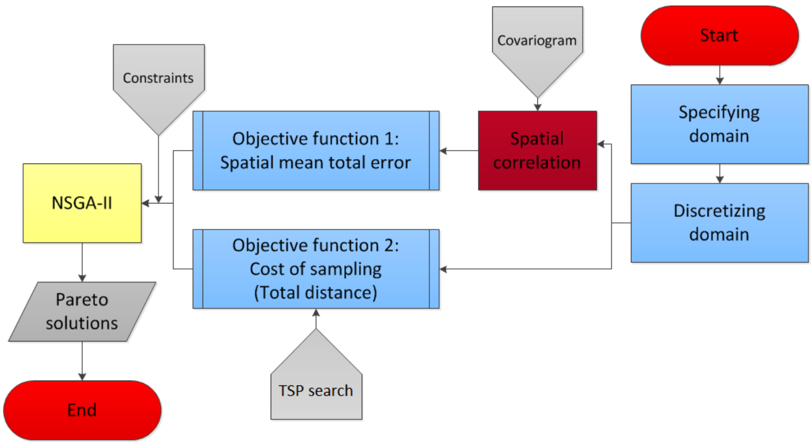

Figure 1: Diagram of the customized NSGA-II for the bi-objective spatial sampling optimization problem.

4. Numerical results

Note that, in Figure 2, each point on the Pareto frontier corresponds to a sample of 50 points in the field. Results show the suitable dispersion and coverage on the Pareto frontier obtained by the customized NSGA-II. It is clear from this frontier that there is a conflict between these two objective functions. For instance, sample 2 shows that minimizing the mean total error requires more cost.

As a sensitivity analysis, we change the parameters of the spherical covariogram. These values are chosen from the literature. In Table 1, in case 3, the mean of mean total errors is smallest, and in case 5, the mean of costs is smallest. Now, the choice of these parameters is also dependent on the decision-makers and their priorities. If they want less total mean error, they must choose parameters in case 3, and for less cost, they choose parameters in case 5.

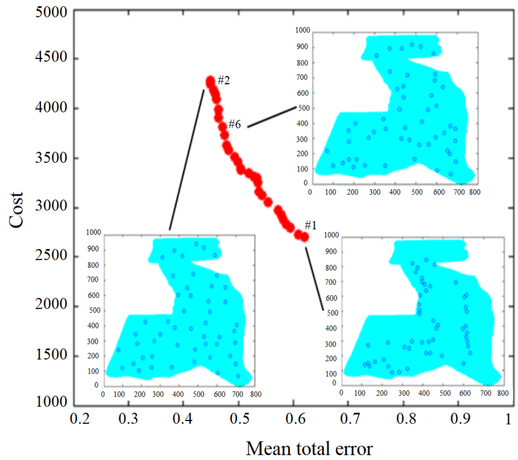

Figure 2: The region of interest and three sample designs from the estimated Pareto frontier obtained by NSGA-II with the parameters shown in the Table 1, case 1.

Table 1: Spherical covariogram parameters c_0, c_1, and a, and the mean of the objective functions values for each optimized sampling scheme.

Case	c_0	c_1	a	mean of $\bar{\sigma}_P^2$	mean of *Cost*
1	0.127	1.0	240.0	0.5173	3438
2	0.127	1.5	240.0	0.6610	3418
3	0.127	0.5	240.0	0.3395	3512
4	0.127	1.0	120.0	0.7675	3490
5	0.127	1.0	90.0	0.8941	3315

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On the growth and fixed points of solutions of linear differential equations with entire coefficients

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Abstract: In this paper, we use a generalized concept of order called the φ-order to investigate the growth and the oscillation of fixed points of solutions of higher order complex linear differential equations with entire coefficients. We describe the relationship between the solutions and the entire coefficients in terms of φ-order and φ-convergence exponent. We extend and improve some earlier results due to Cao, Xu, and Chen, Chyzhykov and Semochko, Kara and Belaïdi.

Resumen: En este trabajo, utilizamos un concepto generalizado de orden llamado φ-orden para investigar el crecimiento y la oscilación de los puntos fijos de las soluciones de las ecuaciones diferenciales lineales complejas de orden superior con coeficientes enteros. Describimos la relación entre las soluciones y los coeficientes enteros en términos del φ-orden y del φ-exponente de convergencia. Ampliamos y mejoramos algunos resultados anteriores de Cao, Xu y Chen, Chyzhykov y Semochko, Kara y Belaïdi.

Keywords: entire function, meromorphic function, φ-order, φ-exponent of convergence, linear differential equation.

MSC2010: 30D35, 34M10.

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1. Introduction

Throughout this paper, we use the standard notations of Nevanlinna value distribution theory such as $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ (see [5]). The term "meromorphic function" will mean meromorphic in the whole complex plane $\mathbb{C}$. For $k \geq 2$, we consider the following linear differential equations:

(1) $$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_0(z)f = 0,$$

(2) $$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_0(z)f = F(z).$$

It is well know that, if the coefficients F and $A_0(z), \ldots, A_{k-1}$ are entire functions, all solutions of (1) and (2) are entire. Cao, Xu, and Chen [1] have investigated the growth of meromorphic solutions of equations (1) and (2) when the coefficients are meromorphic functions of finite iterated order. Chyzhykov and Semochko [2] used a more general concept called the φ-order which can cover an arbitrary growth of fast growing functions. They obtained the precise estimates for φ-order of entire solutions of (1) when the coefficient A_0 strictly dominates the growth of coefficients. Later, the authors [3] investigated equations (1) and (2) when the coefficients are meromorphic functions with finite φ-order.

Definition 1 ([2]). Let φ be an increasing unbounded function on $(0,+\infty)$. The φ-orders of a meromorphic function f are defined by

$$\rho_\varphi^0(f) = \limsup_{r\to+\infty} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r}, \qquad \rho_\varphi^1(f) = \limsup_{r\to+\infty} \frac{\varphi\left(T(r,f)\right)}{\log r}.$$

Definition 2 ([3]). Let φ be an increasing unbounded function on $(0,+\infty)$. We define the φ-convergence exponents of the sequence of zeros of a meromorphic function f by

$$\lambda_\varphi^0(f) = \limsup_{r\to+\infty} \frac{\varphi\left(e^{N(r,1/f)}\right)}{\log r}, \qquad \lambda_\varphi^1(f) = \limsup_{r\to+\infty} \frac{\varphi\left(N\left(r,\frac{1}{f}\right)\right)}{\log r}.$$

Similarly, if we replace $N(r,1/f)$ by $\overline{N}(r,1/f)$, we obtain $\overline{\lambda}_\varphi^0(f)$ and $\overline{\lambda}_\varphi^1(f)$, which denote the φ-convergence exponents of the sequence of distinct zeros of f.

Let Φ denotes the class of positive unbounded increasing functions on $(0,+\infty)$ such that $\varphi(e^t)$ grows slowly, i.e., for all $c>0$ we have $\lim_{t\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1$. For instance, $\log\log(\cdot) \in \Phi$, while $\log(\cdot) \notin \Phi$.

Proposition 3 ([2, 4]). *Let $\varphi \in \Phi$ and let f_1, f_2 be two meromorphic functions. Then, for $j = 0,1$ we have*

$$\max\left\{\rho_\varphi^j(f_1+f_2), \rho_\varphi^j(f_1\,f_2)\right\} \leq \max\left\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\right\}.$$

Moreover, if $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$, then $\rho_\varphi^j(f_1+f_2) = \rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2)$.

Proposition 4 ([3, 4]). *Let $\varphi \in \Phi$ and let f be a meromorphic function. Then,*

(i) *$\rho_\varphi^j(f') = \rho_\varphi^j(f)$ for $j = 0,1$,*

(ii) *if $\rho_\varphi^0(f) < +\infty$, then $\rho_\varphi^1(f) = 0$.*

Theorem 5 ([2]). *Let $\varphi \in \Phi$ and $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying*

$$\max\{\rho_\varphi^0(A_j), j = 1,\ldots,k-1\} < \rho_\varphi^0(A_0).$$

Then, every solution $f \not\equiv 0$ of (1) satisfies $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$.

Theorem 6 ([3]). *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If f is a meromorphic solution of (2) satisfying for $i = 0,1$*

$$\max\left\{\rho_\varphi^i(F), \rho_\varphi^i(A_j) \,:\, j = 0,1,\ldots,k-1\right\} < \rho_\varphi^i(f),$$

then $\overline{\lambda}_\varphi^i(f) = \lambda_\varphi^i(f) = \rho_\varphi^i(f)$.

2. Main results

This paper is concerned with the properties of growth and oscillation of fixed points of entire solutions of equations (1) and (2) involving the concept of φ-order. We list here our main results.

Theorem 7. *Under the hypothesis of Theorem 5, if $A_1(z) + zA_0(z) \not\equiv 0$, then every solution $f \not\equiv 0$ of (1) satisfies $\overline{\lambda}^1_\varphi(f - z) = \rho^1_\varphi(f) = \rho^0_\varphi(A_0)$.*

Proof. Let $f \not\equiv 0$ be an entire solution of (1). Set $g = f - z$. Clearly,

$$\overline{\lambda}^1_\varphi(g) = \overline{\lambda}^1_\varphi(f - z) \quad \text{and} \quad \rho^1_\varphi(g) = \rho^1_\varphi(f - z) = \rho^1_\varphi(f). \tag{3}$$

From (1), we get

$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \ldots + A_1(z)g' + A_0(z)g = -[A_1(z) + zA_0(z)]. \tag{4}$$

By Theorem 5, we have $\rho^1_\varphi(f) = \rho^0_\varphi(A_0)$. Since $A_1(z) + zA_0(z) \not\equiv 0$ is also an entire function, it follows from Proposition 4 that

$$\max\left\{\rho^1_\varphi(-A_1 - zA_0), \rho^1_\varphi(A_j) : j = 0, 1, \ldots, k-1\right\} < \rho^1_\varphi(f) = \rho^1_\varphi(g).$$

Thus, by applying Theorem 6 to (4), we obtain $\overline{\lambda}^1_\varphi(g) = \rho^1_\varphi(g)$. Therefore, $\overline{\lambda}^1_\varphi(f - z) = \rho^1_\varphi(f) = \rho^0_\varphi(A_0)$. ■

By using analogous proofs of Theorem 1 and Theorem 4 in Kara and Belaïdi [3], we can easily obtain the following two results.

Theorem 8. *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying*

$$\max\left\{\rho^0_\varphi(A_j) : j = 0, 1, \ldots, k-1\ (j \neq s)\right\} < \rho^0_\varphi(A_s) < +\infty.$$

Then, every transcendental solution f of (1) satisfies $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s) \le \rho^0_\varphi(f)$. Furthermore, there exists at least one solution satisfying $\rho^1_\varphi(f) = \rho^0_\varphi(A_s)$.

Theorem 9. *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be entire functions satisfying*

$$\max\left\{\rho^1_\varphi(F), \rho^0_\varphi(A_j) : j = 1, \ldots, k-1\right\} < \rho^0_\varphi(A_0) < +\infty.$$

Then, every solution f of (2) satisfies $\overline{\lambda}^1_\varphi(f) = \lambda^1_\varphi(f) = \rho^1_\varphi(f) = \rho^0_\varphi(A_0)$ with at most one exceptional solution satisfying $\rho^1_\varphi(f) < \rho^0_\varphi(A_0)$.

By replacing the dominant coefficient A_0 in Theorem 7 by an arbitrary coefficient A_s ($s \in \{0, 1, \ldots, k-1\}$), we obtain the following result.

Theorem 10. *Under the hypothesis of Theorem 8, if $A_1(z) + zA_0(z) \not\equiv 0$, then every transcendental solution f of (1) such that $\rho^0_\varphi(f) > \rho^0_\varphi(A_s)$ satisfies $\overline{\lambda}^0_\varphi(f - z) = \rho^0_\varphi(f)$. Moreover, there exists at least one solution f_1 satisfying $\overline{\lambda}^1_\varphi(f_1 - z) = \rho^1_\varphi(f_1) = \rho^0_\varphi(A_s)$.*

Proof. By Theorem 8, we have $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s)$. Assume that $\rho^0_\varphi(A_s) < \rho^0_\varphi(f)$ and, since $A_1(z) + zA_0(z) \not\equiv 0$, then

$$\max\left\{\rho^0_\varphi(-A_1 - zA_0), \rho^0_\varphi(A_j) : j = 0, 1, \ldots, k-1\right\} = \rho^0_\varphi(A_s) < \rho^0_\varphi(f).$$

Thus, by using the fact that $\rho^0_\varphi(f) = \rho^0_\varphi(g) = \rho^0_\varphi(f - z)$, and applying Theorem 6 to (4), we obtain $\overline{\lambda}^0_\varphi(g) = \rho^0_\varphi(g)$, i.e., $\overline{\lambda}^0_\varphi(f - z) = \rho^0_\varphi(f)$. Again, by Theorem 8, there exists a solution f_1 of (1) such that $\rho^1_\varphi(f_1) = \rho^0_\varphi(A_s)$. We deduce from Proposition 4 that

$$\max\left\{\rho^1_\varphi(-A_1 - zA_0), \rho^1_\varphi(A_j) : j = 0, 1, \ldots, k-1\right\} < \rho^1_\varphi(f_1) = \rho^1_\varphi(f_1 - z).$$

Hence, it follows from Theorem 6 and (4) that $\overline{\lambda}^1_\varphi(f_1 - z) = \rho^1_\varphi(f_1 - z)$. Therefore, $\overline{\lambda}^1_\varphi(f_1 - z) = \rho^1_\varphi(f_1) = \rho^0_\varphi(A_s)$. ■

Theorem 11. *Under the hypothesis of Theorem 9, if $F(z) - [A_1(z) + zA_0(z)] \not\equiv 0$, then every solution f of (2) such that $\rho^1_\varphi(f) = \overline{\lambda}^1_\varphi(f)$ satisfies $\overline{\lambda}^1_\varphi(f - z) = \rho^1_\varphi(f)$.*

Proof. Let f be a solution of (2) such that $\rho^1_\varphi(f) = \overline{\lambda}^1_\varphi(f)$. Set $g = f - z$. Equation (2) becomes

(5)
$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g = F(z) - [A_1(z) + zA_0(z)]\,.$$

It follows from Theorem 9 that every solution f of (2) satisfies $\rho^1_\varphi(f) = \rho^1_\varphi(A_0) = \overline{\lambda}^1_\varphi(f)$ with at most one exceptional solution satisfying $\rho^1_\varphi(f) < \rho^0_\varphi(A_0)$. Hence, by Proposition 4 and since $F(z) - [A_1(z) + zA_0(z)] \not\equiv 0$, we obtain

$$\max\left\{\rho^1_\varphi(F - A_1 - zA_0), \rho^1_\varphi(A_j) \,:\, j = 0, 1, \dots, k-1\right\} < \rho^1_\varphi(f) = \rho^1_\varphi(g).$$

Thus, by applying Theorem 6 to (5), we obtain $\overline{\lambda}^1_\varphi(g) = \rho^1_\varphi(g)$. Therefore, $\overline{\lambda}^1_\varphi(f - z) = \rho^1_\varphi(f)$. ∎

3. Future aspects

This paper rises many interesting questions, such as the following:

Question 1: Can we obtain similar results if the coefficients of equations (1) and (2) are meromorphic functions?

Question 2: What can be said if we consider equations (1) and (2) with analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} \,:\, |z| < 1\}$?

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On linear generic inhomogeneous boundary-value problems for differential systems in Sobolev spaces

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Abstract: For the systems of ordinary differential equations of an arbitrary order on a compact interval, we study a character of solvability of the most general linear boundary-value problems in the Sobolev spaces W_p^n, with $n \in \mathbb{N}$ and $1 \leq p \leq \infty$. We find the indices of these Fredholm problems and obtain a criterion of their well-posedness. Each of these boundary-value problems relates to a certain rectangular numerical characteristic matrix with kernel and cokernel of the same dimension as the kernel and cokernel of the boundary-value problem. The condition for the sequence of characteristic matrices to converge is found. We obtain a constructive criterion under which the solutions to these problems depend continuously on the small parameter ε at $\varepsilon = 0$, and find the degree of convergence of the solutions. Also applications of these results to multipoint boundary-value problems are obtained.

Resumen: Para los sistemas de ecuaciones diferenciales ordinarias de un orden arbitrario en un intervalo compacto, estudiamos un carácter de solubilidad de los problemas lineales de valor límite más generales en los espacios de Sobolev W_p^n, con $n \in \mathbb{N}$ y $1 \leq p \leq \infty$. Encontramos los índices de estos problemas de Fredholm y obtenemos un criterio de su buena composición. Cada uno de estos problemas de valor límite se relaciona con una cierta matriz característica numérica rectangular con núcleo y cokernel de la misma dimensión que el núcleo y el cokernel del problema de valor límite. Se encuentra la condición para que la secuencia de matrices características converja. Obtenemos un criterio constructivo bajo el cual las soluciones de estos problemas dependen continuamente del pequeño parámetro ε en $\varepsilon = 0$, y encontramos el grado de convergencia de las soluciones. También se obtienen aplicaciones de estos resultados a problemas de valores límite multipunto.

Keywords: differential system, generic boundary-value problem, Sobolev space, operator index, continuity in a parameter.

MSC2010: 34B08.

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1. Generic boundary-value problem

Let a finite interval $[a,b] \subset \mathbb{R}$ and parameters $\{m,n,r,l\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given. By $W_p^{n+r} = W_p^{n+r}([a,b];\mathbb{C}) := \{y \in C^{n+r-1}[a,b] : y^{(n+r-1)} \in AC[a,b], y^{(n+r)} \in L_p[a,b]\}$ we denote a complex Sobolev space and set $W_p^0 := L_p$. This space is a Banach one with respect to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \|y^{(k)}\|_p + \|y^{(n+r)}\|_p,$$

where $\|\cdot\|_p$ is the norm in the space $L_p([a,b];\mathbb{C})$. Similarly, by $(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m)$ and $(W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m})$ we denote Sobolev spaces of vector-valued functions and matrix-valued functions, respectively, whose elements belong to the function space W_p^{n+r}.

We consider the following linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t)y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c, \tag{2}$$

where the matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, the vector-valued function $f(\cdot) \in (W_p^n)^m$, the vector $c \in \mathbb{C}^l$, and the linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^l \tag{3}$$

are arbitrarily chosen, and the vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

We represent vectors and vector-valued functions in the form of columns. A solution to the boundary-value problem (1), (2) is understood as a vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ satisfying equation (1) almost everywhere on (a,b) (everywhere for $n \geq 2$) and equality (2) specifying l scalar boundary conditions. The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the boundary condition (2) with continuous operator (3) is the most general condition for this equation.

It includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and many-point problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives (generally fractional) $y^{(k)}(\cdot)$ with $0 < k \leq n+r$.

For $1 \leq p < \infty$, every operator B in (3) admits a unique analytic representation

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t)y^{(n+r)}(t)\,\mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m,$$

where the matrices $\alpha_k \in \mathbb{C}^{rm\times m}$ and the matrix-valued function $\Phi(\cdot) \in L_{p'}\big([a,b];\mathbb{C}^{rm\times m}\big)$, $1/p + 1/p' = 1$.

For $p = \infty$ this formula also defines an operator $B\colon (W_\infty^{n+r})^m \to \mathbb{C}^{rm}$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

With the generic inhomogeneous boundary-value problem (1), (2), we associate a linear continuous operator in pair of Banach spaces

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \tag{4}$$

Recall that a linear continuous operator $T\colon X \to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If operator T is Fredholm, then its range $T(X)$ is closed in Y and the index

$$\operatorname{ind} T := \dim\ker T - \dim(Y/T(X)) \in \mathbb{Z}$$

is finite.

Theorem 1. *The linear operator (4) is a bounded Fredholm operator with index $mr - l$.*

Theorem 1 allows the next specification.

For each number $k \in \{1, \dots, r\}$, we consider the family of matrix Cauchy problems:

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a, b),$$

with the initial conditions

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \qquad j \in \{1, \dots, r\}.$$

Here, $Y_k(\cdot)$ is an unknown $m \times m$ matrix-valued function, and $\delta_{k,j}$ is the Kronecker symbol.

By $[BY_k]$ we denote the numerical $m \times l$ matrix, in which the j-th column is the result of the action of the operator B on the j-th column of the matrix-valued function $Y_k(\cdot)$.

Definition 2. A block rectangular numerical matrix $M(L, B) := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{mr \times l}$ is characteristic to the inhomogeneous boundary-value problem (1), (2). It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m \times l}$. ◂

Here mr is the number of scalar differential equations of the system (1), and l is the number of scalar boundary conditions.

Theorem 3. *The dimensions of the kernel and cokernel of the operator (4) are equal to the dimensions of the kernel and cokernel of the characteristic matrix $M(L, B)$, respectively.*

Theorem 3 implies a criterion for the invertibility of the operator (4).

Corollary 4. *The operator (L, B) is invertible if and only if $l = mr$ and the matrix $M(L, B)$ is nondegenerate.*

2. Generic boundary-value problem with a parameter

Let us consider, parameterized by number $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, the linear boundary-value problem

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon) y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon), \quad t \in (a,b), \tag{5}$$

$$B(\varepsilon)y(\cdot;\varepsilon) = c(\varepsilon), \tag{6}$$

where for every fixed ε the matrix-valued functions $A_{r-j}(\cdot;\varepsilon) \in (W_p^n)^{m \times m}$, the vector-valued function $f(\cdot;\varepsilon) \in (W_p^n)^m$, the vector $c(\varepsilon) \in \mathbb{C}^{rm}$, $B(\varepsilon)$ is the linear continuous operator $B(\varepsilon)\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$, and the solution (the unknown vector-valued function) $y(\cdot;\varepsilon) \in (W_p^{n+r})^m$.

It follows from Theorem 1 that the boundary-value problem (5), (6) is a Fredholm one with index zero.

Definition 5. A solution to the boundary-value problem (5), (6) depends continuously on the parameter ε at $\varepsilon = 0$ if the following two conditions are satisfied:

- there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0, \varepsilon_1)$ and arbitrary chosen right-hand sides $f(\cdot;\varepsilon) \in (W_p^n)^m$ and $c(\varepsilon) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot;\varepsilon)$ that belongs to the space $(W_p^{n+r})^m$;
- the convergence of the right-hand sides $f(\cdot;\varepsilon) \to f(\cdot;0)$ in $(W_p^n)^m$ and $c(\varepsilon) \to c(0)$ in $\mathbb{C}^{rm}$ as $\varepsilon \to 0+$ implies the convergence of the solutions $y(\cdot;\varepsilon) \to y(\cdot;0)$ in $(W_p^{n+r})^m$. ◂

Consider the following conditions as $\varepsilon \to 0+$:

(i) the limiting homogeneous boundary-value problem

$$L(0)y(t,0) = 0, \quad t \in (a,b), \quad B(0)y(\cdot,0) = 0$$

has only the trivial solution;

(ii) $A_{r-j}(\cdot;\varepsilon) \to A_{r-j}(\cdot;0)$ in the space $(W_p^n)^{m\times m}$ for each number $j \in \{1,\dots,r\}$;

(iii) $B(\varepsilon)y \to B(0)y$ in the space $\mathbb{C}^{rm}$ for every $y \in (W_p^{n+r})^m$.

Theorem 6. *A solution to the boundary-value problem (5), (6) depends continuously on the parameter ε at $\varepsilon = 0$ if and only if this problem satisfies condition (i) and the conditions (ii) and (iii).*

We supplement our result with a two-sided estimate of the error $\|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p}$ of the solution $y(\cdot;\varepsilon)$ via its discrepancy

$$\tilde{d}_{n,p}(\varepsilon) := \|L(\varepsilon)y(\cdot;0) - f(\cdot;\varepsilon)\|_{n,p} + \|B(\varepsilon)y(\cdot;0) - c(\varepsilon)\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\cdot;0)$ as an approximate solution to the problem (5), (6).

Theorem 7. *Suppose that the boundary-value problem (5), (6) satisfies conditions (i), (ii) and (iii). Then, there exist positive numbers $\varepsilon_2 < \varepsilon_1$ and γ_1, γ_2 such that, for any $\varepsilon \in (0,\varepsilon_2)$, the following two-sided estimate is true:*

$$\gamma_1\, \tilde{d}_{n,p}(\varepsilon) \le \|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p} \le \gamma_2\, \tilde{d}_{n,p}(\varepsilon),$$

where the quantities ε_2, γ_1, and γ_2 do not depend on $y(\cdot;\varepsilon)$ and $y(\cdot;0)$.

Thus, the error and discrepancy of the solution $y(\cdot;\varepsilon)$ to the boundary-value problem (5), (6) are of the same degree of smallness.

The results are published in the articles by Atlasiuk and Mikhailets [1, 2].

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What is Sparse Domination and why is it so plentiful?

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Abstract: Many operators in analysis are non-local, in the sense that a perturbation of the input near a point modifies the output everywhere; consider for example the operator that maps the initial data to the corresponding solution of the heat equation.

Sparse Domination consists in controlling such operators by a sum of positive, local averages. This allows to derive plenty of estimates, which are often optimal.

In this work we will introduce this concept, and we will discuss the case of operators that are beyond Calderón–Zygmund theory.

Resumen: Muchos operadores en análisis son no locales, en el sentido de que una perturbación de la entrada cerca de un punto modifica la salida en todas partes; consideremos, por ejemplo, el operador que mapea los datos iniciales a la solución correspondiente de la ecuación del calor.

La dominación dispersa consiste en controlar estos operadores mediante una suma de medias locales positivas. Esto permite derivar multitud de estimaciones que a menudo son óptimas.

En este trabajo introduciremos este concepto y discutiremos el caso de los operadores que están más allá de la teoría de Calderón–Zygmund.

Keywords: sparse domination, weighted estimates, $T(1)$ theorem, elliptic operators.

MSC2010: 42B20, 42B25.

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1. Weighted estimates

In Harmonic Analysis, weighted inequalities allow us to better understand the action of operators on different domains, and they have many applications to PDEs [12], approximation theory, complex analysis and operator theory [1].

We call *weight* a positive, locally integrable function. You might be interested in understanding how your operator depends on the weight in the underlying measure. Given a (sub)linear operator T from L^p to itself, you can start asking the following questions:

(i) For which weights w is the operator bounded from $L^p(w)$ to $L^p(w)$?

(ii) Can we characterise all such weights?

(iii) How does the operator norm depend on the weight?

Since the '90s there have been a lot of efforts towards answering these questions and to quantify this dependence, also in relation to a problem in quasiconformal theory [2, 17].

A key step towards this goal was a representation of the action of the operator in terms of simpler dyadic operators. This representation was first obtained for the Hilbert transform [16], and later for general Calderón–Zygmund operators [13]. Today we know that a *domination*, rather than a representation, is often enough for deriving optimal weighted estimates with less effort. Such domination is popular as *sparse domination*.

Sparse domination is having a tremendous impact on Harmonic Analysis [4, 7–10]. It has simplified the proof of the A_2-conjecture [13], has found applications beyond the classical theory [4], in the discrete setting [10], and has resolved long-standing questions in operator theory [1].

1.1. Maximal operators

The questions posed above were first answered by Benjamin Muckenhoupt [15] for the maximal operator:

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, \mathrm{d}y,$$

where the supremum is taken over cubes Q with sides parallel to the coordinate axis.

If we could control an operator T by the maximal operator M, we could derive weighted estimates for T from the ones for M. Unfortunately, contrarily to maximal operators, singular integral operators are not bounded in L^∞. Thus we cannot hope to control them (pointwise) by a single maximal average.

Sparse domination consists in controlling non-local operators by a *sum* of positive averages. This allows to derive plenty of unweighted, weighted, and vector valued estimates (which are often optimal) from weighted L^p estimates for maximal operators, while the L^∞-norm is still allowed to blow up.

This domination can be performed by constructing a *sparse* family of cubes for a given input function. Roughly speaking, a sparse family is a collection of cubes having a subcollection of sets that are disjoint and not too small. More precisely, for a fixed $\tau \in (0,1)$ we say that:

Definition 1. A collection of dyadic cubes $\mathcal{S}$ is τ-sparse if for any $Q \in \mathcal{S}$ there exists a subset $E_Q \subset Q$ such that $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint and the ratio $|E_Q|/|Q|$ is bounded below by τ. ◂

Given a function f and a cube Q_0, one can construct a sparse collection inside Q_0 by selecting maximal cubes covering the superlevel set:

$$F(Q_0) = \left\{ x \in Q_0 \,:\, Mf(x) > \lambda \frac{1}{|Q_0|} \int_{Q_0} |f| \right\}.$$

The weak boundedness of M ensures that we can choose $\lambda > 0$ so that the measure of the complement $E_{Q_0} \coloneqq Q_0 \setminus F(Q_0)$ is not too small. By iterating this procedure, one obtains a collection of nested cubes organised in generations.

Then, our operator is controlled by the desired average on E_{Q_0}. At each iteration, the remaining area shrinks geometrically, leading to the pointwise domination for $x \in Q_0$

$$|Tf(x)| \le C \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \mathbb{1}_Q(x), \tag{1}$$

where the collection $\mathcal{S}$ is the union of all maximal cubes in each iteration, and it is *sparse* as in definition 1.

The same method can be used to bound bilinear expressions, leading to a domination by a *sparse bilinear form*:

$$\Big| \int_{Q_0} Tf \cdot g \, \mathrm{d}x \Big| \le C \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \left(\frac{1}{|Q|} \int_Q |g| \right) |Q|. \tag{2}$$

The sparse collections $\mathcal{S}$ in (1) and (2) depend on the input functions.

How does one recover bounds in terms of the maximal function? When we integrate sparse operators, the sparseness property allows to reduce the sum over $\mathcal{S}$ to a sum over disjoint sets. The averages are then controlled by the maximal averages. For example, for a $\frac{1}{2}$-sparse family $\mathcal{S}$ we have that $|Q| \le 2|E_Q|$, so

$$\sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) |Q| \le 2 \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) |E_Q| \le 2 \sum_{Q \in \mathcal{S}} \int_{E_Q} Mf \le 2 \int_{Q_0} Mf.$$

In a similar fashion, one recovers L^p estimates from L^p bounds of the maximal operator. One can then take the supremum over all possible $\frac{1}{2}$-sparse collections $\mathcal{S}$ obtaining the same weighted estimates.

2. Further applications

2.1. Sparse $T1$ theorems

In the '80s, David and Journé [11] showed that L^2-boundedness of singular integral operators follows from the uniform boundedness on characteristic functions. This result is known as the "$T(1)$ theorem", as the operator is tested on constant functions. The analogous condition for classical square functions is a Carleson measure condition [6].

These classical results have recently been recast to give a sparse domination [5, 14]. Thus, instead of just L^2 boundedness, these theorems imply all weighted L^p-bounds with optimal dependence on the weight.

2.2. Sparse domination of non-integral operators

Classical operators in Harmonic Analysis come with an integral representation and a kernel. On the other hand, many operators coming from elliptic PDEs are "non-integral", in the sense that they do not possess such an explicit representation.

Recently, the sparse paradigm has been successfully applied also in this context [4], where the usual assumptions on the kernel are replaced by hypotheses on the action of the operator on the semigroup e^{-tL} generated by the elliptic operator L.

For non-integral square functions, optimal weighted estimates are deduced form a different sparse form [3], which reflects the quadratic nature of these operators. This quadratic sparse domination yields estimates for square functions associated with divergence forms and Laplace–Beltrami operators on Riemannian manifolds as particular examples.

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Identities in prime rings

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Abstract: Given a ring, a generalized polynomial identity (GPI) is a polynomial identity in which the coefficients can be taken from the ring. Prime rings are a class of rings very well suited to manage problems related to identities, as for example those coming from Herstein's theory, which is the study of nonassociative objects and structures arising from associative rings. In such study, a particular kind of GPI, that in one variable depending only on the powers of a single element of the ring, often appears. The standard tool for simplifying this kind of GPI, Martindale's lemma, is powerful but not systematic. I present a new method, based on translating the problem to the polynomial setting, which makes the simplification systematic and deals with all field characteristics at once. The proofs will appear elsewhere.

Resumen: Dado un anillo, una identidad polinómica generalizada (GPI) es una identidad polinómica cuyos coeficientes pueden ser tomados del propio anillo. Los anillos primos forman una clase de anillos muy adecuada para tratar problemas relacionados con identidades, como por ejemplo las que surgen de la teoría de Herstein, el estudio de los objetos y estructuras no asociativos construidos a partir de anillos asociativos. En dicho estudio aparece a menudo un tipo especial de GPI, que tiene una única variable y depende solamente de las potencias de un único elemento del anillo. La herramienta estándar para simplificar este tipo de GPI, el lema de Martindale, es potente pero no sistemática. Aquí presento un nuevo método, basado en una traducción del problema al contexto de anillos de polinomios, que produce una simplificación sistemática y considera cuerpos de todas las características al mismo tiempo. Las demostraciones serán publicadas en otro artículo.

Keywords: prime rings, generalized polynomial identities, minimal polynomial, plane curves.

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1. GPIs in one variable in prime rings

Prime rings are the noncommutative counterparts of integral domains. A commutative unital ring R is an integral domain if

$$ab = 0 \text{ implies } a = 0 \text{ or } b = 0, \text{ for } a, b \in R.$$

In the noncommutative setting, elements are replaced by ideals. So, a ring R is *prime* if

$$IJ = 0 \text{ implies } I = 0 \text{ or } J = 0, \text{ for } I, J \text{ ideals of } R.$$

There is also a characterization of primeness by elements. A ring R is prime if and only if

$$axb = 0 \text{ for all } x \in R \text{ implies } a = 0 \text{ or } b = 0, \text{ for } a, b \in R. \tag{1}$$

We can say that prime rings are those in which, as in (1), an identity with one term (axb) in one variable (x) cornered by two fixed expressions (a, b) can be simplified to one of those expressions ($a = 0$ or $b = 0$). What other simplifications of similar identities follow from primeness? It is straightforward to show, using the characterization by elements, the following simplification of an identity with one term and two variables: if R is prime, then

$$axbyc = 0 \text{ for all } x, y \in R \text{ implies } a = 0 \text{ or } b = 0 \text{ or } c = 0, \text{ for } a, b, c \in R.$$

In general an identity of this kind, formed by a linear combination of some expressions of variables cornered by some fixed elements of the ring, is called a *generalized polynomial identity (GPI)*. It is also true, but not that straightforward to show, that primeness allows to simplify any GPI in one variable. Now we need to introduce some technical concepts. Just as an integral domain can be embedded in its field of quotients, a prime ring R has a similar overring, called the Martindale ring of quotients $Q(R)$. Although $Q(R)$ is not a division ring, its center $\mathcal{C} \coloneqq \mathcal{C}(R) \coloneqq Z(Q(R))$ is a field, called the *extended centroid* of R. And although R is not a $\mathcal{C}$-algebra in general, we can work in $\mathcal{C}R + \mathcal{C}$ inside $Q(R)$ and informally consider the elements of $\mathcal{C}$ as scalars for R (see [2, Section 2] for more details). In this paper we will consider GPIs with coefficients in the extended centroid. As stated above, primeness allows to simplify any GPI in one variable (and degree 1 in the variable) [2, Theorem 2.3.4]:

Lemma 1 (Martindale's). *Let R be a prime ring, $a_1, b_1, \ldots, a_n, b_n \in R$ and $\lambda_1, \ldots, \lambda_n \in \mathcal{C}$. If $\lambda_1, b_1 \neq 0$, then*

$$\lambda_1 a_1 x b_1 + \ldots + \lambda_n a_n x b_n = 0 \text{ for all } x \in R \text{ implies } a_1 \in \mathcal{C}a_2 + \ldots + \mathcal{C}a_n. \tag{2}$$

The conclusion of Martindale's lemma is that a_1 is a linear combination of the left elements of the other terms of the GPI.

2. Operator-algebraic elements

We are interested in a special kind of GPI in one variable, in which only powers of a fixed element $a \in R$ appear as coefficients from the ring, with $\lambda_{ij} \in \mathcal{C}$:

$$\lambda_{10}ax + \lambda_{01}xa + \lambda_{20}a^2x + \lambda_{11}axa + \lambda_{02}xa^2 + \lambda_{30}a^3x + \lambda_{21}a^2xa + \ldots = 0 \text{ for all } x \in R.$$

This kind of GPI appears often in Herstein's theory, the study of the nonassociative structures and objects arising from associative rings (see e.g. [3–5, 7]). We write it more concisely as

$$\sum_{i,j=0}^{n} \lambda_{ij} a^i x a^j = 0 \text{ for all } x \in R, \tag{3}$$

with the implicit assumption that $i + j > 0$. If we apply Martindale's lemma to (3), by looking at a fixed term of the form $\lambda_{ij}a^ixa^j$ with $\lambda_{ij} \neq 0$ we find that either $a^j = 0$ (implying a is nilpotent, in particular algebraic), or a^i is a linear combination of the other left powers of a appearing in the GPI, implying that a is algebraic. So Martindale's lemma implies in any case that a is an algebraic element, and thus it must have a minimal polynomial. But given a GPI of this form there can be several different minimal polynomials giving rise to it. The problem we want to solve is: which are the minimal polynomials giving rise to a fixed GPI of the form (3)? We solve the problem by translating it to a polynomial problem in two variables, which we then solve by elementary algebraic geometry.

3. A polynomial problem

Given a GPI of the form (3), we can associate to it a polynomial in two variables in $\mathcal{C}[X, Y]$ by translating it term by term, translating the left power of a as a power of X and the right power of a as a power of Y. For example, the identity $a^2x - 2axa + 3axa^3 = 0$ generates the polynomial $X^2 - 2XY + 3XY^3$. More in general, we get

$$\sum_{i,j=0}^{n} \lambda_{ij} a^i x a^j \mapsto f(X, Y) := \sum_{i,j=0}^{n} \lambda_{ij} X^i Y^j.$$

It can be shown (the proofs will appear elsewhere) that the problem of finding the minimal polynomials is equivalent to this one: which are the polynomials in one variable $p \in \mathcal{C}[X]$ such that the fixed polynomial in two variables $f \in \mathcal{C}[X, Y]$ belongs to the ideal generated by $p(X)$ and $p(Y)$?

This problem can be solved through the Taylor expansion of the polynomial f. If $\operatorname{char}(\mathcal{C}) = 0$, then the coefficients of the expansion are given by evaluations of the partial derivatives of f divided by factorials of some integers. To solve this problem for any field $\mathcal{C}$ of arbitrary characteristic we need to compute the coefficients of the expansion without making any divisions. These coefficients are given by the Hasse-Schmidt partial derivatives, which are linear maps but not derivations in general, and that we succinctly present here for two variables:

$$D_{X^i}(X^m Y^n) := \binom{m}{i} X^{m-i} Y^n, \quad D_{Y^i}(X^m Y^n) := \binom{n}{i} X^m Y^{n-i}, \quad D_{X^i Y^j} := D_{X^i} \circ D_{Y^j}.$$

Now, a version of the combinatorial nullstellensatz [1] accounting for multiplicities [6] solves our polynomial problem:

Theorem 2. *Let $p \in \mathcal{C}[X]$ have root structure $p(X) = \prod_{i=1}^{n} (X - \lambda_i)^{e_i}$ over the algebraic closure of $\mathcal{C}$. Then, $f \in \mathcal{C}[X, Y]$ belongs to the ideal of $\mathcal{C}[X, Y]$ generated by $\{p(X), p(Y)\}$ if and only if for each pair of roots (λ_i, λ_j) we have*

$$D_{X^r Y^s} f(\lambda_i, \lambda_j) = 0$$

for all $0 \le r < e_i$, $0 \le s < e_j$.

From this theorem, we can readily extract an algorithm for determining the minimal polynomials of a given GPI of the form (3) in a prime ring. The conditions on the partial derivatives imply that we can even determine them geometrically, by plotting the two-dimensional curve generated by the zeros of the polynomial in two variables and determining its behaviour over rectangular grids of potential roots.

Example 3. Let us determine the possible minimal polynomials making a satisfy the GPI

$$a^3 x a - 2axa^2 = 0$$

in a prime ring. We consider its associated polynomial in two variables

$$f(X, Y) = X^3 Y - 2XY^2.$$

By Theorem 2, the potential roots of the minimal polynomials must be roots of

$$f(X, X) = X^4 - 2X^3 = X^3(X - 2),$$

so the potential roots are 0 (with multiplicity at most 3) and 2. We may have 0 as the unique root (with some maximal multiplicity e), 2 as the unique root, or both 2 and 0 (with some maximal multiplicity perhaps smaller than e). To determine them, we compute the Hasse-Schmidt derivatives of f:

$$D_X f = 3X^2 Y - 2Y^2, \qquad D_Y f = X^3 - 4XY,$$
$$D_{X^2} f = \binom{3}{2} XY = 3XY, \quad D_{XY} f = 3X^2 - 4Y, \quad D_{Y^2} f = -\binom{2}{2} 2X = -2X,$$
$$D_{X^3} f = \binom{3}{3} Y = Y, \quad D_{X^2 Y} f = 3X, \quad D_{XY^2} f = -2, \quad D_{Y^3} f = 0,$$
$$D_{X^3 Y} f = 1, \qquad D_{X^4} f = D_{X^2 Y^2} f = D_{XY^3} f = 0.$$

(i) Since $D_X f$, $D_Y f$, $D_{XY} f$ have $(0,0)$ as zero, 0 can be found as the unique root of a minimal polynomial with multiplicity 2. To be found with multiplicity 3, we would need also $D_{X^2} f$, $D_{X^2Y} f$, $D_{X^2Y^2} f$, $D_{XY^2} f$, and $D_{Y^2} f$ to have $(0,0)$ as zero; this happens if and only if $\mathrm{char}(\mathcal{C}) = 2$, since $D_{XY^2} f = -2$. We cannot have 0 with multiplicity 4 because $D_{X^3Y} f = 1 \neq 0$ in all characteristics.

(ii) Since $D_X f(2,2) = 2^4$, $D_Y f(2,2) = -2^3$, $D_{XY} f(2,2) = 2^2$, for 2 to be found as the unique root of a minimal polynomial with multiplicity 2 it is necessary and sufficient that $\mathrm{char}(\mathcal{C}) = 2$, in which case we have $2 = 0$ and we are in the previous case.

(iii) To find 0 and 2 together as roots of the same minimal polynomial, it is necessary that $D_X f(0,2) = -2^3 = 0$, so again we would have $\mathrm{char}(\mathcal{C}) = 2$ and, hence, only one root.

In conclusion, the maximal possible minimal polynomials for a are X^2, X^3 if $\mathrm{char}(\mathcal{C}) = 2$, and $X - 2$; so $a^3xa - 2axa^2 = 0$ for all $x \in R$ prime if and only if either $a^2 = 0$; $a^2 \neq 0$, $a^3 = 0$ and $2 = 0$; or $a = 2$. ◂

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Factor of iid Schreier decoration of transitive graphs

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Abstract: A Schreier decoration is a combinatorial coding of an action of the free group F_d on the vertex set of a $2d$-regular graph. We investigate whether a Schreier decoration exists on various countably infinite transitive graphs as a factor of iid.

We show that the square lattice and also the three other Archimedean lattices of even degree and $\mathbb{Z}^d$, $d \geq 3$, have finitary factor of iid Schreier decorations, and exhibit examples of transitive graphs of arbitrary even degree in which obtaining such a decoration as a factor of iid is impossible.

We also prove that non-amenable quasi-transitive unimodular $2d$-regular graphs have a factor of iid balanced orientation, meaning each in- and outdegree is equal to d. This result involves extending earlier spectral-theoretic results on Bernoulli shifts to the Bernoulli graphings of quasi-transitive unimodular graphs. Balanced orientation is also obtained for symmetrical planar lattices.

Resumen: Una decoración de Schreier es una codificación combinatoria de una acción del grupo libre F_d en el conjunto de vértices de un grafo $2d$-regular. Investigamos si existe una decoración de Schreier en varios grafos transitivos numerables infinitos como un factor de iid.

Mostramos que el retículo cuadrado y también los otros tres grafos arquimedianos de grado par y $\mathbb{Z}^d$, $d \geq 3$, tienen decoraciones de Schreier de factor finito de iid, y mostramos ejemplos de grafos transitivos de grado par arbitrario en los que la obtención de tal decoración como factor de iid es imposible.

También demostramos que los grafos $2d$-regulares unimodulares cuasi transitivos no amenables tienen un factor de orientación equilibrada iid, lo que significa que cada grado de entrada y salida es igual a d. Este resultado implica la extensión de los resultados espectrales anteriores sobre los desplazamientos de Bernoulli a los grafos de Bernoulli de los grafos unimodulares cuasi-transitivos. También se obtiene la orientación equilibrada para retículos planos simétricos.

Keywords: transitive graph, factor of iid, Schreier graph, site percolation, Archimedean lattice, planar lattice, graphing, spectral gap.

MSC2010: 05E18, 05C63, 37A30, 60C05.

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1. Introduction, notation, and basics

A *Schreier decoration* of a $2d$-regular graph G is a colouring of the edges with d colours together with an orientation such that, at every vertex, there is exactly one incoming and one outgoing edge of each colour. A partial result towards a Schreier decoration is a balanced orientation of the edges. An orientation of a graph with all degrees even is *balanced* if the indegree of any vertex is equal to its outdegree. We investigate whether an invariant random Schreier decoration or at least balanced orientation can be obtained on infinite transitive graphs as a factor of iid.

Informally, a Schreier decoration is a factor of iid if it is produced by a certain randomised local algorithm. To start with, each vertex independently gets a random label from $[0, 1]$. Then it makes a deterministic measurable decision about the decoration of its incident edges, based on the labelled graph that it sees from itself as a root. Adjacent vertices must make a consistent decision regarding the edge between them.

1.1. Schreier graphs, factors of iid and non-examples

Given a group $\Gamma = \langle S\rangle$ and an action $\Gamma \curvearrowright X$, the *Schreier graph* $\mathrm{Sch}(\Gamma \curvearrowright X, S)$ has X as its vertex set, and for every $x \in X$, $s \in S$, we introduce an oriented s-labelled edge from x to $s \cdot x$. A map $\Phi : X \to Y$ between two Γ-spaces is a *Γ-factor* if it is measurable and $\gamma \cdot \Phi(x) = \Phi(\gamma \cdot x)$ for every $\gamma \in \Gamma$, $x \in X$.

Definition 1. Let G be a graph and u denote the Lebesgue measure on $[0, 1]$. We endow the space $[0, 1]^{V(G)}$ with the product measure $\mathrm{u}^{V(G)}$. A factor of iid Schreier decoration (respectively, balanced orientation) of G is an $\mathrm{Aut}(G)$-factor $\Phi : \left([0, 1]^{V(G)}, \mathrm{u}^{V(G)}\right) \to \mathtt{Sch}(G)$ (respectively, to $\mathtt{BalOr}(G)$). ◂

For simple graphs G_1 and G_2, let the graph $G_1 \times G_2$ be defined by having $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ with vertices (u, v) and (u', v') being adjacent if $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$.

Proposition 2. *Let H be a finite $(2d - 2)$-regular graph with an odd number of vertices. The $2d$-regular graph $H \times P$, where P is the bi-infinite path, has no factor of iid balanced orientation.*

Being quasi-isometric to P is a necessary condition in our non-examples. It is, however, not sufficient.

Proposition 3. *Let H be a finite bipartite $(2d - 2)$-regular graph. Then, the $2d$-regular graph $H \times P$, where P is the bi-infinite path, has a factor of iid Schreier decoration.*

2. Infinite amenable graphs

To obtain Schreier decorations of the Archimedean lattices later in this section, we partition their vertex set V into finite clusters such that, for each cluster C, there is a unique cluster C^+ surrounding it.

Definition 4 (Hierarchy). Let G be a graph and $\mathbf{H}$ a partition of $V(G)$. We say that two distinct parts $C, D \in \mathbf{H}$ are adjacent if and only if there is $u \in C$ and $v \in D$ which are adjacent in G. Then, $\mathbf{H}$ is a *hierarchy on G* if the following hold for every $C \in \mathbf{H}$: 1) C is finite, 2) there is a *unique* $C^+ \in \mathbf{H}$ such that C and C^+ are adjacent and, for all $v \in V(G)$ but finitely many, any path from C to v contains a vertex from C^+, and 3) whenever $B \in \mathbf{H}$ is adjacent to C, either $B = C^+$ or $C = B^+$. ◂

A key feature of our hierarchies is that any two non-adjacent clusters are far from one another. But starting with any hierarchy, we can obtain one in which such clusters are as far from one another as we wish.

Proposition 5. *Let G be a graph and $k \in \mathbb{N}$. A hierarchy $\mathbf{H}$ on G is k-*spaced *if, for all non-adjacent $B, C \in \mathbf{H}$, the graph distance $d(B, C) = \min_{u\in B, v\in C} d_G(u, v)$ is at least k. Suppose there is a factor of iid hierarchy $\mathbf{H}$ on G. Then, for all $k \in \mathbb{N}$, there is a factor of iid k-spaced hierarchy $\mathbf{H}_k$ on G.*

Moreover, for all $c, k \in \mathbb{N}$, there is a factor of iid pair $(J_{c,k}, \eta : J_{c,k} \to [c])$ where $J_{c,k}$ is a k-spaced hierarchy and $\eta : J_{c,k} \to [c]$ is a colouring with c colours such that, for all $C \in J_{c,k}$, if C has colour i, then C^+ has colour $i + 1 \pmod c$.

For a general planar lattice Λ, we wish to use site percolation to obtain a hierarchy.

Theorem 6 ([1]). *Let Λ be a plane lattice with m-fold symmetry for some $m \geq 2$ and $\Lambda^{\times}$ be its matching lattice, i.e., the graph obtained from Λ by adding all diagonals to all faces of Λ. Then, for every $p \in [0,1]$, the probabilities $\theta^s_{\Lambda}(p)$, $\theta^s_{\Lambda^{\times}}(p)$ satisfy that $\theta^s_{\Lambda}(p) = 0$ or $\theta^s_{\Lambda^{\times}}(p) = 0$. Furthermore, $p^s_H(\Lambda) + p^s_H(\Lambda^{\times}) = 1$.*

So if Λ has m-fold symmetry, we can add a vertex to every non-triangular face and connect it to all the vertices of that face. The resulting lattice $\Lambda^{\bullet}$ also has m-fold symmetry and is self-matching, so Theorem 6 tells us that $p^s_H(\Lambda^{\bullet}) = \frac{1}{2}$ and percolation does not occur at criticality. This gives us a hierarchy on Λ as follows. Colour the vertices of Λ yellow or green uniformly at random. For each face, decide randomly whether either all its yellow or all its green vertices will be treated as if they were connected through the face. This results in a hierarchy, which together with Proposition 5 gives basis for the following.

Theorem 7. *Let Λ be a planar lattice with m-fold symmetry, $m \geq 2$, in which all degrees are even. There is a finitary factor of iid which is a balanced orientation of Λ almost surely.*

2.1. Schreier decorations of Archimedean lattices and $\mathbb{Z}^d$, $d \geq 3$, as factors of iid

Theorem 8. *Let Λ be $\mathbb{Z}^d$, $d \geq 3$, or any of the four Archimedean lattices with even degrees: the square lattice, the triangular lattice, the Kagomé lattice or the $(3,4,6,4)$ lattice. There is a finitary factor of iid which is a.s. a Schreier decoration of Λ. Moreover, it has almost surely no infinite monochromatic paths.*

Our approach is the same throughout. We break the lattices into a hierarchy of finite pieces. Then, for each piece independently, we choose an edge-d-colouring scheme such that we can ensure that every monochromatic connected subgraph is a finite cycle. Each cycle will orient itself strongly.

Both in the case of the triangular lattice and $\mathbb{Z}^d$, $d \geq 3$, once we have a spaced enough hierarchy, we reuse the patterns developed for $\Lambda_{\square}$. Unlike in the proofs for Archimedean lattices, we do not use percolation as our starting point for $\mathbb{Z}^d$, $d \geq 3$, but instead the results of Gao, Jackson, Krohne and Seward [3].

Corollary 9. *For every $d \geq 2$, there is a factor of iid which is a proper $2d$-colouring of the edges of $\mathbb{Z}^d$ a.s. Subsequently, there is a factor of iid which is a perfect matching on $\mathbb{Z}^d$ a.s.*

3. Balanced orientation of non-amenable quasi-transitive graphs

Theorem 10. *Every non-amenable quasi-transitive unimodular $2d$-regular graph G has a factor of iid orientation that is balanced almost surely.*

For example, the regular trees are unimodular. For $d > 1$, the $2d$-regular tree T_{2d} is also non-amenable, so it is covered by Theorem 10. For $d > 2$, Theorem 10 allows us to remark the following too.

Proposition 11. *If T_d has a factor of iid proper edge d-colouring, then T_{2d} has a factor of iid Schreier decoration.*

Despite this connection, it remains open whether there is a factor of iid Schreier decoration of T_{2d}.

To prove Theorem 10, we first reduce a balanced orientation of G to a perfect matching in an auxiliary bipartite graph G^*. Then, we extend earlier matching results on Cayley graphs to the case of unimodular quasi-transitive graphs. The key step for us, as it is for the earlier results [2, 4], is to use spectral theory to show stabilisation of an infinite algorithm.

Theorem 12. *Let G be a connected, unimodular, quasi-transitive graph. If G is non-amenable, then its Bernoulli graphing $\mathcal{G}$ has positive spectral gap.*

The interpretation of spectral gap differs depending on the Bernoulli graphing being bipartite or not. See Theorems 14 and 15 for exact statements.

Corollary 13. *Let G be a connected, unimodular, quasi-transitive non-amenable d-regular bipartite graph. Then, G has a factor of iid subset of the edges which is a perfect matching almost surely.*

3.1. Unimodular quasi-transitive graphs, Bernoulli graphings and spectral gap

Let G be a locally finite quasi-transitive graph, $\Gamma = \mathrm{Aut}(G)$. G is *unimodular* if $|\mathrm{Stab}_\Gamma(x) \cdot y| = |\mathrm{Stab}_\Gamma(y) \cdot x|$ for any $x, y \in V(G)$ that are in the same Γ orbit. Let $T = \{o_1, \dots, o_t\}$ be a set of representatives of the orbits of $\Gamma \curvearrowright V(G)$. We set $p(o_i) = \mu(o_i)^{-1}$ and scale such that $\sum_i p(o_i) = 1$.

The notion of unimodularity comes hand in hand with the Mass Transport Principle, which allows us to set up a Markov chain M_T mimicking the transitions of the random walk on G between Γ-orbits. Let its states be T and transition probabilities $p_{M_T}(o_i, o_j) = \frac{|\{(v,o_i)\in E | v\in\Gamma\cdot o_j\}|}{\deg(o_i)}$.

List the eigenvalues of M_T in decreasing order, $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t$. We say M_T is *bipartite* if $\lambda_t = -1$. When M_T is not bipartite, we set $\rho_T = \max(|\lambda_2|, |\lambda_t|)$. When it is, we set $\rho_T = \max(|\lambda_2|, |\lambda_{t-1}|)$.

Let Ω denote the space of $[0,1]$-labelled rooted connected graphs. Elements of Ω are of the form (H, u, ω), where (H, u) is a bounded-degree rooted graph and $\omega : V(H) \to [0,1]$. We connect (H, u, ω) with (H', u', ω') if and only if we can obtain (H', u', ω') from (H, u, ω) by moving the root to one of its neighbours. We denote the resulting edge set by $\mathcal{E}$. To define the probability measure on Ω, pick the rooted graph (G, o_i) with probability $p(o_i)$. Then pick a labelling $\omega \in [0,1]^{V(G)}$ according to $\mathrm{u}^{V(G)}$. Let ν_G denote the distribution of (G, o_i, ω). Then, the *Bernoulli graphing* of G is $\mathcal{G} = (\Omega, \mathcal{E}, \nu_G)$.

The Markov operator $\mathcal{M}$ is a self-adjoint operator on $L^2(\Omega, \nu_{\mathrm{st}})$. Similarly, denote the Markov operator of G on $\ell^2(G, m_{\mathrm{st}})$ as M. Here ν_{st} and m_{st} denote the degree-biased versions of ν and of the counting measure on $V(G)$. The following two theorems deal with the non-bipartite and bipartite cases separately.

Theorem 14. *Let G be as in Theorem 12, and assume also that M_T is not bipartite. Let $\rho < 1$ denote the spectral radius of G on $\ell^2(G, m_{\mathrm{st}})$. Then, the spectral radius of $\mathcal{M}$ on $L^2_0(\Omega, \nu_{\mathrm{st}})$ is at most* $\max(\rho, \rho_T) < 1$.

Theorem 15. *Let G be as in Theorem 12, and assume that M_T is bipartite. Let $\rho < 1$ denote the spectral radius of G on $\ell^2(G, m_{\mathrm{st}})$. The Bernoulli graphing $\mathcal{G}$ is measurably bipartite, with bipartition $X_1 \cup X_2 = V(\mathcal{G})$. Let $L^2_{00}(\Omega, \nu_{\mathrm{st}})$ denote the orthogonal complement of the subspace generated by the functions $\mathbb{1}_X$ and $\mathbb{1}_{X_1} - \mathbb{1}_{X_2}$. Then, the spectral radius of $\mathcal{M}$ on $L^2_{00}(\Omega, \nu_{\mathrm{st}})$ is at most* $\max(\rho, \rho_T) < 1$.

3.2. Perfect matchings and balanced orientations

To prove Theorem 10, we relate balanced orientations of our $2d$-regular G to perfect matchings of a bipartite graph G^*. G^* is constructed from G by local transformations, which makes sure that it remains unimodular. In particular, the bipartite $G^* = (S, T, E^*)$ is obtained by setting $S = E(G)$ and $T = V(G) \times [d]$. Edges of G^* are defined by connecting $e \in S$ to $(v, i) \in T$ if and only if e is incident to v in G.

Lemma 16. *The graph G^* is quasi-isometric to G. If G is unimodular quasi-transitive, then so is G^*. Crucially, any perfect matching M in G^* defines a balanced orientation of G by orienting the edge $e \in S$ towards its endpoint v if and only if e and (v, i) are matched by M for some $i \in [d]$.*

Proof of Theorem 10. We construct G^*, which Lemma 16 tells us is bipartite, unimodular, quasi-transitive, and quasi-isometric to G. As amenability is a quasi-isometry invariant property, G^* is non-amenable. By Corollary 13, G^* has a perfect matching, which by Lemma 16 gives a balanced orientation of G. ■

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On the connection between Hardy kernels and reproducing kernels

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Abstract: Hardy kernels are known for being a useful tool to construct bounded operators on $L^p(\mathbb{R}^+)$ spaces, property which follows from Hardy's inequality. Even more, recently Hardy kernels have also been used to define bounded operators on Hardy spaces on the half plane $H^p_\alpha(\mathbb{C}^+)$. In this work, the range spaces in $L^p(\mathbb{R}^+)$ and $H^p_\alpha(\mathbb{C}^+)$ of such operators are analysed. We focus on the case $p = 2$, where under some circumstances, these range spaces arise as reproducing kernel Hilbert spaces. We show that in the $L^2(\mathbb{R}^+)$ case, the reproducing kernels of these spaces turn out to be Hardy kernels as well, whereas in the $H^2_\alpha(\mathbb{C}^+)$ setting, their reproducing kernels are holomorphic extensions of Hardy kernels. We also present how the Laplace transform connects the real and complex settings of this family of range spaces.

Resumen: Los núcleos de Hardy son conocidos por ser una herramienta útil para construir operadores acotados en los espacios $L^p(\mathbb{R}^+)$, hecho que se sigue de la desigualdad de Hardy. Además, los núcleos de Hardy han sido recientemente utilizados para construir operadores acotados en los espacios de Hardy del semiplano $H^p_\alpha(\mathbb{C}^+)$. En este trabajo, se analizan los espacios rango de dichos operadores en $L^p(\mathbb{R}^+)$ y $H^p_\alpha(\mathbb{C}^+)$. En particular, nos centramos en el caso $p = 2$, en el que, bajo determinadas condiciones, estos espacios rango son de hecho espacios de Hilbert con núcleo reproductor. Demostramos que, en el caso de $L^2(\mathbb{R}^+)$, los núcleos reproductores de dichos espacios son a su vez núcleos de Hardy, y que en el caso de $H^2_\alpha(\mathbb{C}^+)$, los núcleos reproductores vienen dados por extensiones holomorfas de núcleos de Hardy. Por último, mostramos cómo la transformada de Laplace conecta los escenarios real y complejo de esta familia de espacios rango.

Keywords: Hausdorff operators, reproducing kernel Hilbert spaces, Hardy kernels.

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1. Banach algebra of Hardy kernels

Set $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \Re z > 0\}$.

Definition 1. Let $1 \leq p < \infty$ and let $H\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ be a measurable map. H is said to be a Hardy kernel of index p if the following conditions hold.

(i) H is homogeneous of degree -1; that is, for all $\lambda > 0$, $H(\lambda r, \lambda s) = \lambda^{-1} H(r, s)$ for all $r, s > 0$.
(ii) $\int_0^\infty H(1, s) s^{-1/p}\, \mathrm{d}s < \infty$.

Let us denote by $\mathfrak{H}_p$ the set of Hardy kernels of index p. ◂

For $1 \leq p < \infty$, let $L^p(\mathbb{R}^+)$ denote the classical Lebesgue space on the positive real line, and $H_a^p(\mathbb{C}^+)$ the Hardy space on the right complex half plane. Given a Hardy kernel of index p, one can construct bounded operators A_H and D_H on $L^p(\mathbb{R}^+)$ and $H^p(\mathbb{C}^+)$ respectively, which are given by

$$(A_H f)(r) := \int_0^\infty H(r, s) f(s)\, \mathrm{d}s, \quad \text{for a.e. } r > 0,\ f \in L^p(\mathbb{R}^+),$$
$$(D_H F)(z) := \int_0^\infty H(|z|, s) F(s\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}s, \quad z = |z|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}^+,\ F \in H^p(\mathbb{C}^+).$$

The boundedness of $A_H \in \mathcal{B}(L^p(\mathbb{R}^+))$ follows from Hardy's inequality [3, Theorem 319], and the boundedness of $D_H \in \mathcal{B}(H^p(\mathbb{C}^+))$ was shown in the recent work about Hausdorff operators [5]. In fact, these families of operators $(A_H)_{H\in\mathfrak{H}_p}$ and $(D_H)_{H\in\mathfrak{H}_p}$ may be labelled as Hardy-Hausdorff operators since they are a particular case of Hausdorff operators, see the survey article [6] for more details.

It is part of folklore that the family of operators given by $(A_H)_{H\in\mathfrak{H}_p}$ can be described as convolution operators by identifying a Hardy kernel H with a Lebesgue integrable function $g_H \in L^1(\mathbb{R})$, see for example the paper about the spectra of A_H [1]. More precisely, let H be a Hardy kernel of index p, and set $g_H(t) := H(1, \mathrm{e}^{-t})\mathrm{e}^{-t/p'}$ for all $t \in \mathbb{R}$, where p' is such that $1/p + 1/p' = 1$. It is readily seen that $g_H \in L^1(\mathbb{R})$, with $\|g_H\|_1 = \int_0^\infty |H(1, s)| s^{-1/p}\, \mathrm{d}s$. Moreover, if one takes certain equivalence classes on $\mathfrak{H}_p$, it is straightforward to obtain that the mapping $H \mapsto g_H$ is a bijection from $\mathfrak{H}_p$ onto $L^1(\mathbb{R})$, see the forthcoming paper [8] for more details. Therefore, one obtains that this set of equivalence classes of Hardy kernels of index p entails a commutative Banach algebra structure, isomorphic to $L^1(\mathbb{R})$, whose norm and product are given, respectively, by

$$\|H\|_{\mathfrak{H}_p} := \int_0^\infty |H(1, s)| s^{-1/p}\, \mathrm{d}s, \quad H \in \mathfrak{H}_p,$$
$$(H \bullet G)(r, s) := \int_0^\infty H(r, t) G(t, s)\, \mathrm{d}t, \quad r, s > 0,\ H, G \in \mathfrak{H}_p.$$

Notice that the multiplication $\bullet$ resembles typical formulas about the construction of reproducing kernel Hilbert spaces, see for example the expression [9, (2.1)].

For the purposes of this work, two subsets of $\mathfrak{H}_p$ must be pointed out. First, set $\mathcal{I}_p := \{H \in \mathfrak{H}_p \mid g_H \in L^{p'}(\mathbb{R})\}$, which is a dense ideal of $\mathfrak{H}_p$. Second, let $\mathfrak{H}_p^{Hol}$ denote the subspace of $\mathfrak{H}_p$ of Hardy kernels H of index p that admit a (unique) extension H^{Hol} from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{C}^+ \times \mathbb{C}^+$ such that $H^{Hol}(z, w)$ is holomorphic in z and anti-holomorphic in w. The following will also be needed.

Definition 2. Let $1 < p < \infty$, and $H \in \mathfrak{H}_p$. Set $H^t(r, s) := H(s, r)$ and $H^*(r, s) := \overline{H(s, r)}$ for all $r, s > 0$, where $\overline{z}$ denotes the conjugate of a complex number z. ◂

It is readily seen that both H^t, H^* belong to $\mathfrak{H}_{p'}$.

2. Range spaces of Hardy-Hausdorff operators

In this section, we proceed to study the range spaces of Hardy-Hausdorff operators on $L^2(\mathbb{R}^+)$ and $H_a^2(\mathbb{C}^+)$ as reproducing kernel Hilbert spaces, that is, Hilbert spaces of functions for which point evaluations define continuous functionals. This is partly motivated by the forthcoming work [2], where range spaces of fractional Cesàro operators are analysed. Let us recall that if Ω is a reproducing kernel Hilbert space (RKHS from now on) of complex functions with domain X, its reproducing kernel K is given by

$$K(x,y) = k_y(x), \quad x,y \in X,$$

where $k_y \in \Omega$ is such that $f(y) = \langle f | k_y \rangle$ for all $f \in \Omega$. One of the main interesting properties of reproducing kernels is that one can recover the whole Hilbert space Ω from its reproducing kernel K, see for example Chapter I in [7].

2.1. Hardy kernels as reproducing kernels on $\mathbb{R}^+ \times \mathbb{R}^+$

First, we shall study the $L^p(\mathbb{R}^+)$ scenario. Recall that $L^p(\mathbb{R}^+)$ denotes the Banach space of functions f defined a.e. on $\mathbb{R}^+$, such that $\|f\|_{L^p} := \left(\int_0^\infty |f(r)|^p \,\mathrm{d}r\right)^{1/p} < \infty$.

Definition 3. Let $1 \leq p < \infty$, and $H \in \mathfrak{H}_p$. Set the range space $\mathcal{A}(H) := A_H(L^p(\mathbb{R}^+))$, and endow it with the canonical Banach space structure $\mathcal{A}(H) \cong L^p(\mathbb{R}^+)/\ker A_H$. ◂

Notice that since $\mathcal{A}(H) \subset L^p(\mathbb{R}^+)$, one cannot guarantee that point evaluations are well defined on $\mathcal{A}(H)$. Let $C(\mathbb{R}^+)$ denote the set of complex continuous functions on $\mathbb{R}^+$.

Lemma 4. *Let $1 \leq p < \infty$ and let $H \in \mathcal{I}_p \subset \mathfrak{H}_p$. One has that $\mathcal{A}(H) \subset C(\mathbb{R}^+)$, in the sense that, if $f \in \mathcal{A}(H)$, then there is a (unique) continuous function $g \in C(\mathbb{R}^+)$ such that $f = g$ a.e.*

Therefore, if $H \in \mathcal{I}_p$ and $r > 0$, one can define point evaluations on $\mathcal{A}(H)$ by $f(r) := g(r)$, where $f \in \mathcal{A}(H)$ and $g \in C(\mathbb{R}^+)$ are as in the lemma above. The proposition below shows that these are all the Hardy kernels for which one can define continuous point evaluations on $\mathcal{A}(H)$.

Proposition 5. *Let $1 \leq p < \infty$, and $H \in \mathfrak{H}_p$. Then, one can define continuous point evaluations on $\mathcal{A}(H)$ if and only if $H \in \mathcal{I}_p$. If this is the case, it follows that for all $f \in \mathcal{A}(H)$*

$$|f(r)| \leq r^{-1/p} \|g_H\|_{L^{p'}} \|f\|_{\mathcal{A}(H)}, \qquad r > 0.$$

Next we give the reproducing kernel of this family of range spaces with continuous point evaluations.

Theorem 6. *Let $H \in \mathfrak{H}_2$. Then, $\mathcal{A}(H)$ is a RKHS if and only if $H \in \mathcal{I}_2$, and in this case its reproducing kernel K_H is separately continuous and given by*

$$K_H(r,s) = \int_0^\infty H(r,t)\overline{H(s,t)}\,\mathrm{d}t, \quad \text{for } r,s > 0.$$

It follows that K_H defines a Hardy kernel, satisfying $K_H = H \bullet H^*$.

2.2. Hardy kernels as reproducing kernels on $\mathbb{C}^+ \times \mathbb{C}^+$

Now we focus on the Hardy spaces of the half plane $H_a^p(\mathbb{C}^+)$, which are formed by all holomorphic functions F on $\mathbb{C}^+$ such that $\|F\|_{H^p} := \sup_{x>0}\left(\int_{-\infty}^\infty |f(x+\mathrm{i}y)|^p \,\mathrm{d}y\right)^{1/p} < \infty$. It is well known that these spaces present continuous point evaluations, so in particular $H_a^2(\mathbb{C}^+)$ is a RKHS whose reproducing kernel $\mathcal{K}$ is given by $\mathcal{K}(z,w) = (z+\overline{w})^{-1}$ for all $z,w \in \mathbb{C}^+$, see for example Proposition 1.8 in the notes [4]. Notice that, if one restricts $\mathcal{K}$ to $\mathbb{R}^+ \times \mathbb{R}^+$, one obtains the Stieltjes kernel $\mathcal{S}$, which is a Hardy kernel of index 2 given by $\mathcal{S}(r,s) = (r+s)^{-1}$ for all $r,s > 0$.

Definition 7. Let $H \in \mathfrak{H}_p$. Set $\mathcal{D}(H) := D_H(H_a^p(\mathbb{C}^+)) \subset H_a^p(\mathbb{C}^+)$ and endow $\mathcal{D}(H)$ with the canonical structure of a Banach space by $\mathcal{D}(H) \cong H_a^p(\mathbb{C}^+)/\ker D_H$. ◂

It is readily seen that point evaluations are continuous functionals on $\mathcal{D}(H)$ for all $H \in \mathfrak{H}_p$. The following theorem gives the reproducing kernel $\mathcal{K}_H$ of $\mathcal{D}(H)$, where $H \in \mathfrak{H}_2$, and which is given by the holomorphic extension of a Hardy kernel.

Theorem 8. *Let $H \in \mathfrak{H}_2$. One has that $H \bullet \mathcal{S} \bullet H^* \in \mathfrak{H}_2^{Hol}$, and that $\mathcal{D}(H)$ is a RKHS continuously embedded into $H_a^2(\mathbb{C}^+)$ whose reproducing kernel $\mathcal{K}_H$ is given by*

$$\mathcal{K}_H = (H \bullet \mathcal{S} \bullet H^*)^{Hol}.$$

2.3. A Paley-Wiener result

Next, we analyse the connection between the real and complex settings. First of all, recall that the classical Paley-Wiener theorem states that the Laplace transform $\mathcal{L}$, given by $(\mathcal{L}f)(z) := \int_0^\infty e^{-rz} f(r)$ for all $z \in \mathbb{C}^+$, defines an isometric isomorphism from $L^2(\mathbb{R}^+)$ onto $H_a^2(\mathbb{C}^+)$. The results below show how the Laplace transform connects the range spaces presented in subsections above.

Proposition 9. *Let $H \in \mathfrak{H}_2$. It follows that $\mathcal{L}A_H = D_{H^t}\mathcal{L}$.*

Theorem 10. *Let $H \in \mathfrak{H}_2$. The Laplace transform $\mathcal{L}$ restricted to $\mathcal{A}(H)$ is an isometric isomorphism onto $\mathcal{D}(H^t)$, $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H^t)$.*

Corollary 11. *Let $H \in \mathfrak{H}_2$. Either if H is symmetric, that is, $H = H^t$, or if H is real-valued, one obtains that $\mathcal{D}(H) = \mathcal{D}(H^t)$ as RKH spaces. Thus, the Laplace tranform $\mathcal{L}$ restricts to an isometric isomorphism from $\mathcal{A}(H)$ onto $\mathcal{D}(H)$, $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H)$.*

One may ask whether there exists an isometric isomorphism from $\mathcal{A}(H)$ onto $\mathcal{D}(H)$ for a general $H \in \mathfrak{H}_2$. This question is answered in the forthcoming work [8], where two mappings $\mathcal{P}, \mathcal{S} : L^2(\mathbb{R}^+) \to H_a^2(\mathbb{C}^+)$ are given such that they define isometric isomorphisms from $\mathcal{A}(H)$ onto $\mathcal{D}(H)$ for all $H \in \mathfrak{H}_2$.

As a final note, we refer the reader again to the upcoming work [8], where the proofs of all the results presented here, as well as a bunch of new results about this topic, are given in detail.

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Poisson brackets on the space of solutions of first order Hamiltonian field theories

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Abstract: We investigate the existence of a symplectic and, consequently, a Poisson structure on the space of solutions of a first order field theory. We provide an affirmative answer for theories where all constraints can be solved. The analysis for gauge theories is postponed to a more extensive work.

Resumen: Se investiga la existencia de una estructura simpléctica y, en consecuencia, de Poisson en el espacio de soluciones de una teoría de campos de primer orden. Se da una respuesta afirmativa para las teorías en las que se pueden resolver todas las restricciones. El análisis para las teorías gauge se pospone a un trabajo más extenso.

Keywords: multisymplectic geometry, Poisson geometry, global analysis.

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Introduction

We aim to analyse geometrical structures needed to give a description of a classical field theory allowing for a formulation of its quantum counterpart being compatible with special relativity. Now, we motivate how, with this problem in mind, it could be of interest to search for a Poisson structure on the space of solutions of the equations of the motion of the classical field theory.

Given a classical dynamical system, a formulation in terms of Poisson geometry is helpful to give a description of its quantum counterpart, if existing. If a Poisson structure exists on the *phase space*[1], say $\mathcal{M}$, of the dynamical system, then a Poisson bracket on the space of smooth functions on $\mathcal{M}$ is defined. Real valued smooth functions on $\mathcal{M}$ are usually interpreted as the observables of the theory. The so called *Dirac's analogy principle* states that in order for the predictions of the classical and the quantum theory to coincide within the energy scale where they are experimentally indistinguishable, then the Poisson structure on the space of classical observables must "come from" a Lie algebra structure on the space of quantum observables, that are modelled as self-adjoint operators on a Hilbert space. This motivates the search of a Poisson description of the classical theory.

However, the very concept of phase space is not compatible with special relativity. Indeed, often, the phase space is the space of configurations and momenta of the dynamical system for a fixed value of the *time* and the definition of a concept of time requires the introduction of a reference frame which splits the relativistic space-time into *space* and *time*. But, after a particular reference frame is introduced, the covariance of the dynamical system[2] under the Lorentz group can not be manifest. Following an idea of Souriau [7], the space of solutions of the equations of the motion seems to be more suitable for a relativistic description. Indeed, the *relativity group* is a group whose action preserves the equations of the motion and, consequently, maps solutions into solutions. Thus, differently from phase space, the space of solutions is actually covariant with respect to the action of the relativity group. With all this in mind, in order to give a description of the quantum counterpart of a classical field theory which is compatible with special relativity, it is of interest to investigate whether and how a Poisson structure can be given on the space of solutions of the classical theory.

This is what we do in this paper, giving an affirmative answer in the case of field theories where all the constraints can be solved and postponing the analysis of gauge theories to a more extensive work.

1. Multisymplectic formulation of field theories

We refer to [5, 6] for basic notions, notations and conventions about differential geometry and jet bundles. We adopt the so called multisymplectic formulation of field theories [4]. In this formulation the fields of the theory are modelled as sections of a fibre bundle $(E, \pi, \mathcal{M})$ whose base space $\mathcal{M}$ is a space-time with boundary $\partial\mathcal{M}$. A chart on $\mathcal{M}$ will be denoted by $(U_{\mathcal{M}}, \psi_{\mathcal{M}})$, $\psi_{\mathcal{M}}(m) = (x^\mu)_{\mu=0,\dots,d}$, with $d+1$ being the dimension of the space-time and $m \in \mathcal{M}$. An adapted fibered chart on E will be denoted by (U_E, ψ_E), $\psi_E(e) = (x^\mu, u^a)_{\mu=0,\dots,d;\, a=1,\dots,n}$, with n being the dimension of the fibres of E and $e \in E$. Sections of π are the fields of the theory, and we denote them by ϕ^a. The analogue of the phase space of mechanics is the so called *covariant phase space* which is the affine dual of the first order jet bundle of π [4]. It is again a fibre bundle over $\mathcal{M}$, denoted by $(\mathcal{P}(E), \tau_1, \mathcal{M})$, where an adapted fibered chart will be denoted by $(U_{\mathcal{P}}, \psi_{\mathcal{P}})$, $\psi_{\mathcal{P}}(p) = (x^\mu, u^a, \rho^\mu_a)_{\mu=0,\dots,d;\, a=1,\dots,n}$, with $p \in \mathcal{P}(E)$. Sections of τ_1 will be denoted by $\chi = (\phi^a, P^\mu_a)$, where P^μ_a are the momenta fields conjugate with the fields ϕ^a. We take actually a subset of suitably regular section admitting a Banach manifold structure, we refer to them as *dynamical fields* of the theory and we denote them as $\mathcal{F}_{\mathcal{P}}$. The particular field theory under investigation is specified by selecting an Hamiltonian function, namely, a real valued function on $\mathcal{P}(E)$, say $H(x, u, \rho)$. As it is explained in [4], when an Hamiltonian is fixed, the covariant phase space has a canonical $(d+1)$-form denoted by

$$\Theta_H = \rho^\mu_a \, \mathrm{d}u^a \wedge i_\mu vol_{\mathcal{M}} - H \, \nu_{\mathcal{M}},$$

[1] It is a space where each possible configuration of the dynamical system is represented by a point.
[2] We mean the invariance of the equations of the motion as well as the invariance of physical meaningful quantities.

where i_μ denotes the contraction with the vector field $\frac{\partial}{\partial x^\mu}$ and $\nu_{\mathcal{M}}$ is the volume form on $\mathcal{M}$.

The dynamical content of the theory is encoded in the Schwinger-Weiss variational principle. Trajectories are defined to be the critical points of the following action functional

$$\mathcal{S}[\chi] = \int_{\mathcal{M}} \chi^\star \Theta_H.$$

The critical points of $\mathcal{S}$ are those dynamical fields for which the variation of the action along any direction only depends on boundary terms. Let us clarify what we mean for "variation", "direction" and "boundary term". The space $\mathcal{F}_{\mathcal{P}}$ is a space of sections. A tangent vector at some "point" χ is defined [4] as a map $m \mapsto X^{(\chi)} \in \mathbf{V}_{\chi(m)}\mathcal{P}(E)$ for all $m \in \mathcal{M}$, namely, as a section of the pull-back bundle of $\mathbf{V}\mathcal{P}(E)$ via χ. Intuitively it is a collection of τ_1-vertical[3] tangent vectors at $\mathcal{P}(E)$ along the map χ. Let us denote as X an extension of $X^{(\chi)}$ to a (τ_1-vertical) vector field on $\mathcal{P}(E)$ defined on a neighborhood of the image of χ. Denote by F_s^X the local flow of X. Then, $\chi_s := F_s^X \circ \chi$ is a one-parameter family of sections of τ_1. The *variation* of $\mathcal{S}$ along the *direction* $X^{(\chi)}$ is defined to be

$$\delta_{X^{(\chi)}}\mathcal{S}[\chi] = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \int_{\mathcal{M}} \chi_s^\star \Theta_H = \int_{\mathcal{M}} \chi^\star \left(i_X \,\mathrm{d}\Theta_H\right) + \int_{\partial\mathcal{M}} \chi_{\partial\mathcal{M}}^\star \left(i_X \Theta_H\right), \tag{1}$$

where $\chi_{\partial\mathcal{M}} = \chi|_{\partial\mathcal{M}}$ is the dynamical field χ restricted to $\partial\mathcal{M}$[4]. The first term on the right hand side (r.h.s.) of (1) can be interpreted as the contraction of a differential one-form over $\mathcal{F}_{\mathcal{P}}$, that we denote by $\mathbb{EL}$ and call *Euler-Lagrange form*, with the tangent vector $X^{(\chi)}$. The second term in (1) is a *boundary term* in the sense that it only depends on the restriction of the dynamical fields to the boundary. We are going to denote the space of restrictions of dynamical fields to the boundary by $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$. Therefore, the second term on the r.h.s. can be interpreted as the pull-back of a differential form on $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$ via the restriction map $\Pi_{\partial\mathcal{M}} \colon \mathcal{F}_{\mathcal{P}} \to \mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$. We denote such a differential form by $\Pi_{\partial\mathcal{M}}^\star \alpha^{\partial\mathcal{M}}$, where $\alpha^{\partial\mathcal{M}}$ is a differential one-form on $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$. Thus, following the Schwinger-Weiss principle, trajectories are those χ for which the first term on the r.h.s. of (1) vanishes for any direction $X^{(\chi)}$. Then, the fundamental lemma of the calculus of variations implies that χ satisfies the following equations of the motion

$$\mathbb{EL}_\chi(X^{(\chi)}) = 0 \quad \forall X^{(\chi)} \in \mathbf{T}_\chi\mathcal{F}_{\mathcal{P}} \implies \chi^\star \left(i_X \,\mathrm{d}\Theta_H\right) = 0 \quad \forall X \in \mathfrak{X}^v\left(U_{\mathcal{P}}^{(\chi)}\right) \implies \begin{cases} \dfrac{\partial \phi^a}{\partial x^\mu} = \dfrac{\partial H}{\partial \rho_a^\mu}(\chi), \\ \dfrac{\partial P_a^\mu}{\partial x^\mu} = -\dfrac{\partial H}{\partial u^a}(\chi), \end{cases}$$

$U_{\mathcal{P}}^{(\chi)}$ being an open neighborhood of the image of χ. The space of solutions of the equations of the motion will be denoted by $\mathcal{EL}_{\mathcal{M}}$.

Now, we focus on the role of the differential form $\Pi_{\partial\mathcal{M}}^\star \alpha^{\partial\mathcal{M}}$ within the construction of the Poisson bracket on $\mathcal{EL}_{\mathcal{M}}$. First, its differential gives the following two-form on $\mathcal{F}_{\mathcal{P}}$ being, again, the pull-back of a two-form on $\mathcal{F}_{\mathcal{P}}^{\partial\mathcal{M}}$

$$\mathrm{d}\Pi_{\partial\mathcal{M}}^\star \alpha^{\partial\mathcal{M}}(X^{(\chi)}, Y^{(\chi)}) =: \Pi_{\partial\mathcal{M}}^\star \Omega^{\partial\mathcal{M}}(X^{(\chi)}, Y^{(\chi)}) = \int_{\partial\mathcal{M}} \chi_{\partial\mathcal{M}}^\star \left(i_X i_Y \,\mathrm{d}\Theta_H\right).$$

It can be proved that [1, 2]

Proposition 1. *$\mathcal{EL}_{\mathcal{M}}$ is an isotropic manifold for $\Pi_{\partial\mathcal{M}}^\star \Omega^{\partial\mathcal{M}}$.*

On the other hand, if we consider a block $\mathcal{M}_{12}$ in $\mathcal{M}$ whose boundary is made by two hypersurfaces Σ_1 and Σ_2 with opposite orientations, then $\Pi_{\partial\mathcal{M}}^\star \Omega^{\partial\mathcal{M}} = \Pi_{\Sigma_1}^\star \Omega^{\Sigma_1} - \Pi_{\Sigma_2}^\star \Omega^{\Sigma_2}$, where $\Pi_\Sigma^\star \Omega^\Sigma = \int_\Sigma \chi_\Sigma^\star \left(i_X i_Y \,\mathrm{d}\Theta_H\right)$, χ_Σ being the restriction to Σ of a dynamical field. Then, because of proposition 1, we have $\Pi_{\Sigma_1}^\star \Omega^{\Sigma_1}\big|_{\mathcal{EL}_{\mathcal{M}}} = \Pi_{\Sigma_2}^\star \Omega^{\Sigma_2}\big|_{\mathcal{EL}_{\mathcal{M}}}$. The same argument for any couple of hypersurfaces in $\mathcal{M}$ gives that the differential two-form $\Pi_\Sigma^\star \Omega^\Sigma$ on the space of restrictions of dynamical fields to a hypersurface Σ does not depend on the particular Σ if it is evaluated on solutions of the equations of the motion. We are going to denote the equivalence class of all these equivalent $\Pi_\Sigma^\star \Omega^\Sigma$ on $\mathcal{EL}_{\mathcal{M}}$, as $\Pi^\star \Omega$.

[3] They must be vertical in order to ensure their flow to lie in the space of sections.

[4] It is actually a section of the pull-back bundle of τ_1 via $\mathfrak{i}_{\partial\mathcal{M}}$, $\mathfrak{i}_{\partial\mathcal{M}}$ being the canonical immersion of $\partial\mathcal{M}$ into $\mathcal{M}$.

2. Construction of the bracket

The crucial point to obtain a Poisson bracket on $\mathcal{EL}_{\mathcal{M}}$ is that $\Pi^\star\Omega$ is a symplectic structure for theories where all constraints can be solved[5].

Proposition 2. *$\Pi^\star\Omega$ is a symplectic structure if all constraints can be solved.*

Proof sketch. Since $\Pi^\star\Omega$ does not depend on Σ, one can consider as Σ a hypersurface of Cauchy data for the equations on the motion. If all constraints can be solved, an existence and uniqueness theorem for the equations of the motion holds and, thus, the space of Cauchy data is diffeomorphic with the space of solutions, the diffeomorphism denoted by $\Phi_{\mathcal{EL}_{\mathcal{M}}}$. The structure $\Pi^\star\Omega$ restricted to the space of Cauchy data of the equations of the motion is $\Phi^\star_{\mathcal{EL}_{\mathcal{M}}}\Pi^\star_\Sigma\Omega^\Sigma$. Via a direct computation it is easy to prove that $\Phi^\star_{\mathcal{EL}_{\mathcal{M}}}\Pi^\star_\Sigma\Omega^\Sigma$ is symplectic. Therefore, the structure $\Pi^\star\Omega$ on the space of solutions is the pull-back via a diffeomorphism of a symplectic structure, thus, it is symplectic. ∎

With the symplectic structure $\Pi^\star\Omega$ in hand, a Poisson bracket on $\mathcal{EL}_{\mathcal{M}}$ can be defined in the usual way as

$$\{F, G\} = \Pi^\star\Omega(X_F, X_G) = \mathfrak{L}_{X_F}G,$$

F and G being functions on $\mathcal{EL}_{\mathcal{M}}$ and X_F being the Hamiltonian vector field associated with F w.r.t. $\Pi^\star\Omega$, i.e., the one satisfying $i_{X_F}\Pi^\star\Omega = \mathrm{d}F$.

We conclude by mentioning that an easier way to compute the Hamiltonian vector field , and, thus, the bracket, exists[6]. Indeed, the functional F can be restricted to the space of Cauchy data to $f = \Phi^\star_{\mathcal{EL}_{\mathcal{M}}}F$. To f a Hamiltonian vector field, say X_f, can be associated via the symplectic structure $\Phi^\star_{\mathcal{EL}_{\mathcal{M}}}\Pi^\star\Omega$, and this is much easier from the computational point of view. Then, it can be proved that the Hamiltonian vector field associated with the original functional F with respect to the structure $\Pi^\star\Omega$ can be recovered by solving the linearization of the equations of the motion with X_f as Cauchy datum.

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[5] For gauge theories the road to obtain a symplectic structure from this two-form is more involved and can not be treated here [3].
[6] We will give an extended proof in a more extensive work [3].

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A comparative study of robust regularization methods based on minimum density power and Rényi divergence losses

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Abstract: Over the last decades several regularization methods have been developed for sparse high-dimensional regression models. The influence of outliers is particularly awkward in the high dimensional context and so certain robust methods have been considered. Regularization methods simultaneously perform the model selection and the estimation of regression coefficients, merging a loss function based on the residuals and a penalty function inducing sparsity. Different penalties have been proposed, such as LASSO or Adaptive LASSO, a variant which improves the oracle model selection property, or non-concave penalties such as SCAD or MCP, which demostrably overcome the bias problem of the LASSO. We propose to examine robust losses with the various proposals for the penalties, leading to the differents estimating methods, namely the minimun density power divergence (DPD) and Rényi psedudodistance (RP) estimator penalized with LASSO, adaptative LASSO and SCAD. We develop an estimating algorithm for each method, focusing on their differences and similarities. Finally, we study the performance of the methods throught a simulation study.

Resumen: En las últimas décadas se han desarrollado varios métodos de regularización para el modelo lineal de regresión con datos de alta dimensión. La influencia de los datos atípicos en la estimación es particularmente perjudicial en el contexto de datos de alta dimensión, y por tanto se han considerado métodos robustos de estimación. Los métodos de regularización llevan a cabo simultáneamente la selección de variables y la estimación paramétrica mediante la combinación de una función de pérdida, basada en los residuos del modelo, y una función de penalización que induce la selección de variables. Han sido propuestas distintas penalizaciones como las penalizaciones LASSO y LASSO Adaptativo, una variante que mejora las propiedades oráculo del estimador, o penalizaciones no cóncavas como SCAD o MCP, que resuelven el problema de sesgo que presenta la penalización LASSO. Se propone examinar las pérdidas robustas con distintas funciones de penalización, dando lugar a distintos estimadores, a saber, el estimador de mínima potencia (DPD) y de mínima pseudodistancia de Rényi penalizado con LASSO, LASSO adaptativo y SCAD . Se desarrolla un algoritmo de estimación para cada método, señalando sus diferencias y similitudes. Por último, se estudia el comportamiento de los métodos a través de un estudio de simulación.

Keywords: high-dimensional linear regression models, adaptive LASSO estimator, density power divergence loss, Rényi pseudodistance, variable selection.

MSC2010: 62F35.

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1. Introducción

We consider the high-dimensional linear regression model (LRM) given by

$$Y_i = \boldsymbol{X}_i^T \boldsymbol{\beta} + U_i, \quad i = 1, \dots, n, \tag{1}$$

where $\boldsymbol{X}_i = (X_{i1}, \dots, X_{ip})^T$ are the explanatory variables or covariates, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^T \in \mathbb{R}^p$ is the vector of unknown regression coefficients and the U_is are random noise with $\boldsymbol{U} = (U_1, \dots, U_n) \in \mathbb{R}^n$ being normally distributed with null mean vector and variance covariance matrix $\sigma^2 \boldsymbol{I}_n$.

The term high-dimensional data is used when the number of explanatory variables, p, is greater than the number of observations by nonpolynomial dimensionality. On the other hand, sparse models are those whose number of true non-zero regression parameters is very low respect to the total number of covariates. This situation is accurate to real-life problems in several areas, such as genetics and genomic, bioinformatics, neuroimaging or chemometrics. Finally, it is known that contaminated data could worsen the estimation of the regression parameters. To avoid this issue, we need to develop robust estimating procedures. In this line, we are following the ideas by Castilla et al. (2020) [1] and Ghosh et al. (2020) [2].

The main awkward of the high dimensional regression models is the variable selection. As the number of possible models grows exponentially, information criteria are not suitable to choose the best model. Hence, regularization methods are clearly more convenient in these settings. Regularization methods introduce a penalty term, which penalizes the absolute value of the regression coefficients, on the objective function to achieve simultaneously model selection and parameter estimation. Regularization methods for sparse high-dimensional data analysis are characterized by loss functions measuring data fits and penalty terms constraining model parameters. In LRM, we estimate the parameter vector $(\boldsymbol{\beta}, \sigma) \in \mathbb{R}^{p+1}$ by minimizing an objective function of the form

$$Q_{n,\lambda}(\boldsymbol{\beta}, \sigma) = L_n(\boldsymbol{\beta}, \sigma) + \textstyle\sum_{j=1}^p p_{\lambda_n}\left(|\beta_j|\right), \tag{2}$$

which consists of a data fit functional $L_n(\boldsymbol{\beta}, \sigma)$, called loss function, and a penalty function $\sum_{j=1}^p p_{\lambda_n}\left(|\beta_j|\right)$, assessing the physical plausibility of $\boldsymbol{\beta}$ and controlling the complexity of the fitted model in order to avoid overfitting. A regularization parameter λ_n ($\lambda_n \geq 0$) regulates the penalty. From a practical point of view, the regularization parameter is chosen using some information criterion or by cross-validation.

The most common penalties are $p_{\lambda_n}(s) = s^2$ for Ridge estimator and $p_{\lambda_n}(s) = |s|$ for the LASSO estimator. The first one does not achieve model selection as it is unable to detect the null regression coefficients, but is more convenient when there is multicolinearity. Further, there have been several generalizations of the LASSO penalty yielding consistent estimator of the active set under much weaker conditions. In this vein, we also consider the Adaptative-LASSO and the SCAD (smoothly clipped absolute deviation) penalties.

Respect to the loss function, the most common is the least squares function obtained by the maximum likelihood criterion. The lack of robustness of this quadratic function is known, so it must be replaced by a robust loss so as to limit the impact of contamination in the data.

2. Robust losses

Let us consider the linear regression model (1) on which we assume that $Y|\boldsymbol{X} = \boldsymbol{x}$ follows a normal $\mathcal{N}\left(\boldsymbol{x}^T\boldsymbol{\beta}, \sigma^2\right)$ depending on the regression parameter, and let us consider a random sample $(Y_i, \boldsymbol{X}_i)_{1,\dots,n}$ from the model whose empirical distribution is G_n.

The minimum distance approach aims to minimize "some kind of measure of the distance or the divergence" between the proposed distribution of $Y|\boldsymbol{X} = \boldsymbol{x}$ and its empirical version. We use two of these measures of proximity between two distributions, namely the density power divergence (DPD) and the Rényi's pseudodistance (RP). These two measures take the following form for the linear regression model:

$$L_{n,\alpha}^{\mathrm{DPD}}(\boldsymbol{\beta}, \sigma) = \frac{1}{(2\pi)^{\alpha/2}\sigma^\alpha}\left(\frac{1}{\sqrt{\alpha+1}} - \frac{\alpha+1}{\alpha}\frac{1}{n}\sum_{i=1}^n \exp\left\{-\alpha\frac{\left(y_i - \boldsymbol{x}_i^T\boldsymbol{\beta}\right)^2}{2\sigma^2}\right\}\right) + \frac{1}{\alpha}, \tag{3}$$

$$L_{n,\alpha}^{\mathrm{RP}}(\boldsymbol{\beta}, \sigma) = \frac{1}{n}\sum_{i=1}^n -\sigma^{\frac{-\alpha}{\alpha+1}} \exp\left(-\frac{\alpha}{2}\left(\frac{y_i - \boldsymbol{x}_i^T\boldsymbol{\beta}}{\sigma}\right)^2\right), \tag{4}$$

where $f_{\boldsymbol{x}^T\boldsymbol{\beta},\sigma^2}$ denotes the normal density with mean $\boldsymbol{x}^T\boldsymbol{\beta}$ and variance σ^2. Note that both depend on a tunning parameter $\alpha > 0$ which controls the trade-off between efficiency and robustness. The minimun DPD estimator (MDPDE) $(\hat{\boldsymbol{\beta}}_\alpha^{\mathrm{DPD}}, \hat{\sigma}_\alpha^{\mathrm{DPD}})$ and the minimun RP estimator (MRPE) $(\hat{\boldsymbol{\beta}}_\alpha^{\mathrm{RP}}, \hat{\sigma}_\alpha^{\mathrm{RP}})$ are defined as the values $(\boldsymbol{\beta}, \sigma)$ minimizing (3) and (4), respectively. Even more, both measures can be defined at $\alpha = 0$ as the log-likelihood function taking continuous limit in α. Hence, both approaches include the maximum likelihood estimator (MLE) for the value $\alpha = 0$. From a practical point of view, the main difference between these measures lies in the estimation of σ^2.

3. Penalized MDPDE and MRPE.

The regularization methods based on DPD and RP losses are constructed by including a penalty term to the objective function so as to achieve simultaneously model selection and parameter estimation. Therefore, our objective function is $Q_{n,\alpha,\lambda}(\boldsymbol{\beta}) = \tilde{L}_{n,\alpha,\lambda}(\boldsymbol{\beta}) + \sum_{j=1}^p p_\lambda(|\beta_j|)$ for a robust loss $\tilde{L}_{n,\alpha,\lambda}(\boldsymbol{\beta})$ (DPD or RP loss) and a penalty function $p_\lambda(\cdot)$. We are considering three different penalties to compare their performance, namely LASSO, Adaptive LASSO and a non-concave penalty SCAD.

- LASSO panalty: $p_\lambda(\beta_j) = \lambda \sum_{j=1}^p |\beta_j|$.
- Adaptive LASSO penalty: $p_\lambda(\beta_j) = \lambda \sum_{j=1}^p \frac{1}{|\tilde{\beta}_j|} \cdot |\beta_j|$, where $\tilde{\boldsymbol{\beta}}$ is a robust estimate of $\boldsymbol{\beta}$.
- Non-concave penalty SCAD: $p_\lambda(|\beta_j|) = \begin{cases} \lambda|\beta_j| & \text{if } |\beta_j| \leq \lambda, \\ \frac{2a\lambda|\beta_j| - |\beta_j|^2 - \lambda^2}{2(a-1)} & \text{if } \lambda < |\beta_j| \leq a\lambda, \\ \frac{(a+1)\lambda^2}{2} & \text{if } a\lambda < |\beta_j|, \end{cases}$ where $a = 3.7$.

3.1. Robustness of the proposed estimators

Local robustness of an estimator can be studied through its influence function (IF). The IF measures the possible asymptotic bias in the estimation due to an infinitesimal contamination, and an estimator is said robust if its IF is bounded. We can verify that the IF of the proposed estimators is bounded for $\alpha > 0$ and non-bounded for $\alpha = 0$ corresponding to the MLE. Figure 1 shows the IF of the MDPDEs and MRPEs for univariate linear regression with $\sigma_0 = 1$, $x_t = 1$ and $\mathbb{E}[x^2] = 1$. The abscissa axis corresponds to the perturbation $u = y - x\beta$ and the ordinate axis corresponds to the IF value.

4. Estimating algorithm

The basic idea of our proposed algorithm is to iteratively minimize the objective $Q_n(\boldsymbol{\beta}, \sigma)$ in two steps: we first update the current solution of the regression parameter $\boldsymbol{\beta}$ and then we minimize the error deviance σ. For the first step, we combine MM-algorithm and coordinate descent algorithm, adapted to each situation, so as to update $\boldsymbol{\beta}$. As mentioned before, this update is similar for both proposed losses, DPD and RP. For the second step, we approximate a solution of the estimating equations of σ, obtained by equating the first derivative of the objective function to zero.

5. Simulation Study

We finally carry out a simulation study so as to evaluate the robustness and efficiency of the proposal penalized MDPDE MNPRPE under the LRM. We also estimate the regression parameters $(\boldsymbol{\beta}, \sigma)$ using other existing robust and non-robust methods of high-dimensional LRM to compare their performances with our proposed method. For each one of the estimators, we calculate the mean square error (MSE) for the true non-zero and zero coefficients separately, Absolute Prediction Bias using an unused test sample generated in the same way as train data, True Positive proportion, True Negative proportion and Model Size of the estimated regression coefficient $\hat{\boldsymbol{\beta}}$, and Estimation Error of the estimate $\hat{\sigma}$.

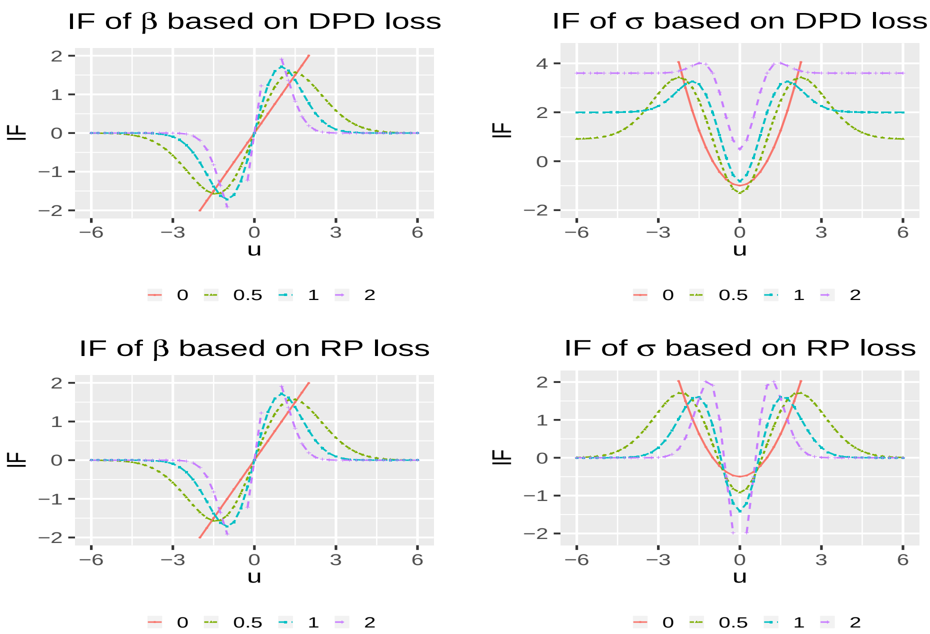

Figure 1: IF of the MDPDE for beta (upper left) and sigma (upper right), and IF of the MRPE for beta (bottom left) and sigma (bottom right).

Further, in order to examine the efficiency loss against non-robust methods in absence of any contamination, as well as compare the performance in the presence of contamination in the data, we consider different scenarios for data contamination, besides the pure data setting, including contaminating data in the responde variable Y and the explanatory variables $\boldsymbol{X}$.

The simulation results show the gain in robustness when the parameter α increases, as well as the improvement that the Adaptative LASSO and SCAD penalty entail for the variable selection. We conclude that the proposed estimators are very competitive to the classical MLE, and moreover, they perform better with contaminated data.

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3rd *BYMAT Conference* (2020)

Challenges in the nowadays, disabled people, race, income and COVID-19, under the mathematical and probabilistic

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Abstract: The term disability means a physical, intellectual or sensory permanent or temporary nature, which limits the ability to perform one or more activities. According to the WHO, it is estimated that about 15% of the world's population lives with some type of disability (2010 estimates). This number is lower than estimates from IBGE, of the same year, which indicate that approximately 23.9% of Brazil's population. Race is a social construction used to distinguish people in terms of one or more physical marks, among them color. Social inequality is the phenomenon that differentiates among people in the same society. As an aggravation of this whole scenario, in this year of 2020, the COVID-19 pandemic, caused by the SARS-CoV-2, with social, economic and health impacts unprecedented. The estimate of infected and dead competes directly with the impact on health systems, with the populations exposure, vulnerable groups and the economical system support. In mathematical and statistical terms, we intend to describe in this work relations between COVID-19 and different social inequality factors in moments before, during and after this pandemic.

Resumen: El término discapacidad significa una naturaleza física, intelectual o sensorial permanente o temporal, que limita la capacidad para realizar una o más actividades. Según la OMS, se estima que alrededor del 15 % de la población mundial vive con algún tipo de discapacidad (estimaciones de 2010). Esta cifra es inferior a las estimaciones del IBGE, del mismo año, que indican que aproximadamente el 23,9 % de la población brasileña. La raza es una construcción social que se utiliza para distinguir a las personas en términos de una o más marcas físicas, entre ellas el color. La desigualdad social es el fenómeno que diferencia a las personas de una misma sociedad. Como agravante de todo este escenario, en este año de 2020, la pandemia COVID-19, provocada por el SARS-CoV-2, con impactos sociales, económicos y de salud sin precedentes. La estimación de infectados y muertos compite directamente con el impacto en los sistemas de salud, con la exposición de las poblaciones, los grupos vulnerables y el apoyo al sistema económico. En términos matemáticos y estadísticos, pretendemos describir en este trabajo las relaciones entre COVID-19 y diferentes factores de desigualdad social en momentos antes, durante y después de esta pandemia.

Keywords: disability, COVID-19, demographic census, exploratory data, race.

MSC2010: 62P25.

Acknowledgements: The author thanks Professor Dr. Júlia Maria Pavan Soler for proposing the theme and IBGE for the availability of census data.

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1. Introduction

Worldwide, disabled people have worse health prospects, lower education level, lower economic participation and higher poverty rates compared to people without disabilities. This is partly due to the fact that disabled people face barriers to access services that many of us have long considered guaranteed, such as health, education, employment, transport and information. Such difficulties are exacerbated in the poorest communities [4].

Race can be understood as a social construct, used to distinguish people in terms of one or more physical marks. In other words, race is a category used to refer to a group of people whose physical marks are considered socially significant. Perceptions and conceptions of race can affect people's social lives, being mainly responsible for creating and maintaining a system of social inequality [2].

People's participation in the workforce is important for reasons such as maximizing human resources by increasing individual well-being, as well as promoting human dignity and social cohesion.

According to experts, the first cases of COVID-19 originated in the seafood market in the city of Wuhan located in China in December 2019 and the incidence increased exponentially in the first weeks.

This virus is believed to host certain species of bats and the pangolin; its incubation period is estimated at around 4 to 14 days; the rate of transmission is 2.75 individuals, and finally, the disease has an overall lethality of 3.4%. Infection with this virus has high rates of contagion with very rapid spread of cases, hospitalization in highly complex hospitals and high mortality rates.

In mathematical and statistical terms, it is intended, with the use of techniques of data analysis, to describe relationships between disabled people, race, income, decent work and COVID-19. In mathematical and statistical terms, probability is the study of the chances of obtaining each result of a random experiment. These chances are assigned real numbers in the range between 0 and 1. Conditional probability refers to the probability of an event A knowing that another B occurred.

To continue this work, in section 2 we will present materials and methods, in section 3 results and discussions, and finally, in section 4 the conclusions of this research.

2. Materials and methods

2.1. Motivation

In order to assess the effects of the epidemiology of COVID-19 in relation to factors such as disabled people, work, income, race and sex, among others, we will use the data from the 2010 IBGE Demographic Census of 20800804 respondents from the sample and aggregates for the 5565 municipalities together with the 2013 UNDP data.

In statistical terms, there are few published works that describe the effects of the COVID-19 epidemic in areas such as education, disability, income, economy, work, sex and social inequality, among others.

2.2. Social inequality

Social inequality, also called economic inequality, is a social problem present in all countries of the world and it is an economic difference that exists between certain groups of people within the same society. It stems mainly from the poor distribution of income and the lack of investment in the social area, such as education and health. In this way, the majority of the population is at the mercy of a minority that owns the resources, which generates inequalities.

The main causes of inequality are lack of investment in social, cultural, health and education areas; mismanagement of resources, poor distribution of income; market logic and, finally, corruption. Among the consequences generated by inequality it is possible to mention an increase in the rate of violence and crime; poverty and misery; delay in economic progress; famine, destruction and child mortality;

marginalization of young people; increase in the unemployment rate and, finally, formation of different socials classes.

The expansion of the COVID-19 epidemic in the slums, peripheries and interiors opened up the social and economic inequality between the naturalized social classes that are accepted by a large part of society and State institutions, which represents a barrier to the recommendations of basic hygiene, social detachment and staying at home.

According to health experts, the main problems pointed out are the need for special protection for groups in situations of vulnerability or at risk, such as people on the street, suffering or mental disorder, with disabilities, living with AIDS/HIV, LGBTI, indigenous, black, riverside, informal market workers; the lack of sanitation and housing conditions in the face of the pandemic; recommendations such as the use of alcohol gels and masks, hand hygiene and not leaving home are measures that come up against realities in the country, or in the absence of basic rights such as employment, health and housing, and, finally, it is not possible to develop the economy in the country without effective control of the pandemic.

3. Results and discussion

Before the COVID-19 pandemic, in regards of adequate housing, it was already possible to notice worrying scenarios with a group of vulnerable people on the rise formed by groups such as disabled people, black, brown and indigenous races, and joining these groups came the groups of people most prone to counter the COVID-19 virus such as the elderly, diabetics, hypertension, heart, cancer and respiratory diseases; greater worsening in economic terms with rising unemployment and greater difficulties in meeting the recommendations of health authorities in terms of prevention due to unavailability of resources such as water, electricity, gas, food, health and adequate housing conditions.

Figure 1 shows the distribution of people without disabilities by age group (I), with disabilities by age group (II), mortality by age group (III), clinical status of COVID-19 (IV), people without disabilities (V), people with disabilities (VI). The graphs in Figure 1 also show that COVID-19 affects more disabled people, older people and people working in worse conditions.

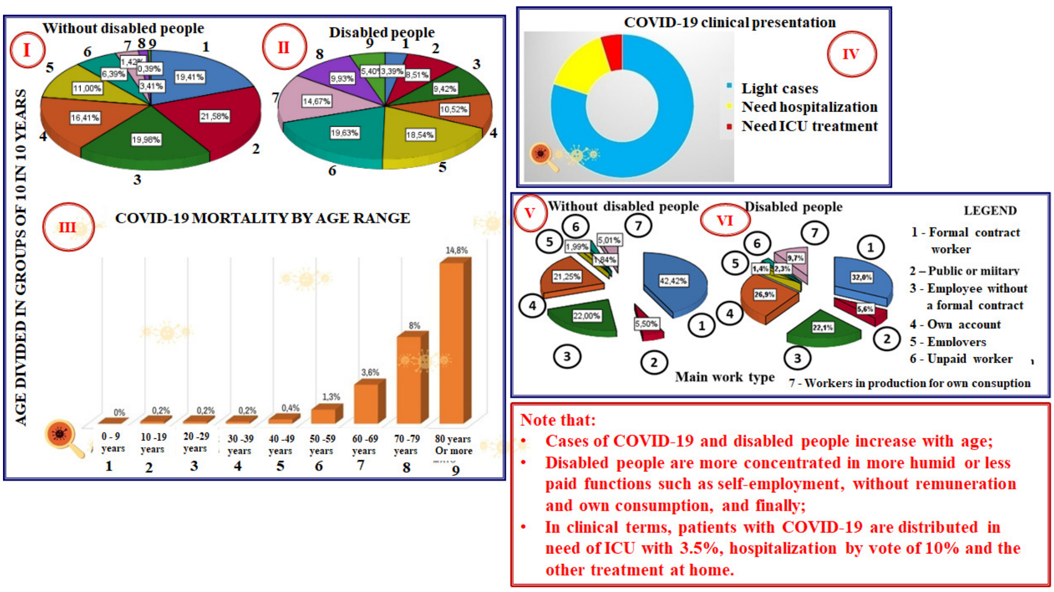

Figure 1: Disabled people distribution, age, work type and COVID-19 clinical internship.

Following, in Figure 2 it shows the distribution of residents by dormitories (1), access to drinking water (2), education level (3) and number of children (4). Analyzing the data in Figure 2, it appears that, proportionally, disabled people are more exposed to COVID-19 than people without disabilities. And first-time mothers have a greater number of children, a lower level of education, less access to drinking water, and a greater number of people per bedroom.

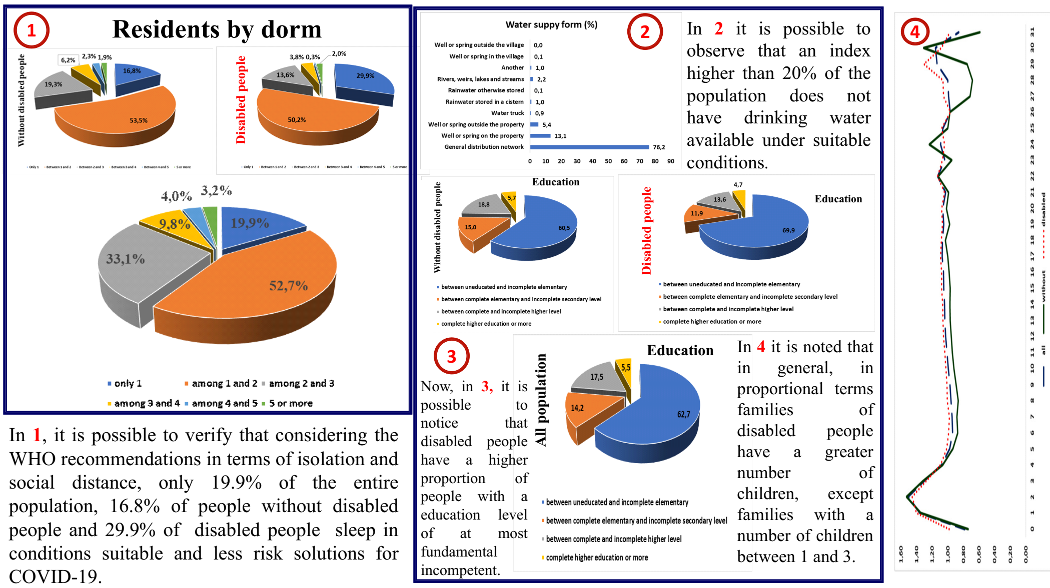

Figure 2: COVID-19, disabled and housing conditions.

4. Conclusions

Data from the 2010 IBGE Census show the disabled people predominance and of black, brown and indigenous races in worse working conditions, income, education and employment. With the emergence of the COVID-19 pandemic and the isolation policies as a solution to combat it, it became even more precarious for a good part of the population, further expanding economic and social inequality due to the increase in unemployment, mainly in informal and closing of many small and medium-sized companies, increasing the mass of vulnerable people in the population according to the results of studies carried out by several specialists in several areas, such as public health, economics, statistics, history, or medicine, among others. These results also show the mismanagement by authorities in establishing a more effective combat plan. These results are confirmed by the results of analyses made with data from the IBGE Demographic Census and with the results of several surveys carried out in Brazil and abroad.

Among the possible results stand out disabled people and other groups of people in more vulnerable situations were already in worse situations on issues such as education, health, housing conditions, income, work, leisure and many other things. These situations were aggravated after the beginning of the COVID-19 pandemic, since, in addition to these factors mentioned above, there is the need for isolation, greater hygiene, and having to satisfy new protocols developed by WHO.

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Distribution functions and probability measures on linearly ordered topological spaces

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Abstract: In this work we describe a theory of a cumulative distribution function (in short, cdf) on a separable linearly ordered topological space (LOTS) from a probability measure defined in this space. This function can be extended to the Dedekind-MacNeille completion of the space where it does make sense to define the pseudo-inverse. Moreover, we study the properties of both functions (the cdf and the pseudo-inverse) and get results that are similar to those which are well-known in the classical case.

Resumen: En este trabajo describimos una teoría sobre la función de distribución acumulada en un espacio topológico linealmente ordenado separable a partir de una medida de probabilidad definida en este espacio. Esta función se puede extender a la completación Dedekind-MacNeille del espacio donde tiene sentido definir la pseudo-inversa. Además, estudiamos las propiedades de ambas funciones (la función de distribución y la pseudo-inversa) y obtenemos resultados similares a los conocidos en el caso clásico.

Keywords: probability, measure, Dedekind-MacNeille completion, cumulative distribution function, linearly ordered topological space.

MSC2010: 60E05, 60B05.

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1. Introduction

This work collects some results on a theory of a cumulative distribution function (cdf) on a separable linearly ordered topological space (LOTS).

In [2], we described a theory of a cumulative distribution function on a separable linearly ordered topological space. Moreover, we showed that this function plays a similar role to that played in the classical case and studied its pseudo-inverse, which allowed us to generate samples of the probability measure that we used to define the distribution function. In [3], we extended a cdf defined on a separable linearly ordered topological space, X, to its Dedekind-MacNeille completion, $DM(X)$. That completion is, indeed, a compactification. Moreover, we proved that each function satisfying the properties of a cdf on $DM(X)$ is the cdf of a probability measure defined on $DM(X)$. Indeed, if X is compact, a similar result can be obtained in this context. Finally, the compactification $DM(X)$ lets us generate samples of a distribution in X. By following this research line, the next step is exploring some conditions on X such that, given a function F with the properties of a cdf, we can ensure that there exists a unique probability measure on X such that its cdf is F. This is completely developed in [4].

For further reference about the classical measure theory see, for example, [5].

2. Preliminaries: linearly ordered topological spaces

First, we recall the definition of a linear order:

Definition 1. A partially ordered set $(X, \leq)$ (that is, a set X with the binary relation $\leq$ that is reflexive, antisymmetric and transitive) is totally ordered if every $x, y \in X$ are comparable, that is, $x \leq y$ or $y \leq x$. In this case, the order is said to be total or linear. ◂

For further reference about partially ordered spaces see, for example, [1].

From the previous definition, we can talk about linearly ordered topological spaces:

Definition 2. A linearly ordered topological space (in short, LOTS) is a triple $(X, \tau, \leq)$ where $(X, \leq)$ is a linearly ordered set and where τ is the topology of the order. ◂

The topology of the order is defined as follows:

Definition 3. Let X be a set which is linearly ordered by $<$. We define the order topology τ on X by taking the sub-basis $\{\{x \in X : x < a\} : a \in X\} \cup \{\{x \in X : x > a\} : a \in X\}$. ◂

In the rest of this work, we will assume that X is a separable LOTS, μ is a probability measure on the Borel σ-algebra of X, $\sigma(X)$, and τ is the order topology in X.

3. Defining the cumulative distribution function

First we give the definition of the cumulative distribution function of a probability measure defined on X.

Definition 4. Given a probability measure μ on X, its cumulative distribution function (in short, cdf) is the function $F\colon X \to [0, 1]$ defined by $F(x) = \mu(\{a \in X : a \leq x\})$, for each $x \in X$. ◂

This function satisfies some properties that we collect next, and that are similar to those which are well-known in the classical theory of distribution functions:

Proposition 5. *Let μ be a probability measure on X, and F its cdf. Then,*

(i) *F is monotonically non-decreasing.*

(ii) *F is right τ-continuous.*

(iii) $\sup F(X) = 1$.

(iv) *If there does not exist* $\min X$*, then* $\inf F(X) = 0$.

Once we have defined a cdf and studied its properties, it does make sense to ask ourselves if, given a function, F, satisfying the properties collected above, there exists a probability measure μ on X such that its cdf, F_μ, is F. This is a question we will answer with the help of a structure we analyse in the next section.

Indeed, if we work with F we can get the measure of each interval in X.

Proposition 6. *If F is the cdf of a probability measure μ on X, then $\mu(\{x \in X : a < x \leq b\}) = F(b) - F(a)$, for each $a, b \in X$ such that $a < b$.*

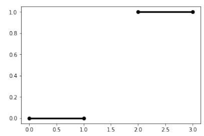
The next result is about the continuity of a cdf:

Proposition 7. *Let $x \in X$, μ be a probability measure on X and F its cdf. If $\mu(\{x\}) = 0$, then F is τ-continuous at x.*

However, the converse is not true. To show that, we include an example where the cdf of a probability measure on a space is a step cdf which is continuous with respect to τ.

Example 8. Let $X = [0,1] \cup [2,3]$ with the usual order. If μ is a probability measure defined on X by $\mu(\{2\}) = 1$, then its cdf is the function $F\colon X \to [0,1]$ defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{if } x \geq 2. \end{cases}$$



Note that F is a step cdf and it is τ-continuous. ◂

4. The extension of a cdf to the Dedekind-MacNeille completion

Definition 9. The Dedekind-MacNeille completion of a partially ordered set X is defined to be $DM(X) = \{A \subseteq X : A = (A^u)^l\}$ ordered by inclusion ($A \leq B$ if and only if $A \subseteq B$), where A^u (resp. A^l) is the set of upper (resp. lower) bounds of A. ◂

We also define $\phi\colon X \to DM(X)$ as the embedding given by $\phi(x) = (\{a \in X : a \leq x\})$, for each $x \in X$. For further reference about cuts and the Dedekind-MacNeille completion see, respectively, [6] and [7].

The next result gives us that each cdf on a LOTS can be naturally extended to its Dedekind-MacNeille completion.

Proposition 10. *$DM(X)$ is, indeed, a compactification of X and F can be extended to a cdf, $\widetilde{F}$, on $DM(X)$ by defining $\widetilde{F}\colon DM(X) \to [0,1]$ by $\widetilde{F}(A) = \inf F(A^u)$, for each $A \in DM(X)$.*

5. The pseudo-inverse of a cdf

Definition 11. Let F be a cdf. We define the pseudo-inverse of F by $G\colon [0,1] \to DM(X)$ given by $G(r) = \{x \in X : F(x) \geq r\}^l$, for each $r \in [0,1]$. ◂

This function satisfies some properties which are similar to those which are well-known in the classical case for the pseudo-inverse of a cdf, as we can see next:

Proposition 12. *The following hold:*

- (i) *G is monotonically non-decreasing.*
- (ii) *G is left τ-continuous.*
- (iii) *$G(r) \leq \phi(x)$ if and only if $r \leq F(x)$, for each $x \in X$ and each $r \in [0,1]$.*

6. Relationship between a probability measure and its cdf

Once we have defined and studied the main properties of a cdf and its pseudo-inverse, we answer the question made in Section 3 about the univocal relationship between a probability measure and its cdf in the context of separable LOTS. For that purpose, the main result is the next one:

Theorem 13. *Let X be a separable LOTS such that $DM(X) \setminus \phi(X)$ is countable and $F\colon X \to [0,1]$ be a monotonically non-decreasing and right τ-continuous function satisfying* $\sup F(X) = 1$ *and* $\sup F(A) = \inf F(A^u)$*, for each $A \in DM(X)$. Moreover,* $\inf F(X) = 0$ *if there does not exist the minimum of X. Then, there exists a unique probability measure on X, μ, such that $F = F_\mu$.*

What is more, the pseudo-inverse of a cdf is also univocally determined by its probability measure:

Theorem 14. *Let X be a separable LOTS such that $DM(X)\setminus\phi(X)$ is countable and let $G\colon [0,1] \to DM(X)$ be a monotonically non-decreasing and left τ-continuous function such that* $\sup G^{-1}(< A) = \inf G^{-1}(> A)$*, for each $A \in DM(X) \setminus \phi(X)$, $G(0) = \min DM(X)$, $G^{-1}(\max DM(X)) \subseteq \{1\}$ if there does not exist the maximum of X and $G^{-1}(\min DM(X)) = \{0\}$ if there does not exist the minimum of X. Then, there exists a unique probability measure on X, μ, such that G is the pseudo-inverse of F_μ.*

7. Applications

7.1. Generating samples

First, we can get the measure of each subset in the Borel σ-algebra of X from the pseudo-inverse of a cdf.

Proposition 15. *Let μ be a probability measure. Then, $\mu(A) = l(G^{-1}(A))$ for each $A \in \sigma(X)$, where l is the Lebesgue measure.*

That procedure lets us generate samples of a distribution, similarly to the classical procedure for distribution functions in the real line.

Remark 16. We can also calculate integrals with respect to μ, so, for $g : X \to \mathbb{R}$, it holds that

$$\int g(x)\,\mathrm{d}\mu(x) = \int g(G(t))\,\mathrm{d}t.$$ ◂

7.2. A goodness-of-fit test

In this subsection, we give a goodness-of-fit test whose idea is similar to the one followed by the Kolmogorov-Smirnov test, but in a more general context. Suppose that we are given a random sample on a separable LOTS according to a certain cumulative distribution function. Our purpose is testing if that distribution comes from a certain cdf F. Let us denote by F_n the empirical cumulative distribution function of the sample and define the statistic $D_n = \sup_{x\in X}|F_n(x) - F(x)|$, then the next statement holds.

Theorem 17. *Given a separable LOTS, X, and $n \in \mathbb{N}$, the distribution of D_n is the same for each cdf, F_μ, satisfying that $\mu(\{x\}) = 0$, for each $x \in X$.*

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3rd *BYMAT Conference* (2020)

Computing rotation numbers in the circle with a new algorithm

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Abstract: We present an efficient algorithm to compute rotation intervals of circle maps of degree one. It is based on the computation of the rotation number of a monotone circle map of degree one with a constant section. The main strength of this algorithm is that it computes *exactly* the rotation interval of a natural subclass of the continuous non-invertible degree one circle maps.

We also compare our algorithm with other existing ones by plotting the Devil's Staircase of a one-parameter non-differentiable family of maps, which is out of reach for the existing algorithms that are centred around differentiable maps.

Resumen: Presentamos un algoritmo eficiente para calcular el intervalo de rotación para aplicaciones en el círculo de grado 1. Está basado en el cálculo del número de rotación de aplicaciones en el círculo de grado 1 monótonas que tengan una sección constante. El punto fuerte de este algoritmo es que calcula el intervalo de rotacion de formula *exacta* para una subclasse natural de aplicaciones en el círculo continuas y no invertibles.

También compararemos nuestro algoritmo con otros existentes para dibujar la *Devil's Staircase* de una familia dependiente de un parametro no-diferenciable, fuera del alcance de los algoritmos existentes, centrados en funciones diferenciables.

Keywords: rotation number, circle maps, algorithm.

MSC2010: 37E10, 37E45, 68W25.

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1. Statement of the problem

This extended abstract basically summarizes the results in [1]. Most of the preliminary results can be found in [2].

We want to efficiently compute rotation intervals for degree one circle maps, the reason being the theoretical importance it plays on combinatorial dynamics. Many results, ranging from the exact set of periods of the maps to their entropy, use the rotation interval strongly. Now we will introduce the notion of rotation number and interval, and give some important properties relating degree one circle maps and their rotation numbers or intervals. First, let us introduce the notion of degree one map.

Definition 1 (degree one maps). Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map and let $F : \mathbb{R} \to \mathbb{R}$ be such that $\exp(2\pi x)\circ f = F\circ \exp(2\pi x)$. We will say that F *is a lifing of* f. We say that f is of degree 1 if $F(1)-F(0) = 1$. ◂

Note that there may be many liftings, but if F and F' are liftings of f, then $F = F' + k$, with $k \in \mathbb{Z}$, hence the property $F(1) - F(0)$ is independent of the choice of lifting. Now let us introduce the stars of the show, the rotation number and rotation interval.

Definition 2 (rotation number and rotation interval). Let f be a map of degree 1 and let F be a lifting. We will define the *rotation number* of F on $x \in \mathbb{R}$ as

$$\rho_F(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}.$$

Note that this number is dependent on x. Moreover we will define the *rotation set of F as*

$$\mathrm{Rot}(F) = \{\rho_F(x) \mid x \in \mathbb{R}\} = \{\rho_F(x) \mid x \in [0, 1]\},$$

which is an interval [3]. ◂

Now, let us study some some ways to infer the rotation number from the properties of F.

Lemma 3. *Let $F \in \mathcal{L}_1$. Then, x is an n-periodic* (mod 1) *point of F if and only if there exists $k \in \mathbb{Z}$ such that $F^n(x) = x + k$ but $F^j(x) - x \notin \mathbb{Z}$ for $j = 1, 2, \dots, n-1$. In this case,*

$$\rho_F(x) = \lim_{m\to\infty} \frac{F^m(x) - x}{m} = \frac{k}{n}.$$

Proposition 4. *Let $F \in \mathcal{L}_1$ be non-decreasing. Then, for every $x \in \mathbb{R}$ the limit*

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

exists and is independent of x. In this case we denote the rotation number of the map by ρ_F.

Using this proposition we will compute the rotation interval by just computing the rotation number of two non decreasing maps. However, first we need to introduce these special maps.

Definition 5. We set

$$F_l(x) = \inf\{F(y) : y \geq x\},$$
$$F_u(x) = \sup\{F(y) : y \leq x\},$$

where u stands for *upper* and l for *lower.* ◂

In Figure 1a we show an example of the upper and lower maps. Finally we can show a result relating the rotation interval with the well defined rotation number of two maps.

Theorem 6. *Let F be of degree 1. Then,*

$$\mathrm{Rot}(F) = [\rho_{F_l}, \rho_{F_u}].$$

2. Main result and new algorithm

For a real number x, we will denote the floor of x as $\lfloor x \rfloor$ and the decimal part function as $\{x\}$.

A *constant section of a lifting* F of a circle map is a closed non-degenerate subinterval K of $\mathbb{R}$ such that $F|_K$ is constant. In the special case when $F \in \mathcal{L}_1$, we have that $F(x+1) = F(x) + 1 \neq F(x)$ for every $x \in \mathbb{R}$. Hence, the length of K is less than 1.

The algorithm we propose is based on Lemma 8 but, especially, on the following simple proposition which allows us to compute *exactly* the rotation number of a non-decreasing lifting from $\mathcal{L}_1$ that has a constant section, provided that $F^n(K) \cap (K + \mathbb{Z}) \neq \varnothing$.

Proposition 7. *Let $F \in \mathcal{L}_1$ be non-decreasing and have a constant section K. Assume that there exists $n \in \mathbb{N}$ such that $F^n(K) \cap (K + \mathbb{Z}) \neq \varnothing$, and that n is minimal with this property. Then, there exists $\xi \in \mathbb{R}$ such that $F^n(K) = \{\xi\} \subset K + m$ with $m = \lfloor \xi - \min K \rfloor \in \mathbb{Z}$, ξ is an n-periodic* (mod 1) *point of F, and $\rho_F = \frac{m}{n}$.*

Proof. Since K is a constant section of F, $F(K)$ contains a unique point, and hence there exists $\xi \in \mathbb{R}$ such that $F^n(K) = \{\xi\}$. Then, the fact that $F^n(K) \cap (K + \mathbb{Z}) \neq \varnothing$ implies that $\xi \in K + m$ with $m = \lfloor \xi - \min K \rfloor \in \mathbb{Z}$. Set $\tilde{\xi} := \xi - m \in K$. Then, $\{F^n(\tilde{\xi})\} = F^n(K) = \{\tilde{\xi} + m\}$. Moreover, the minimality of n implies that $F^j(\tilde{\xi}) - \tilde{\xi} \notin \mathbb{Z}$ for $j = 1, 2, \dots, n-1$. So, Lemma 3 tells us that $\tilde{\xi}$ (and hence ξ) is an n-periodic (mod 1) point of F. Thus, $\rho_F = \frac{m}{n}$ by Proposition 4. ∎

Notice that this proposition gives us the backbone for an algorithm to compute rotation numbers for non-decreasing maps with a constant section. What remains to be checked is what happens if the iteration of the constant part K never falls again inside $K + \mathbb{Z}$, or the number of iterates that are required is too large to make it computationally practical. For this, we may use the following lemma.

Lemma 8. *For every non-decreasing lifting $F \in \mathcal{L}_1$ and $n \in \mathbb{N}$ we have*

$$\left| \rho_F - \frac{F^n(x) - x}{n} \right| < \frac{1}{n},$$

for every $x \in \mathbb{R}$.

2.1. Algorithm

From Proposition 7 and Lemma 8 we can obtain the following algorithm:

(i) Decide the maximum number of iterates $\mathsf{N} = \text{ceil}\left(\frac{1}{\texttt{error}}\right)$ to perform in the worst case (i.e., when Proposition 7 does not work).

(ii) Re-parametrize the lifting F so that it has a maximal constant section of the form $[0, \beta]$.

(iii) Initialize $x = 0$ and $m = 0$.

(iv) Compute iteratively $x = \{F^n(0)\}$ and $m = \lfloor F^n(0) \rfloor$ (so that $F^n(0) = x + m$) for $n \leq \mathsf{N}$.

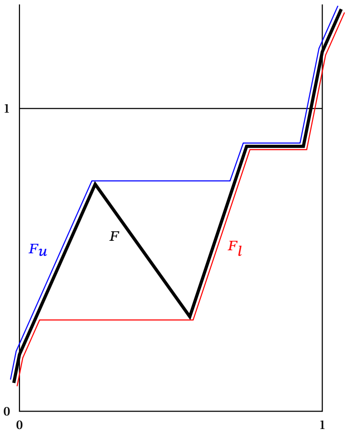
(v) Check whether $x \leq \beta$. On the affirmative we apply the previous proposition, and thus, $\rho_F = \frac{m}{n}$; ⇒ "exact" rotation number.

(vi) If we reach N iterates with $x > \beta$ for every n then, by the Lemma 8

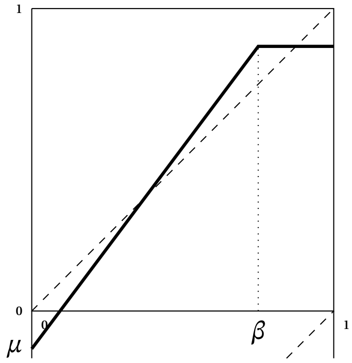
$$\left| \rho_F - \frac{m + x}{\mathsf{N}} \right| = \left| \rho_F - \frac{F^n(0)}{\mathsf{N}} \right| < \frac{1}{\mathsf{N}},$$

and the algorithm returns $\frac{m+x}{\mathsf{N}}$ as an estimate of ρ_F with $\frac{1}{\mathsf{N}}$ as the estimated error bound.

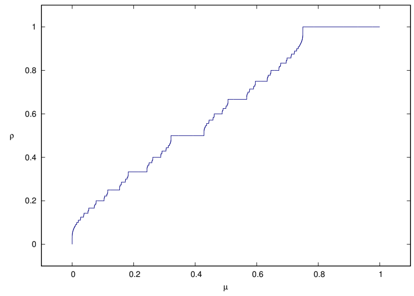
In [1], one can find a slightly more nuanced presentation of the algorithm, taking into account machine and rounding errors, but in spirit they are the same.



(a) An example of a map $F \in \mathcal{L}_1$ with its lower map F_l in red and its upper map F_u in blue.



(b) Plot of F_μ for a general μ



(c) Devil's Staircase plotted using the proposed algorithm

Figure 1: All the figures of the paper.

Table 1: Time taken by both the algorithms studied

Method	Time (s)
Classic	132.418015
Proposed Algorithm	0.003307

3. Testing of the algorithm

To test the algorithm we have plotted the *Devil's Staircase* for the one-parametric family of maps

$$F_\mu(x) = F_\mu|_{[0,1]}(\{x\}) + \lfloor x \rfloor.$$

See Figure 1b for a schematic plot. The so-called Devil's staircase is the result of plotting the rotation number as a function of the parameter μ. It can be proven that this plot will have constant sections for any rational rotation number, hence the "Staircase" in the name.

To conduct the test, we have plotted the Devil's Staircase for F_μ using the proposed algorithm and the algorithm stemming from Lemmas 4 and 8, which tells us that in the non decreasing case we can get the rotation number just by iterating and allow us to control the error. In Figure 1c one can find the plot of the Devil's Staircase plotted with our algorithm and in Table 1 the times each algorithm required to plot such figures. Moreover, the Arnol'd Tongues and the Rotation Intervals have also been computed in [1].

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Covariant reduction by fiberwise actions in classical field theory

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Abstract: Symmetries represent a central tool in the geometric analysis of mechanical systems. When a group of symmetries acts on the configuration space of a Lagrangian system, the quotient by this action of both the space and the variational principle is known as reduction. In the case of mechanics, this produces the well-known Lagrange-Poincaré equations, which have many applications in the literature.

In the realm of field theories, similar results have also been obtained. However, the typical nature of symmetries involved in the most relevant classical field theories is local and has not been addressed so far. In this case, symmetries are given by fiberwise actions of Lie group fiber bundles. The main instance of this situation are gauge theories.

The goal of this contribution is to determine the reduction procedure when a first order Lagrangian is invariant by a certain type of gauge group.

Resumen: El estudio de las simetrías constituye una herramienta fundamental en el análisis geométrico de los sistemas mecánicos. Cuando un grupo de simetrías actúa en el espacio de configuración de un sistema lagrangiano, el cociente por esta acción tanto del espacio, como del principio variacional es conocido como reducción. En el caso de la mecánica, esto da lugar a las ecuaciones de Lagrange-Poincaré, de las que se pueden encontrar muchas aplicaciones en la literatura.

En el contexto de las teorías de campos, se han obtenido resultados similares. Sin embargo, las simetrías involucradas en las teorías de campos más importantes son locales y no se han tratado todavía. En este caso, las simetrías están dadas por acciones fibradas de fibrados de grupos de Lie. El principal ejemplo de esta situación son las teorías gauge.

El objetivo de esta contribución es determinar el procedimiento de reducción cuando un lagrangiano de primer orden es invariante por un cierto tipo de grupo gauge.

Keywords: field theory, gauge group, Lagrangian, local symmetries, reduction.

MSC2010: 37J37, 53C05, 58D19.

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1. Introduction

In the context of Lagrangian mechanics, continuous (global) symmetries of physical systems emerge mathematically as actions of Lie groups on the configuration spaces. The reduction method consists of transferring the variational principle to the quotient space by the symmetry group, yielding the reduced equations when applied to specific Lagrangians. This has been thoroughly treated in the literature [2]. More recently, these ideas have been extended to field theories [1, 3].

The goal of the present contribution is to consider local symmetries of Lagrangian systems instead of global ones. In such case, the Lie group is replaced by an appropriate Lie group fiber bundle. Namely, we focus our attention on a particular type of symmetries whose derivatives are constrained to a prescribed Lie subalgebra. The ideas outlined here will be treated in detail and extended to more general symmetries in a forthcoming paper.

In the following, every manifold or map is assumed to be smooth in the sense of C^∞ unless otherwise stated. Likewise, every fiber bundle is assumed to be locally trivial. The superscript * will denote the dual bundle of the corresponding vector bundle.

2. Geometric setting

2.1. Actions of Lie group bundles and generalized principal connections

A Lie group bundle is a fiber bundle $\pi_{\mathcal{G},X}: \mathcal{G} \to X$ with typical fiber a Lie group G such that for any point $x \in X$ the fiber $\mathcal{G}_x = \pi_{\mathcal{G},X}^{-1}(\{x\})$ is equipped with a Lie group structure and there exist a neighborhood $\mathcal{U} \subset X$ and a diffeomorphism $\mathcal{U} \times G \to \pi_{\mathcal{G},X}^{-1}(\mathcal{U})$ preserving the Lie group structure fiberwisely. We denote by 1_x the identity element of $\mathcal{G}_x$ for each $x \in X$. Any Lie group bundle defines a Lie algebra bundle $\pi_{\mathfrak{g},X}: \mathfrak{g} \to X$ as the vector bundle whose fiber at each $x \in X$ is $\mathfrak{g}_x = T_{1_x}\mathcal{G}_x$, the Lie algebra of $\mathcal{G}_x$.

Definition 1. A *(right) fibered action* of a Lie group bundle $\mathcal{G} \to X$ on a fiber bundle $\pi_{Y,X}: Y \to X$ is a bundle morphism $\Phi: Y \times_X \mathcal{G} \to Y$ covering the identity id_X such that $\Phi(y, hg) = \Phi(\Phi(y, h), g)$ and $\Phi(y, 1_x) = y$, for all $(y, g), (y, h) \in Y \times_X \mathcal{G}$, $\pi_{Y,X}(y) = x$, where $\times_X$ denotes the fibered product. ◂

We denote the fibered action by $\Phi(y, g) = y \cdot g$ and the corresponding quotient by $Y/\mathcal{G}$. The action is said to be *free* if $y \cdot g = y$ for some $(y, g) \in Y \times_X \mathcal{G}$ implies that $g = 1_x$, $x = \pi_{Y,X}(y)$. In the same way, it is said to be *proper* if the bundle morphism $Y \times_X \mathcal{G} \ni (y, g) \mapsto (y, y \cdot g) \in Y \times_X Y$ is proper.

Proposition 2. *If $\mathcal{G} \to X$ acts on $Y \to X$ freely and properly, then $Y/\mathcal{G}$ admits a unique smooth structure such that $Y \to Y/\mathcal{G}$ is a fiber bundle with typical fiber G and $Y/\mathcal{G} \to X$ is a fibered manifold, i.e., a surjective submersion.*

Note that an Ehresmann connection (see [7]) on a Lie group bundle $\mathcal{G} \to X$ may be regarded as a map $\nu: T\mathcal{G} \to \mathfrak{g}$. It is natural to impose a compatibility of ν with the algebraic structure of $\mathcal{G}$.

Definition 3. A *Lie group bundle connection* on $\pi_{\mathcal{G},X}$ is an Ehresmann connection ν on $\pi_{\mathcal{G},X}$ satisfying:

(i) $\ker \nu|_{T_{1_x}\mathcal{G}} = (d1)_x(T_x X)$ for each $x \in X$, where $1: X \to \mathcal{G}$ is the unit section.

(ii) For every $(g, h) \in \mathcal{G} \times_X \mathcal{G}$ and $(U_g, U_h) \in T_g\mathcal{G} \times_{T_x X} T_h\mathcal{G}$, $x = \pi_{\mathcal{G},X}(g)$, then

$$\nu\left((dM)_{(g,h)}(U_g, U_h)\right) = \nu(U_g) + Ad_g\left(\nu(U_h)\right),$$

where $M: \mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$ is the multiplication map and $Ad_g: \mathfrak{g}_x \to \mathfrak{g}_x$ is the adjoint representation. ◂

In the same vein, we ask connections on $Y \to Y/\mathcal{G}$ to be equivariant. Note the analogy with principal connections (see [6]).

Definition 4. Let ν be an Lie group bundle connection on $\mathcal{G} \to X$. A *generalized principal connection* on $Y \to Y/\mathcal{G}$ associated to ν is a 1-form $\omega \in \Omega^1(Y, \mathfrak{g})$ satisfying:

(i) (Complementarity) $\omega_y(\xi_y^*) = \xi$ for every $(y,\xi) \in Y \times_X \mathfrak{g}$, where $\xi_y^* = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} y \cdot \exp(t\xi)$ is the *infinitesimal generator* of ξ at y.

(ii) (Ad-equivariance) For each $(y,g) \in Y \times_X \mathcal{G}$ and $(U_y, U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$, $x = \pi_{Y,X}(x)$, then

$$\omega_{y\cdot g}\left((d\Phi)_{(y,g)}(U_y, U_g)\right) = Ad_{g^{-1}}\left(\omega_y(U_y) + \nu(U_g)\right). \qquad ◂$$

Recall that the *curvature* of ω (see for example [7, §9.4]) is the 2-form $\Omega \in \Omega^2(Y,\mathfrak{g})$ defined as

$$\Omega(U_1, U_2) = -\omega\left([U_1 - \omega(U_1)^*, U_2 - \omega(U_2)^*]\right), \qquad U_1, U_2 \in \mathfrak{X}(Y).$$

2.2. Geometry of the reduced configuration space

Let $\mathcal{G} \to X$ be a Lie group bundle. Then, $\pi_{J^1\mathcal{G},X} : J^1\mathcal{G} \to X$ is also a Lie group bundle (see [4, §3, Th. 1]). We take a *Lie group subbundle* of $\pi_{J^1\mathcal{G},X}$, that is, a Lie group bundle $\pi_{H,X} : H \to X$ such that

(i) H is a submanifold of $J^1\mathcal{G}$,

(ii) H_x is a Lie subgroup of $J^1_x\mathcal{G}$ for each $x \in X$.

We also assume that H_x is closed in $J^1_x\mathcal{G}$ for every $x \in X$, $\pi_{J^1\mathcal{G},\mathcal{G}}(H) = \mathcal{G}$ and $\pi_{H,\mathcal{G}}$ is an affine subbundle of $\pi_{J^1\mathcal{G},\mathcal{G}}$. A Lie group connection ν on $\mathcal{G} \to X$ gives an identification of the first jet bundle $J^1\mathcal{G}$ with the vector bundle modelling it, $J^1\mathcal{G} \simeq \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$. Under this identification, we suppose that $H = \mathcal{G} \times_X (T^*X \otimes \mathfrak{h})$, for certain Lie algebra subbundle $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}_x$ is an ideal of $\mathfrak{g}_x$ for every $x \in X$.

It can be seen that $\mathcal{G} \times_X \mathfrak{h} \to X$ acts fiberwisely, freely and properly on the right on $Y \times_X \mathfrak{g} \to X$. Moreover, the corresponding quotient $ad_{\mathfrak{h}}(Y) = (Y \times_X \mathfrak{g})/(\mathcal{G} \times_X \mathfrak{h})$ is a vector bundle over $Y/\mathcal{G}$, whose elements are denoted by $[\![y,\xi]\!] \in ad_{\mathfrak{h}}(Y)$.

On the other hand, the first jet extension of the fibered action, i.e., $\Phi^1 : J^1Y \times_X J^1\mathcal{G} \to J^1Y$, turns out to be a right fibered action of $J^1\mathcal{G} \to X$ on $J^1Y \to X$ that can be restricted to H.

Theorem 5. *In the above conditions, let $\omega \in \Omega^1(Y,\mathfrak{g})$ be a generalized principal connection on $Y \to Y/\mathcal{G}$ associated to ν. Then, the following map is a fibered isomorphism over $Y/\mathcal{G}$:*

$$J^1Y/H \ni \left[j^1_x s\right]_H \longmapsto \left(j^1_x\left(\pi_{Y,Y/\mathcal{G}} \circ s\right), [\![s(x), (s^*\omega)_x]\!]\right) \in J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} \left(T^*X \otimes ad_{\mathfrak{h}}(Y)\right).$$

The connections ω and ν induce a linear connection $\nabla^{\mathfrak{h}}$ on $ad_{\mathfrak{h}}(Y) \to Y/\mathcal{G}$. Provided a linear connection ∇^X on TX, we get a linear connection ∇ on $T^*X \otimes ad_{\mathfrak{h}}(Y) \to Y/\mathcal{G}$. Likewise, a torsion free linear connection on $T(Y/\mathcal{G}) \to Y/\mathcal{G}$ projectable to ∇^X (see [5]) induces an affine connection on $J^1(Y/\mathcal{G}) \to Y/\mathcal{G}$. Hence, we obtain an affine connection on the reduced space.

3. Reduction of the variational principle

A *(first order) Lagrangian density* on a fiber bundle $Y \to X$ is a bundle morphism $\mathcal{L} : J^1Y \to \bigwedge^n T^*X$ covering the identity on X, where $n = \dim X$. Assuming that X is orientable and $v \in \Omega^n(X)$ is a volume form, we can write $\mathcal{L} = L\,v$ for certain $L : J^1Y \to \mathbb{R}$ called *Lagrangian.*

Let $\mathcal{G} \to X$ be a Lie group bundle acting freely and properly on $Y \to X$ and $H \subset J^1\mathcal{G}$ be a Lie subbundle as in Section 2.2. If the Lagrangian L is *H-invariant*, that is, $L\left(\Phi^1\left(j^1_x s, j^1_x\eta\right)\right) = L\left(j^1_x s\right)$ for each $\left(j^1_x s, j^1_x \eta\right) \in J^1Y \times_X H$, then the *reduced Lagrangian*, $l : J^1Y/H \to \mathbb{R}$, is well defined. Using a generalized principal connection, Theorem 5 enables us to regard l as defined on $J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} \left(T^*X \otimes ad_{\mathfrak{h}}(Y)\right)$.

The principle of stationary action used to obtain the Euler-Lagrange equations can be transferred to the reduced configuration space. When applied to l, the so-called reduced equations are obtained.

Theorem 6 (Reduced field equations). *Let $\mathcal{U} \subset X$ be an open set such that $\overline{\mathcal{U}}$ is compact. Let $s \in \Gamma\left(\overline{\mathcal{U}}, \pi_{Y,X}\right)$ and $\sigma_s = \pi_{Y,Y/\mathcal{G}} \circ s$, and consider the* reduced section $\overline{s} = [\![s, s^*\omega]\!]$. *Then, the following assertions are equivalent:*

(i) *The variational principle $\delta \int_{\mathcal{U}} L(j^1 s)\, v = 0$ holds for vertical variations of s such that $\delta s|_{\partial \mathcal{U}} = 0$.*

(ii) *The section s satisfies the Euler-Lagrange equations for L, i.e., $\mathcal{EL}(L)\left(j^1 s\right) = 0$ (see* [1, Section 2.4]*).*

(iii) *The variational principle $\delta \int_{\mathcal{U}} l(j^1\sigma_s, \overline{s})\, v = 0$ holds for variations of the form*

$$\left(\delta\overline{s}\right)^v = \overline{\nabla}^{\mathfrak{h}} \overline{\eta} - \left[\overline{s}, \overline{\eta}\right] + \sigma_s^* \widetilde{\Omega}\left(\delta\sigma_s, \cdot\right),$$

with $\overline{\eta} \in \Gamma\left(\overline{\mathcal{U}}, \pi_{ad_{\mathfrak{h}}(Y),X}\right)$ arbitrary such that $\pi_{ad_{\mathfrak{h}}(Y),Y/\mathcal{G}} \circ \overline{\eta} = \sigma_s$ and $\overline{\eta}|_{\partial\mathcal{U}} = 0$, and $\delta\sigma_s$ arbitrary vertical variation of σ_s such that $\delta\sigma_s|_{\partial\mathcal{U}} = 0$.

(iv) *The reduced section $\overline{s}$ satisfies the following* reduced field equations

$$\frac{\delta l}{\delta \sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1\sigma_s}\right) = \left\langle \frac{\delta l}{\delta \overline{s}}, \widetilde{\Omega}\left(d\sigma_s, \cdot\right)\right\rangle, \qquad \operatorname{div}^{\mathfrak{h}}\left(\frac{\delta l}{\delta \overline{s}}\right) - \operatorname{ad}^*_{\overline{s}}\left(\frac{\delta l}{\delta \overline{s}}\right) = 0.$$

To conclude, let us define the objects that appear in the equations. First, $\widetilde{\Omega} \in \Omega^2(Y/\mathcal{G}, ad_{\mathfrak{h}}(Y))$ is the *reduced curvature* of ω, which is given by $\widetilde{\Omega}_{[y]_\mathcal{G}}\left(u_1, u_2\right) = [\![y, \Omega_y\left(U_1, U_2\right)]\!]$ for each $[y]_\mathcal{G} \in Y/\mathcal{G}$ and $u_1, u_2 \in T_{[y]_G}(Y/\mathcal{G})$, where $U_1, U_2 \in T_y Y$ project to u_1, u_2, respectively. On the other hand, $\operatorname{div}^{\mathfrak{h}}$ and $\operatorname{div}^{Y/\mathcal{G}}$ are the *divergence* of $\nabla^{\mathfrak{h}}$ and $\nabla^{Y/\mathcal{G}}$, respectively, that is, minus the adjoint of those linear connections. In the same manner, the coadjoint representation of $\mathfrak{g}$ is naturally extended to a map

$$\operatorname{ad}^* : \Gamma(T^*X \otimes ad_{\mathfrak{h}}(Y)) \times \Gamma(TX \otimes ad_{\mathfrak{h}}(Y)^*) \longrightarrow \Gamma(ad_{\mathfrak{h}}(Y)^*).$$

At last, the *partial derivatives* of the reduced Lagrangian are the sections

$$\frac{\delta l}{\delta \sigma_s} \in \Gamma\left(T^*(Y/\mathcal{G})\right), \quad \frac{\delta l}{\delta j^1\sigma_s} \in \Gamma\left(TX \otimes V^*(Y/\mathcal{G})\right), \quad \frac{\delta l}{\delta \overline{s}} \in \Gamma\left(TX \otimes ad_{\mathfrak{h}}(Y)^*\right).$$

The first one is the horizontal derivative using the affine connection on the reduced space, whereas the latter ones are fiber derivatives.

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Recent results and open problems in spectral algorithms for signed graphs

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Abstract: In signed graphs, edges are labeled with either a positive or a negative sign. This small modification greatly enriches the representation capabilities of graphs. However, their spectral properties undergo significant changes, introducing new challenges in related optimization problems. In this extended abstract we discuss recent results in spectral methods for signed graph partitioning and community detection, and propose open problems arising in this context.

Resumen: En grafos con signos, las aristas se etiquetan con un signo positivo o un signo negativo. Esta pequeña modificación enriquece notablemente las capacidades de representación de los grafos. Sin embargo, sus propiedades espectrales sufren cambios significativos, lo que introduce nuevos desafíos en problemas de optimización relacionados. En este resumen extendido hablaremos de resultados recientes en métodos espectrales para la partición y detección de comunidades en grafos con signos, y propondremos problemas abiertos que surgen en este contexto.

Keywords: spectral graph theory, signed graphs.

MSC2010: 05C85.

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1. Introduction

A signed graph can be characterized by the triple $G = (V, E, \sigma)$, with vertex set V, edge set $E \subseteq V \times V$ and signature $\sigma : E \to \{-, +\}$. We consider G to be undirected, and thus $(i, j) \in E$ if and only if $(j, i) \in E$ and $\sigma(i, j) = \sigma(j, i)$ (we omit parentheses for clarity of exposition).

We focus on the spectral analysis of signed graphs, which concerns associated matrices. Given a signed graph $G = (V, E, \sigma)$, we define the adjacency matrix $A = (a_{ij})$, $i, j \in V$, where $a_{ij} = 0$ if $(i, j) \notin E$. Otherwise, $a_{ij} = 1$ if $\sigma(i, j) = +$ and -1 if $\sigma(i, j) = -1$.

We will discuss two recent results from the computer science literature, involving the detection of subgraphs with particular characteristics. The problems, as formulated, are hard to optimize and reveal key differences with respect to their unsigned counterparts. This leads to further questions which we formulate as open problems.

2. Results in spectral signed graph partitioning

2.1. Partitioning and community detection

The first result we discuss was found by Bonchi et al. [1], and it involves a randomized algorithm to simultaneously find and partition a dense subgraph with approximation guarantees. In particular, the problem in question is formulated as follows:

Problem 1. Given a signed graph with $n \times n$ adjacency matrix A find

$$\max_{x \in \{-1,0,1\}^n \setminus \{\mathbf{0}\}} \frac{x^T A x}{x^T x},$$

where $\mathbf{0}$ denotes the null vector in $\mathbb{R}^n$. This problem is akin to finding the densest subgraph, measured by average degree. However, we aim not only to find a subgraph with high average degree, but also that we can approximately partition in accordance to the edge signs; i.e., so that most edges traversing the boundary of the partition are negative. Contrary to its unsigned counterpart, this problem is NP-hard [1].

The main result of the cited work can be stated as follows:

Theorem 2. *There exists a randomized polynomial-time algorithm that outputs a vector $x \in \{-1, 0, 1\}^n \setminus \{\mathbf{0}\}$ satisfying*

$$\frac{x^T A x}{x^T x} \geq \Omega(n^{-1/2}) \lambda_1,$$

where λ_1 is the largest eigenvalue of A.

The result is tight, as there exist graphs where the gap between the optimal vector and λ_1 matches the bound [2].

2.2. Partitioning into an arbitrary number of groups

The next result we discuss is an extension of the work mentioned above —which is limited to a two-way partition of the detected subgraph— to handle an arbitrary number of subgraphs. This extension was accomplished in subsequent work by Tzeng et al. [2]. By defining the sets $E_+(G) = \{e \in E : \sigma(e) = +\}$ and $E_-(G) = \{e \in E : \sigma(e) = -\}$, we can simultaneously quantify the density and the quality of a partition using the following function, mapping collections of k disjoint vertex subsets to the reals:

$$f(S_1, \dots, S_k) = \frac{\sum_{h \in [k]} (|E_+(S_h)| - |E_-(S_h)|) + \frac{1}{k-1} \sum_{h \neq l \in [k]} (|E_-(S_h, S_\ell)| - |E_+(S_h, S_\ell)|)}{|\bigcup_{h \in [k]} S_h|}.$$

We abuse the notation and use vertex subsets in lieu of the corresponding induced subgraphs. $E_+(S_i, S_j)$ (resp. $E_-(S_i, S_j)$) denotes the set of positive (resp. negative) edges with one endpoint in S_i and the other in S_j. $[k]$ is the set $\{1, \dots, k\}$.

We can thus formulate the problem as follows. Given a signed graph G with n vertices, the goal is to find k disjoint vertex subsets, for a given $0 \le k \le n$, attaining the following optimum:

$$\max_{S_1,\dots,S_k} f(S_1, \dots, S_k). \tag{1}$$

Some manipulations reveal that the numerator of the above problem is equivalent to the following quantity:

$$\frac{\langle A, XL_kX^T\rangle}{k-1},$$

where

- $\langle A, B\rangle$ is the Frobenius product of matrices A and B,
- $L_k = kI_k - J_k$,
- I_k is the identity matrix of order k,
- J_k is the all-ones square matrix of order k, and
- $X \in \{0, 1\}^{n\times k}$ is a vertex-subset indicator matrix, so that $x_{ij} = 1$ if $i \in S_j$, $x_{ij} = 0$ otherwise.

The key insight now is that L_k has a $(k-1)$-dimensional invariant subspace, which enables the design of effective algorithms. This is because we can choose the eigenvectors so that they resemble a discrete structure. In particular, let $L_k = UDU^T$, $Y = XU$. Then, U can be chosen as follows:

$$\begin{array}{lll} (U_{:,1})^T = 1/\sqrt{k}\,[1, \dots, 1], & & (U_{:,2})^T = c_1\,[k-1, -1, \dots, -1], \\ (U_{:,3})^T = c_2\,[0, k-2, -1, \dots, -1], & \dots & (U_{:,k})^T = c_{k-1}\,[0, \dots, 0, 1, -1]. \end{array} \tag{2}$$

A similar analysis of the cardinality of the union of the chosen sets leads to the final formulation:

$$\max_{Y\in\mathbb{R}^{n\times(k-1)}\setminus\{0\}} \frac{\mathrm{Tr}(Y^TAY)}{\mathrm{Tr}(Y^TY)} \quad \text{subject to} \quad Y_{i,j} = \begin{cases} c_j(k-j) & \text{if } i \in S_j, \\ 0 & \text{if } i \in \cup_{h=1}^{j-1} S_h \text{ or } i \notin \cup_{h\in[k]} S_h, \\ -c_j & \text{if } i \in \cup_{h=j+1}^{k} S_h. \end{cases}$$

This can be approximately optimized with approximation guarantees. In particular, the authors provide the next result [2]:

Theorem 3. *There exists a randomized polynomial-time algorithm that outputs a collection of vertex subsets $S_1, \dots, S_k$ satisfying $f(S_1, \dots, S_k) \ge \Omega\left((k\sqrt{n})^{-1}\right) OPT$,*

where OPT is the maximum of (1) over all collections of k disjoint vertex subsets.

3. Open problems

The above results highlight interesting differences arising in spectral theory when signs are introduced in graphs. Most notably, what is arguably a natural extension of the densest subgraph problem formulation becomes hard to optimize. This suggests the problem of identifying the conditions under which Problem 1 can be solved in polynomial time. We thus propose the following problem:

Problem 4. Characterize signed graphs for which Problem 1 can be solved in polynomial time. ◂

Since unsigned graphs can be seen as a special case of signed graphs, we know that the above family is non-empty. We can further extend this family by simply taking into account the spectral equivalence between unsigned and balanced graphs [3].

An additional question arises from empirical results presented in the works cited above. Despite the hardness of the problem formulations and the tightness of the given bounds, spectral algorithms usually work well in practice. This suggests the existence of a family of problem instances in which the value of the objective is not too far detached from the maximum eigenvalue. Thus, we formulate the following question:

Problem 5. Characterize signed graphs for which

$$\max_{x\in\{-1,0,1\}^n\setminus\{\mathbf{0}\}} \frac{x^TAx}{x^Tx} = \frac{\lambda_1}{o\left(\sqrt{n}\right)}.$$ ◂

That is, we aim to identify the graphs that allow us to attain approximations significantly better than those described above.

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Global stability and sensitivity of an SQIR model with infectivity during quarantine

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Abstract: To measure the impact of isolating individuals infected with direct transmission diseases a compartmental model was obtained and validated mathematically with respect to qualitative behavior.

First, the basal number R_0 was determined through Van den Driessche-Watmough's method, because that parameter sets the stability of the two equilibrium points. Then, using Liapunov's theory, it was proven that when $R_0 < 1$ the disease-free equilibrium is globally asymptotic stable and the endemic equilibrium is globally asymptotic stable when $R_0 > 1$, has a forward bifurcation in $R_0 = 1$ and is unstable otherwise.

Finally, sensitivity analysis was done through numerical simulations, changing the parameters and analyzing the curve of infected. Increasing the number of infected that enter quarantine and reducing the contagion rate both lead to a significant reduction in the number of cases, however the curve was flattened only in the second case, therefore it is expected to be more effective.

Resumen: Para medir el impacto de aislar individuos infectados con enfermedades de transmisión directa se obtuvo un modelo compartimental y se validó matemáticamente con respecto al comportamiento cualitativo.

En primer lugar, se determinó el número basal R_0 mediante el método de Van den Driessch-Watmough, ya que ese parámetro establece la estabilidad de los dos puntos de equilibrio. Luego, utilizando la teoría de Liapunov, se demostró que cuando $R_0 < 1$ el equilibrio libre de enfermedad es globalmente asintótico estable y el equilibrio endémico es globalmente asintótico estable cuando $R_0 > 1$, tiene una bifucación hacia adelante en $R_0 = 1$ y es inestable en caso contrario.

Por último, se ha realizado un análisis de sensibilidad mediante simulaciones numéricas, cambiando los parámetros y analizando la curva de infectados. Tanto el aumento del número de infectados que entran en cuarentena como la reducción de la tasa de contagio conducen a una reducción significativa en el número de casos; sin embargo, la curva se aplanó solo en el segundo caso, por lo que se espera que sea más eficaz.

Keywords: SQIR model, global stability, sensitivity analysis.

MSC2010: 34D23.

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1. Introduction

Although compartmental models have been a fundamental tool in epidemiology for decades, the necessity for development in this area has become very evident recently due to the COVID-19 pandemic. One tendency in recent works concerning such models is to incorporate different types of isolation into the classical models. For example, some results where made in [2] to cases where the isolation is given as a function of the distancing between susceptibles. More recently, [1] considered non-lethal diseases in which a portion of the susceptible is isolated and the incubation period is infectious.

In this work, we will be concerned with the isolation of the infected while they are in treatment. The Centers for Disease Control and Prevention of the USA recommend this practice for cases of HIV, Tuberculosis, Rubella, Chickenpox, among others [3] and the World Health Organization and many other agencies adopted this method to handle the COVID-19 [4]. Most of these diseases are lethal and some health professionals and patients can end up infected by those isolated patients due to flexibilization of isolating norms.

2. SQIR model

For the modeling, the main aspects were that individuals might recover or die after being infected, some of them get in quarantine while receiving medical care and that the quarantine is imperfect, changing recovery and deaths rate. The following model was obtained considering all this.

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \nu - \beta_1 SQ - \beta_2 SI - \mu S, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \gamma I - (\mu + \alpha_1 + \rho_1)Q, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta_1 SQ + \beta_2 SI - (\mu + \gamma + \alpha_2 + \rho_2)I, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \rho_1 Q + \rho_2 I - \mu R. \end{cases}$$

The parameters are values between 0 and 1. Their meaning is shown in Table1.

Table 1: Summary of all the parameters.

α_1	mortality rate in quarantine
α_2	mortality rate without quarantine
β_1	infection rate in quarantine
β_2	infection rate without quarantine
γ	rate of infected people that go to quarantine
μ	natural deaths (not caused by the disease)
ν	natality and migration rate
ρ_1	recovery rate in quarantine
ρ_2	recovery rate without quarantine

The model dynamic is determined mainly by the domain of the functions S, Q, I, R and the basic reproduction number R_0, which tell us how many people one infected individual will contaminate.

A feasible positively invariant region for the model is

$$D = \left\{ (S, Q, I, R) \in \mathbb{R}^4 \,:\, 0 < S \leq \frac{\nu}{\mu}, 0 \leq Q, 0 \leq I, 0 \leq R \right\}.$$

To determine a suitable R_0, first the substitutions $\delta = \mu + \alpha_1 + \rho_1$, $\omega = \mu + \gamma + \alpha_2 + \rho_2$ were made, then the compartments with the disease were divided into two matrices F and V such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} Q \\ I \end{bmatrix} = F - V = \begin{bmatrix} \gamma I \\ \beta_1 SQ + \beta_2 SI \end{bmatrix} - \begin{bmatrix} \delta Q \\ \omega I \end{bmatrix}.$$

Then, the value of R_0 was established using Van der Driessch-Watmough's method.

$$R_0 = \rho(J(F)J^{-1}(V)) = \frac{\nu(\gamma\beta_1 + \delta\beta_2)}{\mu\omega\delta};$$

here, $\rho(X)$ and $J(X)$ represent the spectral radius and the Jacobian matrix of X, respectively.

3. Results

Through the calculations, it was found that, when $R_0 > 1$, there are two stationary points P_1 and P_2, the first occurs if $I(0) = 0$ in ∂D and the second if $I(0) > 0$ in the interior of D. However, when $R_0 \leq 1$ only P_1 exists. $P_1 = (\frac{\nu}{\mu}, 0, 0, 0)$ reflects the situation where there is no disease to spread and the endemic equilibrium is given by

$$P_2 = \begin{cases} S_2 = \dfrac{\delta\omega}{\beta_1\gamma + \beta_2\delta} = \dfrac{\nu}{\mu R_0}, \\ Q_2 = \dfrac{\nu\gamma}{\mu\delta^3\omega} - \dfrac{\gamma}{\beta_1\gamma + \beta_2\delta}, \\ I_2 = \dfrac{\nu}{\mu\delta^2\omega} - \dfrac{\delta}{\beta_1\gamma + \beta_2\delta} = \dfrac{\delta}{\gamma}Q_2, \\ R_2 = \dfrac{1}{\mu}(\rho_1 Q_2 + \rho_2 I_2) = \dfrac{Q_2}{\mu}\left(\rho_1 + \dfrac{\rho_2\delta}{\gamma}\right). \end{cases}$$

Defining convenient Lyapunov and anti-Lyapunov functions, it was proven that P_1 is globally asymptotic stable when $R_0 < 1$, it has a forward bifurcation in $R_0 = 1$ and is unstable when $R_0 > 1$. Also P_2 is globally asymptotic stable when $R_0 \geq 1$. Therefore it meets the expected qualitative behavior from the disease, i.e., when $R_0 < 1$ it eventually vanishes, no matter how many infected are in the population, and when $R_0 > 1$ it spreads until almost all the people had contact and are now immunized or dead.

The model was also investigated quantitatively using numerical simulations and sensitivity analysis was done on the parameters γ and β_2. The results for $\beta_1 = 0.01$, $\alpha_1 = 0.01$, $\alpha_2 = 0.02$, $\rho_1 = 0.99$, $\rho_2 = 0.98$, $\nu = 0.03$ and $\mu = 0.00005$ fixed are presented below in Figures 1 and 2.

Figure 1: Infected curves for different values of γ.

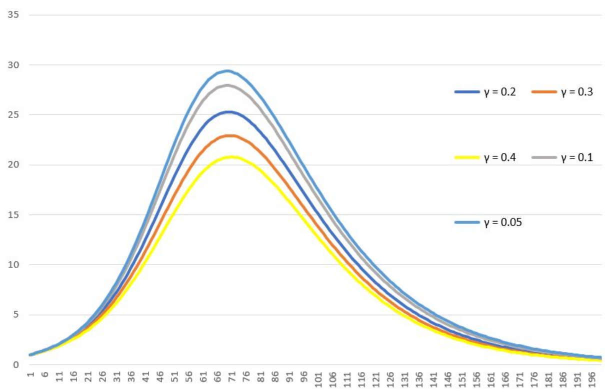
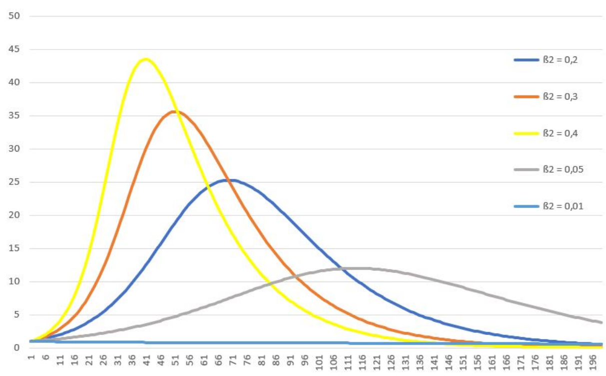


Figure 2: Infected curves for different values of β_2.



As observed, increasing the number of infected that enter quarantine there was a significant reduction in the number of cases, however the curve maintained its shape, and reducing the contagion rate the number of infected decreased and the curve was flattened, retarding the pinnacle of cases. Therefore, contagion reducing actions, such as washing hand, using masks, etc. are expected to be more effective than this type of isolation.

4. Conclusion

The new coronavirus (SARS-CoV-2) identified in December 2019 in Wuhan in China was a great motivator of recent work in epidemiology, and many of these have isolation as their object of study. Following this trend, a model was obtained with the population divided into susceptible, quarantined, infected and recovered, the isolation is done between infected individuals and is imperfect, that is, some infections come from the encounter between quarantined and susceptible people.

It has been analytically proven that the model follows the expected asymptotic behavior of a disease like COVID-19, which should stabilize and eventually disappear if there are no mutations. Although they have this tendency, it is known that this is a very damaging process for society, causing losses and deaths. It is then necessary to intervene through preventive measures to prevent the rapid increase in the number of infected individuals.

In practice, it is not always possible to implement several measures at the same time, so it is interesting to know which ones are more effective. In order to compare between increasing the isolation proposed by this SQIR model with actions that decrease the contagion rate, it was verified from the numerical simulations with random parameters, a decrease in the number of infected with both, but the flattening of the curve only happened in the second case, and therefore should be prioritized.

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Model categories and homotopy theories

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Abstract: Model categories are a category theoretic tool defined by Daniel Quillen with the aim of generalizing the homotopy theory built for topological spaces. The main goal of this text is to give an introduction to them, following an article by Dwyer and Spalinski. Just by its definition, it is almost inmediate that we can generalize analogues of well-known notions such as cylinder spaces, path spaces and homotopies. We use these tools to build a homotopy theory on a model category. Moreover, we will give some examples of different model structures over some categories, such as the expected category of topological spaces and the category of chain complexes of modules over a ring. Concerning the second one, we will also speak a little about spectral sequences and about the related model structure for filtered chain complexes.

Resumen: Las categorías de modelos son un concepto categórico teórico que fue definido por Daniel Quillen con el objetivo de generalizar la teoría de homotopía ya existente para espacios topológicos. Tan solo a partir de su definición, es casi inmediato que podemos generalizar nociones bien conocidas como son los espacios cilíndricos, los espacios de caminos y las homotopías. Utilizaremos estas herramientas para construir una teoría de homotopía en una categoría de modelos. Además, daremos algunos ejemplos de diferentes estructuras de modelos para diversas categorías, como es la esperada categoría de espacios topológicos o como la categoría de complejos de cadenas de módulos sobre un anillo. Con respecto a éste último, también hablaremos un poco sobre sucesiones espectrales y, en relación con ellas, sobre una estructura de modelos para complejos de cadenas filtrados.

Keywords: homotopy theories, homotopy categories, model categories fibrations, cofibrations, weak equivalences, spectral sequences, spectral systems.

MSC2010: 18N40.

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1. Introduction

Model categories were introduced by Daniel Quillen in [8], looking for a generalization of the classic homotopic tools that we knew for topological spaces. As he explains there, he did that in sight of some work by Dold and Kan where some sort of "homotopic methods" were used succesfully in the context of derived categories.

To generalize it, he defined what he called *model categories*. A model structure over a category is defined by distinguishing some classes of maps and imposing some axioms over them. These axioms, which resemble basic homotopy properties, turn out to be enough to define an equivalence relation for the maps of this category. This relation is called *homotopy relation*, and it is what will give us the homotopy theory (also called rational homotopy theory). From then on, several authors have proved different categories to fulfill the axioms for some classes of maps, and also have found different structures for a particular category.

2. The definition of model categories

The first step to define a model structure over a category $\mathcal{C}$ (following [2, Section 3]) is to distinguish three classes of maps, all of them closed under composition:

- *Weak equivalences*, of which we may think of as weak homotopy equivalences (maps that induce isomorphisms over all the homotopy groups).
- *Fibrations*, which can correspond to Serre fibrations and include covering maps.
- *Cofibrations*, which are dual to fibrations, and in the case of topological spaces can correspond to retracts of maps that obtain a space from another one by attaching cells.

Also, we ask them to fulfill the following axioms:

MC1 Finite limits and colimits exist in $\mathcal{C}$.

MC2 If f and g are maps in $\mathcal{C}$ such that its composition $g \circ f$, and two out of the three of them are weak equivalences, then so is the third one.

MC3 If f is a retract of g and g is a fibration, a cofibration or a weak equivalence, then so is f.

MC4 Let us consider the commutative diagram on the right. If i is a cofibration and p is a fibration and a weak equivalence (called acyclic fibration), or if i is an acyclic cofibration and p a fibration, then there exists a lift for the diagram (that is, a map $l\colon B \to X$ that commutes with the other arrows of the diagram).

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

MC5 Any morphism f can be factored (maybe with a functorial factorization) as $f = pi$, where i is a cofibration and p is an acyclic fibration, or where i is an acyclic cofibration and p is a fibration.

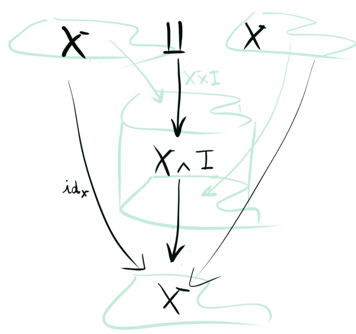
MC1 is purely technical, and is related to the existence of initial and terminal objects. MC2 tells us about the good behaviour of weak equivalences with respect to compositions. MC3 and MC4 ask our classes to behave well with respect to retracts (of maps), extensions and liftings. We notice that those two and MC2 resemble topological spaces, homotopy liftings and composition of weak homotopy equivalences.

To understand MC5, we have to introduce the so called *cofibrant* and *fibrant* objects. These are objects for which, respectively, the map from the initial object is a cofibration and the map to the terminal object is a fibration. In the case of topological spaces, all objects can be fibrants, whereas the cofibrant objects can be the retracts of cell-complexes. Using now MC5, and given an object X, we can factor those maps as

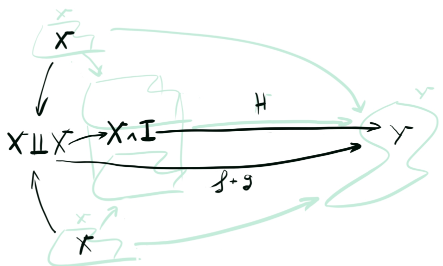
$$\begin{tikzcd} \varnothing \arrow[r, "i"] & X \arrow[r, dashed] & \varnothing \arrow[r] & F \arrow[r, "\sim"] & X \\ X \arrow[r, "p"] & * \arrow[r, dashed] & X \arrow[r, "\sim"] & C \arrow[r] & * \end{tikzcd}$$

In other words, MC5 means that we can find some sort of CW-approximation for any object in our category. These kinds of objects are important because they behave very well with respect to the homotopy relations.

Next, we define a *cylinder object* on a model category $\mathcal{C}$ to be an object $X \wedge I$ that factors the map $\mathrm{id}_X + \mathrm{id}_X : X \coprod X \to X$ in such a way that the map $X \wedge I \to X$ is a weak equivalence. Looking at the diagram on the right, we can see that this definition tries to capture the topological idea of cylinder, with inclusion of both bases into a topological cartesian product $X \times [0,1]$ together with the projection onto X. However, it throws away all the geometric or topological information, and keeps only the maps.



Now, if we take two maps, $f, g : X \to Y$, and a cylinder object for X, $X \wedge I$, then we define a *left homotopy* between f and g via $X \wedge I$ to be a map $H : X \wedge I \to Y$ that extends the sum $f + g : X \coprod X \to Y$. In that case, we say that f and g are homotopic, and we call this relation "left homotopy relation". This obviously reminds of the usual notion of homotopy, as illustrated by the diagram below.



We can also define dual notions of cylinders and left homotopies, which are called *path spaces* and *right homotopies*. The key point here is that not only fibrant and cofibrant objects give us the desired lifting properties for homotopies, but also make left and right homotopy relations equivalent.

Using MC5 as we did before, we take those CW-approximations and define with them a unique homotopy relation. Therefore, the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the category with the same objects of $\mathcal{C}$ and with morphisms the equivalence classes of maps, between the fibrant and cofibrant replacements, by the homotopy relation previously defined. Specifically, this means that we can work there "up to homotopy", and that we have a functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ that inverts all the maps that we distinguished as weak equivalences.

3. Examples

As we mentioned previously, one can easily find different examples of homotopy theories over different categories. We will comment the ones that are mentioned in [2] and that we studied in [6].

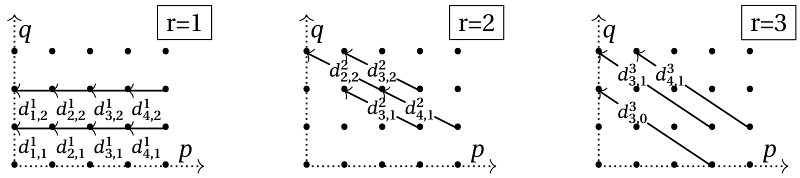
As we have been mentioning previously, *topological spaces* admit a model structure taking the class of weak equivalences to be weak homotopy equivalences, the class of fibrations to be Serre fibrations and the class of cofibrations to be the retracts of maps that attach cells on a given space.

However, this is not the only way to do this. If we look at the class of weak equivalences, we could ask ourselves if it is possible to build a model structure where the weak equivalences are the homotopy equivalences. Strom, in [9], answered this question by building such a model structure, taking Hurewicz fibrations and closed Hurewicz cofibrations. The difference between these structures lies in the fact that there are maps that are weak homotopy equivalences but not homotopy equivalences (see for example [6, Section 3.1]). However, they are the same for CW-complexes, as Whitehead's Theorem states.

There are several categories of *chain complexes* that admit a model structure, and several ways to define one over each of them ([3, Chapter 2]). In particular, we have the so-called projective model structure, which is built by taking as weak equivalences the maps that induce isomorphisms between the homology groups, as cofibrations the monomorphisms with projective kernel and as fibrations the epimorphisms. Also, it is worth mentioning that there are model structures that take as weak equivalences the usual chain homotopy equivalences (called Hurewicz model structure).

Filtered chain complexes. Spectral sequences

Spectral sequences are families $(E^r, d^r)_{r\geq 1}$ of bigraded modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ for each r (the number r is called page). The $d^r_{p,q}$ are maps of bidegree $(-r, r-1)$ that are called differentials (see [5] for more about them). We can obtain each page computing the homology of the previous one.



Given a filtered chain complex $(F_k C_*, d)_{p\in\mathbb{Z}}$, one defines its associated spectral sequence (which is a progressive approximation of homology groups by pages) by taking the quotient of the so called almost-cycles $(Z^r_{p,q})$ and almost-boundaries $(B^r_{p,q})$ as follows:

$$Z^r_{p,q} = \frac{A^r_{p,q} + F_{p-1}C_n}{F_{p-1}C_n}, \quad B^r_{p,q} = \frac{\mathrm{d}(A^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_n}{F_{p-1}C_n} \quad \text{and} \quad E^r_{p,q} := \frac{Z^r_{p,q}}{B^r_{p,q}},$$

where $n = p + q$, $A^r_{p,q} = \{c \in F_pC_n \mid \mathrm{d}(c) \in F_{p-r}C_{n-1}\}$, and the differentials are induced by the ones of the complex. Noticing that a map of filtered chain complexes induces a map of spectral sequences, and looking at the previous example of model structure, one could ask if we can take as weak equivalences the maps that induce a spectral sequences isomorphism from a certain page. The answer is positive, and it is given by Joana Cirici [1]. Moreover, there exists a generalization of spectral sequences for generalized filtered chain complexes, called *spectral systems*, and introduced in [4]. An open problem is to define a model structure for generalized filtered chain complexes by taking the class of weak equivalences to be the maps that induce isomorphisms between certain terms of the associated spectral system.

4. Conclusion

There are more examples that we could mention, such as the classic Kan complexes and the category of simplicial sets. However, there exist more "unexpected" examples, such as [7], concernig schemes. One can apply this in many different areas, and work with generalized homotopy notions that can be thought intuitively but that have also proved themselves useful. Consequently, its study is really encouraging.

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Additivities of the families of Darboux-like functions

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Abstract: It is well known that every continuous function from $\mathbb{R}$ to $\mathbb{R}$ maps connected sets to connected sets. However, the converse is not true in general, that is, the family of real functions that map connected sets to connected sets (known as Darboux functions) strictly contains the family of continuous functions. This method of considering necessary but not sufficient conditions for continuous functions lead us to obtain the families of functions known as Darboux-like functions. In this expository paper we will study the main results related to the inclusions and set operations between the classical families of Darboux-like functions, and also analyze the cardinal coefficient known as additivity of these families.

Resumen: Es bien conocido que toda función continua de $\mathbb{R}$ en $\mathbb{R}$ lleva conjuntos conexos en conjuntos conexos. Sin embargo, el recíproco no es cierto en general, es decir, la familia de funciones que llevan conjuntos conexos en conjuntos conexos (conocidas como funciones Darboux) contiene estrictamente a la familia de funciones continuas. Este método de considerar condiciones necesarias pero no suficientes para las funciones continuas nos lleva a obtener las familias de funciones conocidas como funciones de tipo Darboux. En este artículo expositivo estudiaremos los resultados principales relacionados con las inclusiones y operaciones de conjuntos entre las familias clásicas de funciones de tipo Darboux, y también analizaremos el coeficiente cardinal conocido como aditividad de estas familias.

Keywords: Additivity, Darboux-like functions, lineability.

MSC2010: 26A15, 54C08, 54A25, 15A03.

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1. Introduction and preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. We will denote for the rest of this paper the set of functions from $\mathbb{R}$ to $\mathbb{R}$ and the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ by $\mathbb{R}^{\mathbb{R}}$ and C, respectively.

Let us begin with some historical background. The Intermediate Value Theorem (in its classical formulation) is a well-known result on continuous functions proven by Bolzano in 1817. It states that if $f \in C$, then f maps intervals to intervals, that is, f maps connected sets to connected sets. We say that a function $f \in \mathbb{R}^{\mathbb{R}}$ satisfies the Intermediate Value Property (IVP) if f maps connected sets to connected sets. Around 1875, famous mathematician Darboux studied the IVP proving, for instance, that the derivative of every differentiable function in $\mathbb{R}^{\mathbb{R}}$ satisfies the IVP. However, not every function that maps connected sets to connected sets is continuous, as shown by the following classical example:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We say that $f \in \mathbb{R}^{\mathbb{R}}$ is *Darboux* (in honor of Jean-Gaston Darboux) if f maps connected sets to connected sets. We will denote by $\mathcal{D}$ the family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ that map connected sets to connected sets. Notice that $C \subsetneq \mathcal{D}$. This idea of considering necessary but not sufficient conditions for continuous functions has been thoroughly studied throughout the past century by many mathematicians, leading to what are known as *Darboux-like* functions (or generalized continuous functions).

In this paper we will study how the classical families of Darboux-like functions are related by inclusions and by set operations, showing that they form an algebra of sets, and also we will show the cardinal coefficient known as additivity of these families. The paper is arranged as follows. In Section 2 we will define all the classical families of Darboux-like functions as well as how they are related by inclusions and intersections. This section will also show that these families form an algebra of sets, providing the atoms that generate the algebra as well. In Section 3 we will define the concept of additivity of a family of real functions and its relation with the field of lineability. We will finish this section by providing the known additivities of (i) the classical families of Darboux-like functions, (ii) the complements of the classical families of Darboux-like functions and (iii) the atoms that form the algebra of sets.

To finish this section we will introduce standard notations and definitions from set theory that will be used for the rest of this paper. The symbol $|X|$ will denote the cardinality of the set X. If $f \in \mathbb{R}^{\mathbb{R}}$ and $X \subseteq \mathbb{R}$, then $f \restriction X$ denotes the restriction of f to X. The successor of a cardinal number λ and its cofinality will be denoted by λ^+ and $\operatorname{cof}(\lambda)$, respectively. We say that a cardinal number λ is regular if $\operatorname{cof}(\lambda) = \lambda$. Given a set X and a cardinal number λ, we denote by $[X]^{<\lambda}$ and $[X]^{\lambda}$ the sets of all subsets of X of cardinality less than λ and equal to λ, respectively. Let $\omega_1 = |\mathbb{N}|$, $\omega_2 = \omega_1^+$ and $\mathfrak{c} = |\mathbb{R}|$. We define also $\mathfrak{c}_-$ as $\varkappa$ when $\varkappa = \mathfrak{c}^+$ and as $\mathfrak{c}$ otherwise. Finally, let T be a theory and A an additional axiom. Then, A is consistent with T (or A is relatively consistent with T) if it can be proved that if T is consistent (does not entail contradiction), then $T + A$ is consistent.

2. Darboux-like functions and their relations

There are eight classical families of Darboux-like functions (one of them being $\mathcal{D}$). They are defined and denoted as follows:

PC - family of all *peripherally continuous* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that for every $x \in \mathbb{R}$, there exist two sequences $s_n \nearrow x$ and $t_n \searrow x$ with $\lim_{n\to\infty} f(s_n) = f(x) = \lim_{n\to\infty} f(t_n)$.

Conn - family of all *connectivity* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that the graph of f restricted to any connected $C \subseteq \mathbb{R}$ is a connected subset of $\mathbb{R}^2$.

AC - family of all *almost continuous* functions $f \in \mathbb{R}^{\mathbb{R}}$ (in the sense of Stallings), that is, such that every open subset of $\mathbb{R}^2$ containing the graph of f contains also the graph of function in C.

PR - family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ with *perfect road*, that is, such that for every $x \in \mathbb{R}$ there exists a perfect $P \subseteq \mathbb{R}$ having x as a bilateral limit point such that $f \restriction P$ is continuous at x.

CIVP - family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ with the *Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set C between p and q such that $f[C] \subseteq K$.

SCIVP - family of all functions $f \in \mathbb{R}^{\mathbb{R}}$ with the *Strong Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set C between p and q such that $f[C] \subseteq K$ and $f \restriction C$ is continuous.

Ext - family of all *extendable* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that there exists a connectivity function $g\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ with $f(x) = g(x, 0)$ for all $x \in \mathbb{R}$.

We will denote by $\mathbb{D}$ the set of all classical families of Darboux-like functions. The families in $\mathbb{D}$ are related in terms of sets by inclusions. Figure 1 shows all the strict inclusions of the families in $\mathbb{D}$. Moreover, the families in $\mathbb{D}$ still have the containment relations as in Figure 1 even when we consider the intersections between the families. We refer the interested reader to [2] and the references therein for the proofs of the containment relations.

$$
\begin{tikzcd}
 & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathcal{D} \arrow[dr] & \\
\mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[ur] &
\end{tikzcd}
$$

Figure 1: All strict inclusions, indicated by arrows, among the families in $\mathbb{D}$.

Therefore, the families in $\mathbb{D}$ form an algebra of sets, which will be denoted by $\mathcal{A}(\mathbb{D})$, generated by the following 17 sets: Ext, PC $\setminus$ (PR $\cup$ $\mathcal{D}$), PR $\setminus$ (CIVP $\cup$ $\mathcal{D}$), CIVP $\setminus$ (SCIVP $\cup$ $\mathcal{D}$), SCIVP $\setminus$ $\mathcal{D}$, $\mathcal{D}$ $\setminus$ (PR $\cup$ Conn), $\mathcal{D}\cap$PR$\setminus$(CIVP$\cup$Conn), $\mathcal{D}\cap$CIVP$\setminus$(SCIVP$\cup$Conn), $\mathcal{D}\cap$SCIVP$\setminus$Conn, Conn$\setminus$(PR$\cup$AC), Conn$\cap$PR$\setminus$(CIVP$\cup$AC), Conn$\cap$CIVP$\setminus$(SCIVP$\cup$AC), Conn$\cap$SCIVP$\setminus$AC, AC$\setminus$PR, AC$\cap$PR$\setminus$CIVP, AC$\cap$CIVP$\setminus$SCIVP and AC$\cap$SCIVP$\setminus$Ext.

3. Additivities of the algebra of Darboux-like functions

We begin this section by defining the additivity of a family of real functions.

Definition 1 (Additivity). Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$. The additivity of $\mathcal{F}$ is the following cardinal number:

$$\mathrm{A}(\mathcal{F}) = \min\left(\{|F| : F \subset \mathbb{R}^{\mathbb{R}},\ \forall g \in \mathbb{R}^{\mathbb{R}},\ g + F \nsubseteq \mathcal{F}\} \cup \{(2^{\mathfrak{c}})^{+}\}\right).$$ ◂

Although the additivity is interesting from the point of view of set theoretical real analysis, it can also be used in the field of lineability to find vector spaces of certain dimension. For more information about this field we refer the reader to [1].

Definition 2 (α-lineable). Let X be a vector space, A a subset of X and α a cardinal number. We say that A is α-lineable if $A \cup \{0\}$ contains a vector space of dimension α. ◂

Theorem 3 (Gámez, Muñoz and Seoane [3]). *Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be star-like, that is, $a\mathcal{F} \subseteq \mathcal{F}$ for all $a \in \mathbb{R}$. If $\mathfrak{c} < \mathrm{A}(\mathcal{F}) \leq 2^{\mathfrak{c}}$, then $\mathcal{F}$ is $\mathrm{A}(\mathcal{F})$-lineable.*

Now, one of the properties that additivity has is the following (see [2]): given $\mathcal{F}, \mathcal{G} \subseteq \mathbb{R}^{\mathbb{R}}$ with $\mathcal{F} \subseteq \mathcal{G}$, we have that $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}(\mathcal{G})$. Hence, notice that if we know the additivities of the atoms of $\mathcal{A}(\mathbb{D})$, then we have the lower bounds of all the additivities of the families in $\mathcal{A}(\mathbb{D})$. We proceed to show the main results of this expository paper. We first show the additivities of the families in $\mathbb{D}$ and their complements (see [2] and the references therein). Let us define the following cardinal numbers:

$$e_\mathfrak{c} = \min\{|F| : F \subset \mathbb{R}^\mathbb{R}, \forall g \in \mathbb{R}^\mathbb{R}, \exists f \in F \text{ such that } |f \cap g| < \mathfrak{c}\},$$
$$d_\mathfrak{c} = \min\{|F| : F \subset \mathbb{R}^\mathbb{R}, \forall g \in \mathbb{R}^\mathbb{R}, \exists f \in F \text{ such that } |f \cap g| = \mathfrak{c}\}, \text{ and}$$
$$d_\mathfrak{c}^* = \min\{|F| : F \subset \mathbb{R}^\mathbb{R}, \forall G \in [\mathbb{R}^\mathbb{R}]^\mathfrak{c}, \exists f \in F \text{ such that } \forall g \in G,\ |f \cap g| = \mathfrak{c}\}.$$

Theorem 4 (Ciesielski, Miller, Gámez, Muñoz, Seoane, Mazza, Recław, Jordan, Natkaniec - 1994/95, 1996/97, 2010, 2017). *We have the following results:*

(a) $\mathfrak{c}^+ \leq \mathrm{A(AC)} = \mathrm{A(Conn)} = \mathrm{A}(\mathcal{D}) = e_\mathfrak{c} \leq 2^\mathfrak{c}$, *and this is all that can be proved in ZFC.*

(b) $\mathrm{A(Ext)} = \mathrm{A(PR)} = \mathfrak{c}^+$.

(c) $\mathrm{A(PC)} = 2^\mathfrak{c}$.

(d) $\mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathrm{PC}) = \omega_1$.

(e) $\mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathrm{Ext}) = \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathrm{PR}) = 2^\mathfrak{c}$.

(f) $d_\mathfrak{c} \leq \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathcal{D}) \leq \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathrm{Conn}) \leq \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathrm{AC}) \leq d_\mathfrak{c}^*$.
If $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$, *then* $\mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathcal{D}) = \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathrm{Conn}) = \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathrm{AC}) = d_\mathfrak{c} = d_\mathfrak{c}^*$.
If $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$ *and* $\mathfrak{c} = \lambda^+$, *then* $d_\mathfrak{c} \leq e_\mathfrak{c}$.
Moreover, $\mathfrak{c}^+ \leq d_\mathfrak{c} \leq 2^\mathfrak{c}$ *and*

(f$_1$) *For every cardinals* $\lambda \geq \kappa \geq \omega_2$ *such that* $\mathrm{cof}(\lambda) > \omega_1$ *and* κ *is regular, it is relatively consistent with ZFC+CH that* $2^\mathfrak{c} = \lambda$ *and* $d_\mathfrak{c} = e_\mathfrak{c} = \kappa$. *In particular,* $\mathfrak{c}^+ < d_\mathfrak{c} = \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathcal{D}) = \mathrm{A}(\mathcal{D}) = e_\mathfrak{c} < 2^\mathfrak{c}$ *is consistent with ZFC+CH.*

(f$_2$) *For every cardinal* $\lambda > \omega_2$ *such that* $\mathrm{cof}(\lambda) > \omega_1$, *it is relatively consistent with ZFC+CH that* $\mathfrak{c}^+ = \omega_2 = \mathrm{A}(\mathbb{R}^\mathbb{R} \setminus \mathcal{D}) = d_\mathfrak{c} < e_\mathfrak{c} = \mathrm{A}(\mathcal{D}) = 2^\mathfrak{c} = \lambda$.

Finally we present the known additivities of the atoms of $\mathcal{A}(\mathbb{D})$ (for more details, see [2]).

Theorem 5 (Ciesielski, Natkaniec, Rodríguez, Seoane [2]). *We have the following results:*

(a) $d_\mathfrak{c} \leq \mathrm{A}(\mathrm{PC} \setminus (\mathrm{PR} \cup \mathcal{D})) \leq d_\mathfrak{c}^*$.

(b) $\mathrm{A}(\mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathcal{D})) = \mathrm{A}(\mathrm{CIVP} \setminus (\mathrm{SCIVP} \cup \mathcal{D})) = \mathrm{A}(\mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}) = \mathrm{A}(\mathrm{AC} \cap \mathrm{CIVP} \setminus \mathrm{SCIVP}) = \mathfrak{c}^+$.

(c) $\mathrm{A}(\mathrm{SCIVP} \setminus \mathcal{D}) = \mathrm{A}(\mathcal{D} \cap \mathrm{SCIVP} \setminus \mathrm{Conn}) = \mathrm{A}(\mathrm{Conn} \cap \mathrm{SCIVP} \setminus \mathrm{AC}) = 2$.

(d) $\mathrm{A}(\mathrm{AC} \setminus \mathrm{PR}) = e_\mathfrak{c}$.

(e) $\omega_1 \leq \mathrm{A}(\mathrm{Conn} \setminus (\mathrm{PR} \cup \mathrm{AC})),\ \mathrm{A}(\mathrm{Conn} \cap \mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathrm{AC})),\ \mathrm{A}(\mathrm{Conn} \cap \mathrm{CIVP} \setminus (\mathrm{SCIVP} \cup \mathrm{AC})) \leq \mathfrak{c}$.

(f) $\mathfrak{c}_- \leq \mathrm{A}(\mathcal{D} \setminus (\mathrm{PR} \cup \mathrm{Conn})),\ \mathrm{A}(\mathcal{D} \cap \mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathrm{Conn})),\ \mathrm{A}(\mathcal{D} \cap \mathrm{CIVP} \setminus (\mathrm{SCIVP} \cup \mathrm{Conn})) \leq \mathfrak{c}$.

(g) $2 \leq \mathrm{A}(\mathrm{AC} \cap \mathrm{SCIVP} \setminus \mathrm{Ext}) \leq \mathfrak{c}$.

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A graph equation between the line graph and the edge-complement graph

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Abstract: From a graph G related graphs can be constructed, such as its line graph $L(G)$ and its edge-complement graph $\overline{G}$. After showing how properties of G imply properties of $L(G)$, we ask how different the concepts of the line graph $L(G)$ and that of the edge-complement graph $\overline{G}$ are, by solving the equation $L(G) \simeq \overline{G}$. We show that the equation has only two solutions. The proof uses an argument on the degree of the vertices of a graph that allows to reduce the number of possible solutions until they can be checked algorithmically. This gives an alternative proof to the one by Aigner [1].

Resumen: A partir de un grafo G se pueden construir grafos relacionados, como su grafo de líneas $L(G)$ y su grafo complemento de aristas $\overline{G}$. Después de mostrar cómo las propiedades de G implican propiedades de $L(G)$, nos preguntamos cuán diferentes son los conceptos del grafo lineal $L(G)$ y el del grafo complemento de aristas $\overline{G}$, resolviendo la ecuación $L(G) \simeq \overline{G}$. Demostramos que la ecuación tiene solo dos soluciones. La prueba utiliza un argumento sobre el grado de los vértices de un grafo que permite reducir el número de posibles soluciones hasta poder comprobarlas algorítmicamente. Esto da una prueba alternativa a la de Aigner [1].

Keywords: line graph, edge-complement graph, graph equation.

MSC2010: 05C76.

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1. Introduction

Line graphs as well as edge-complement graphs allow to restate graph questions in sometimes easier versions. In the following section, some relations between the properties of a graph and the respective properties of its line graph are shown. The last section studies the graph equation $L(G) \simeq \overline{G}$ in order to compare the line graph with the edge-complement graph. We find that there are exactly two graphs whose line graphs and edge-complement graphs coincide. This result was first shown by Aigner [1], whose argument uses the existence of a unique cycle in a possible solution. Here, we present an alternative proof that is based on the degree of the vertices of a solution G. The possible degrees of vertices restrict the number of vertices of a graph that is a solution to $L(G) \simeq \overline{G}$. The finite number of remaining cases are checked in an algorithmic way, resulting in exactly two graphs whose line graphs and edge-complement graphs are isomorphic.

2. Line graph: definition and properties

Definition 1. Let G be a graph. The *line graph* $L(G)$ of G is the graph with vertex set $V(L(G)) = E(G)$ and two vertices $u, v \in V(L(G))$ are connected by an edge in $L(G)$ if and only if their corresponding edges share a common vertex in G. ◂

Example 2. A graph G (left) and its line graph $L(G)$ (right) are shown in Figure 1. Edges of G and their corresponding vertices in $L(G)$ are shown in the same colour. ◂

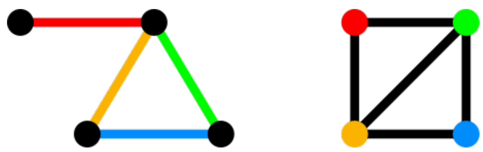

Figure 1: A graph G and its line graph $L(G)$.

The following proposition is an immediate consequence of Definition 1. It relates the number of edges $|E(L(G))|$ and vertices $|V(L(G))|$ in $L(G)$ to the number of edges $|E(G)|$ and vertices $|V(G)|$ in G.

Proposition 3. *Let G be a graph. The* degree *of a vertex is the number of edges attached to that vertex. It holds that $|V(L(G))| = |E(G)|$ and $|E(L(G))| = \frac{1}{2}\sum_{v\in V(G)} \deg v^2 - |E(G)|$.*

Definition 4. A property $\mathcal{P}$ is *preserved under the line graph operation* if it follows from the graph G having property $\mathcal{P}$ that its line graph $L(G)$ also has property $\mathcal{P}$. ◂

The following proposition shows that several properties of graphs are preserved under the line graph operation. We refer to the first chapter of the book [2] for the definitions.

Proposition 5. *Let G be a graph. The following implications are true:*

(i) *If G is connected, then $L(G)$ is connected.*

(ii) *If G is a k-regular graph, then $L(G)$ is a $2(k-1)$-regular graph.*

(iii) *Assume that G and H are two simple graphs. If H is a graph quotient of G via the action of a group $\mathcal{A}$, then $L(H)$ is a graph quotient of $L(G)$ via the action of the same group $\mathcal{A}$.*

Proof. The proofs of the first two statements are direct consequences of Definition 1. For the third statement, note that, by Definition 1, the vertices of $L(G)$ are the edges of G. From this and the assumption that the graph G is simple, it follows that the group $\mathcal{A}$ acts freely on $L(G)$. On the other hand, the assumption that H is a simple graph implies that the action of $\mathcal{A}$ on G and on $L(G)$ is essentially the same. Therefore, a graph morphism is defined between $L(H)$ and $L(G)/\mathcal{A}$. It is straightforward to prove that the morphism is indeed a graph isomorphism. ■

3. The graph equation $L(G) \simeq \overline{G}$

In this section we compare the line graph with the edge-complement of a graph. We find that, except for two graphs, the line graph is different from the edge-complement.

Definition 6. Let G be a simple graph. The *edge-complement* graph $\overline{G}$ of G is the graph that has the same vertex set as G and two vertices $u, v \in V(\overline{G})$ are connected by an edge in $\overline{G}$ if and only if they are not connected by an edge in G. ◂

Example 7. Figure 2 shows an example of a graph G (left) and its edge-complement $\overline{G}$ (right). ◂

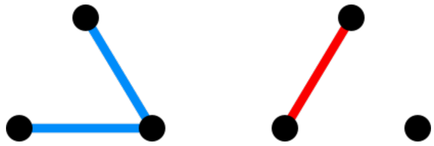

Figure 2: A graph G and its edge-complement $\overline{G}$.

A result similar to Proposition 3 is the following, whose proof follows directly from Definition 6.

Proposition 8. *Let G be a simple graph such that $|V(G)| = n$. If G is not connected, then $\overline{G}$ is connected. Moreover, $|E(\overline{G})| = \binom{n}{2} - |E(G)|$.*

In order to study the relations that exist between the line graph and the edge-complement operations, we focus our attention on the following question: do there exist graphs G with non-empty vertex set which satisfy the equation

$$L(G) \simeq \overline{G}\,? \tag{1}$$

The set of solutions for (1) is not empty, since it is easily found that $G = C_5$, which is the cycle with 5 vertices, is isomorphic to both its line graph and its edge-complement (see Figure 3). In fact, $G = C_5$ is the only regular graph that is a solution to (1).

Figure 3: The graph C_5 fulfills $L(C_5) \simeq C_5 \simeq \overline{C_5}$.

Theorem 9. *The only solutions to the graph equation $L(G) \simeq \overline{G}$ are $G = C_5$ and the graph with six vertices that is drawn left in Figure 4.*

Proof. It follows from the properties of propositions 3, 5 and 8 that a candidate G for a solution to (1) must be connected and must have as many vertices as edges, say $|V(G)| = |E(G)| = n$. That is, if the number of vertices of G grows, the edge-complement graph $\overline{G}$ will have a high number of edges, while the line graph $L(G)$ will not. Thus, focusing on the degrees of vertices of G allows to limit the maximum number of vertices and edges that a G that satisfies (1) is allowed to have.

Indeed, it follows from Definition 6 that G cannot have vertices of degree $n - 1$. If we assume G to be k-regular, from propositions 3 and 8 we obtain that $k^2 n = n(n-1)$ and $nk = 2n$. These two equations are satisfied only if $k = 2$ and $n = 5$. Therefore, the only regular graph which is solution to (1) is C_5.

Thus, we can assume that G is not regular. This assumption implies that G must contain at least one vertex of degree 1. Indeed, if it was not the case, then G would have all vertices of degree at least 2 and at least one vertex of degree at least 3 (because G cannot be regular). This is a contradiction to the handshake lemma (see [2, Theorem 1.1.1]).

The existence of at least one vertex of degree 1 in G implies that $L(G)$ must have at least one vertex of degree $n-2$. For $L(G)$ to contain a vertex of degree $n-2$, there must exist an edge in G with endpoints u and w such that $\deg u + \deg w = n$. However, G has only n edges and vertices, and each vertex is at least of degree 1 since G is connected. Therefore, we are left with two cases: either G has $n-3$ vertices of degree 1 and one vertex of degree 3 in addition to the vertices u and w, or G has $n-4$ vertices of degree 1 and two vertices of degree 2 in addition to u and w. In both cases, since G cannot have vertices of degree $n-1$, it is impossible for G to have more than 7 vertices. This leaves us with a finite number of graphs that are potential solutions and they can be checked individually as shown in the next section for graphs with 6 vertices. It is found that the graph that is drawn left in Figure 4 is the only non-regular solution to equation (1). ■

3.1. An algorithm for the remaining cases

Graphs with n vertices that are solutions to equation (1) can be found algorithmically as outlined below.

For $L(G) \simeq \overline{G}$ to hold, the number of edges of the graphs must be equal. Propositions 3 and 8 obtain $|E(L(G))|$ and $|E(\overline{G})|$ from $|E(G)|$. Equating these two expressions, one obtains that G must satisfy the following equation:

$$\sum_{v \in V(G)} \binom{\deg v}{2} = \frac{n^2 - 3n}{2}. \tag{2}$$

From Definition 1, it follows that a vertex of degree d in G corresponds to $\binom{d}{2}$ edges in the line graph $L(G)$. This observation together with formula (2) allows to list all combinations of degrees of vertices that respect (2) for a fixed n. It is then easy to check whether the resulting graphs are solutions to equation (1).

Example 10. This example shows how the algorithm works for $n = 6$. This case will give the only other solution besides $G = C_5$ to the graph equation (1). First, we determine combinatorially the fourteen degree combinations of 6 vertices that satisfy (2), i.e., $\sum_{i=1}^{6} \frac{(\deg v_i)^2 - \deg v_i}{2} = 9$. Among these, it is possible to remove immediately all combinations which contain a zero, as G must be a connected graph (as argued in the proof of Theorem 9). Hence, only four possible combinations of degrees are left:

$$(1): \{4,3,1,1,1,1\} \quad (2): \{4,2,2,2,1,1\} \quad (3): \{3,3,3,1,1,1\} \quad (4): \{3,3,2,2,2,1\}.$$

There is no graph with vertices of the degrees of (1) or (4), because these would require an odd number of vertices of odd degree. From the remaining two cases, only the graph with degrees of (3) is a solution to equation (1). The graph is shown in Figure 4. ◂

Figure 4: The graph G on the left, its edge-complement $\overline{G}$ in the middle, its line graph $L(G)$ on the right. This is the only graph on six vertices with $L(G) \simeq \overline{G}$.

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Recent progress on the vortex filament equation for regular polygons

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Abstract: Due to its simplicity and geometric structure, the vortex filament equation (VFE) secures a unique place in fluid literature. The equation is a model for the dynamics of a vortex filament (e.g., smoke rings, tornadoes, etc.) in a three-dimensional inviscid incompressible fluid. In this work, we describe recent progress on its behaviour for the polygonal-shaped filaments curves. More precisely, we concentrate on the evolution of VFE for regular polygons as the initial data. Besides problem formulation, addressing it using theoretical and numerical techniques, we discuss the time evolution of a single point located on the curve which, in turn, follows a multifractal trajectory. Simultaneously, we also consider the corresponding problem in the hyperbolic 3-space.

Resumen: Debido a su simplicidad y estructura geométrica, la *vortex filament equation* (VFE) ocupa un lugar único en la literatura de fluidos. La ecuación modela la dinámica de un filamento de vórtice (p. ej., anillos de humo, tornados, etc.) en un fluido incompresible no viscoso tridimensional. En este trabajo, describimos los avances recientes en su comportamiento para las curvas de filamentos de forma poligonal. Más precisamente, nos concentramos en la evolución de la VFE para polígonos regulares como datos iniciales. Además de la formulación del problema, abordando el mismo mediante técnicas teóricas y numéricas, se comenta la evolución temporal de un único punto ubicado en la curva que, a su vez, sigue una trayectoria multifractal. Simultáneamente, también consideramos el problema correspondiente en el 3-espacio hiperbólico.

Keywords: vortex filament equation, Schrödinger map equation, numerical methods for PDEs, multifractality, Talbot effect.

MSC2010: 35Q55, 35Q56, 35Q35.

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1. Introduction

The vortex filament equation (VFE) is a simplified model that describes the dynamics of an ideal fluid whose vorticity is concentrated on a curve called vortex filament, i.e., smoke rings, tornadoes, etc. Given by Da Rios in his PhD thesis in 1906, for an arc-length parametrized curve $\mathbf{X}$ representing a vortex filament in three-dimensions, the VFE is expressed as [8]

$$\mathbf{X}_t = \mathbf{X}_s \wedge \mathbf{X}_{ss}, \qquad s \in \mathbb{R}, t \in \mathbb{R}, \tag{1}$$

where $\wedge$ is the usual cross-product, s arc-length and t time parameter, and subscripts denote the partial derivatives. The tangent vector $\mathbf{T} = \mathbf{X}_s$ solves the so-called Schrödinger map equation onto the sphere

$$\mathbf{T}_t = \mathbf{T} \wedge \mathbf{T}_{ss}. \tag{2}$$

Due to its geometrical structure and properties, in the simplest form, (2) allows $\mathbf{T}$ to take its value on the Euclidean unit sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$, or, a hyperbolic one, i.e., $\mathbb{H}^2 = \{(x_1, x_2, x_3) : -x_1^2 + x_2^2 + x_3^2 = -1, x_1 > 0\}$. Note that, when $\mathbf{T} \in \mathbb{H}^2$, $\mathbf{X}$ lies in the Minkowski 3-space $\mathbb{R}^{1,2} = \{(x_1, x_2, x_3) : \mathrm{d}s^2 = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3\}$, and the cross product in (1)–(2) is given by $\mathbf{a} \wedge \mathbf{b} = (-(a_2b_3 - a_3b_2), a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$. Moreover, with the curvature $\varkappa$, torsion τ, the tangent $\mathbf{T}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ vectors of $\mathbf{X}$ form an orthonormal system and solve the Frenet–Serret formulas

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_s = \begin{pmatrix} 0 & \varkappa & 0 \\ \mp\varkappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \tag{3}$$

where the minus sign refers to the Euclidean and the plus sign to the hyperbolic cases. With this, in 1972, Hasimoto developed a relationship between (1)–(2) and the cubic nonlinear Schrödinger (NLS) equation where the unknown is the wave function $\psi(s,t) = \varkappa(s,t)\mathrm{e}^{\int_0^s \tau(s',t)\,\mathrm{d}s'}$. Thanks to this connection, any advancement in the direction of (1)–(2) is equivalent to that for the NLS equation as well.

Apart from the explicit solutions of VFE, i.e., circle, straight line and helix, another important class is the one-parameter family of the self-similar solutions which are characterized by a parameter $c_0 > 0$. In both Euclidean and hyperbolic cases, for a given time $t > 0$, the curve $\mathbf{X}$ has a curvature $c_0/\sqrt{t}$ and a torsion $s/2t$ and it has been shown that, as the time t tends to zero, it develops a corner and turns into two non-parallel straight lines meeting at $s = 0$. This implies that, at $t = 0$, the corresponding tangent vector is a Heaviside-type function and the initial solution of the NLS equation is a Dirac delta located at $s = 0$. The so-called *one-corner problem* has been well studied by Gutierrez et. al. and Banica et. al. theoretically, and by Buttke and de la Hoz numerically in their PhD theses.

2. Some polygonal solutions of (1)–(2)

Motivating from curves with one corner otherwise smooth, it is natural to address the evolution of (1)–(2) for curves with several corners. In this direction, we consider the simplest case of regular planar polygons in both Euclidean and hyperbolic spaces followed by their extension to respective non-planar ones.

2.1. Regular planar polygons

The evolution of (1)–(2) for $\mathbf{X}(s,0)$ as a regular planar polygon is equivalent to that of the NLS equation with initial datum $\psi(s,0) = c_0 \sum_{k\in\mathbb{Z}} \delta(s - k\Delta s)$, where Δs is the side-length of the initial polygonal curve which is equal to $2\pi/M$ for an M-sided polygon in the Euclidean space and $l > 0$ for a hyperbolic polygon and c_0 depends on the initial configuration of the curve [3, 5]. By assuming uniqueness and using the Galilean invariance of the NLS equation, followed by algebraic calculations, the time evolution of $\mathbf{X}$ and $\mathbf{T}$ can be described up to a rigid movement for the rational multiples of the time-period. The numerical experiments confirm that depending on the (denominator of) rationals, the polygonal curve develops more number of sides, a behaviour reminiscent of the Talbot effect in optics. For the numerical computations, due to the

2π spatial periodicity of $\mathbf{T}$, a pseudo-spectral discretization is used in the Euclidean case; however, for the hyperbolic case, a finite difference scheme with Dirichlet boundary conditions on $\mathbf{T}$ is employed; a fourth-order Runge–Kutta method is used for the time evolution in both cases.

Furthermore, the time evolution of a single point, i.e., $\mathbf{X}(0,t)$ lies in a plane. This is displayed in Figure 1, for an equilateral triangle which also shows that with a vertical translation at half time-period $t = \pi/M^2$, the triangle appears upside down and reappears at the end of the time-period $t = 2\pi/M^2$. The latter is recorded as the axis-switching phenomenon in fluid literature, for example, non-circular jets (for a qualitative comparison see evolutions of an `equilateral triangle`, and a `vortex filament`). The right-hand side of each subfigure in Figure 1 shows the projection of $\mathbf{X}(0,t)$ onto $\mathbb{C}$ and the same after removing the vertical height, denoted by $z_M(t)$ (or, $z_l(t)$ in the hyperbolic case). Nonetheless, as M becomes larger (or l smaller), $z_M(t)$ (or $z_l(t)$), converges to the so-called Riemann's non-differentiable, given by the real part of $\phi(t) = \sum_{k=1}^{\infty} \frac{e^{i\pi k^2 t}}{i\pi k^2}$, $t \in [0,2]$, see Figure 2. Due to its multifractal character, the function ϕ is an important object whose properties were studied by Jaffard in [6], and recently by Eceizabarrena in [2].

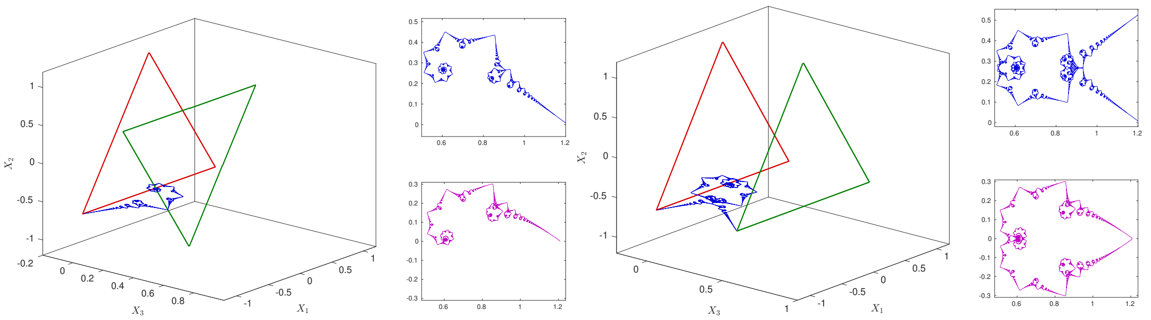

Figure 1: For $M = 3$, $\mathbf{X}(s,0)$ (red), $\mathbf{X}(s,t)$ (green), $\mathbf{X}(0,t_0)$, $t_0 \in [0,t]$ (blue) at $t = \frac{\pi}{M^2}$ (left), $\frac{2\pi}{M^2}$ (right).

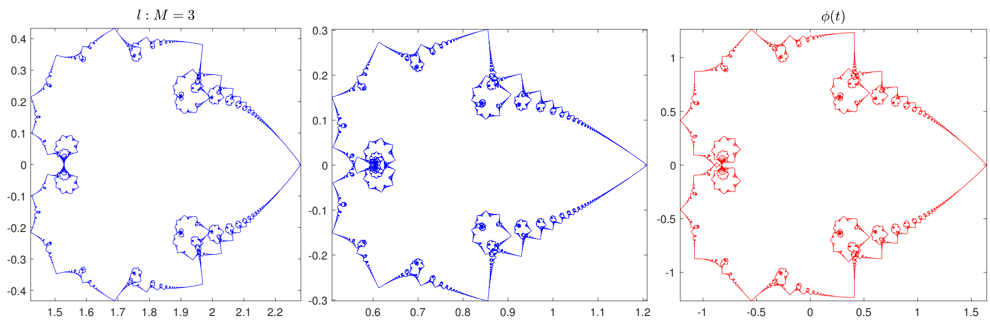

Figure 2: $z_l(t)$, $z_M(t)$, for $M = 3$ (see [3, 5] for their precise definition), and $\phi(t)$.

2.2. Regular polygons with a nonzero torsion

For the arc-length parameterized $\mathbf{X}$, the nonzero torsion can be introduced with the parameter b. In the Euclidean case, $b \in [-1,1]$ corresponds to the third component of the tangent vector, whereas $b \in (-\infty,-1] \cup [1,\infty)$ as the first component of $\mathbf{T} \in \mathbb{H}^2$, and $b \in (-\infty,\infty)$ as the third component result in the *circular helix* and *hyperbolic helix*, respectively [4, 7]. Note that $b = 0$ reduces back to the planar case discussed above. In both settings, together with the parameters M or l (giving side-length), b determines the curvature angle ρ_0 and torsion angle θ_0. Moreover, by denoting ψ as ψ_θ when $\theta_0 > 0$, we have

$$\psi_\theta(s,0) = c_{\theta,0} e^{i\gamma s} \sum_{k\in\mathbb{Z}} \delta(s - k\Delta s) = \frac{c_{\theta,0}}{c_0} e^{i\gamma s} \psi(s,0), \tag{4}$$

with $\gamma = \theta_0/\Delta s$, and $c_{\theta,0} > 0$, Δs, suitably chosen as mentioned above. Besides the algebraic solution, with the numerical simulations, we detect the aperiodic movement of a corner initially at $s = 0$ (e.g., see the link) and categorize it as *Galilean shift* and *phase shift* which also implies that $\mathbf{X}(0,t)$ is non-planar. With a Fourier analysis of $\mathbf{X}(0,t)$ at a numerical level, different variants of $\phi(t)$ have been found whose structure, in turn, depends on the initial torsion. More precisely, for $\theta_0 = \pi c/d$, $c, d \in \mathbb{N}$, $\gcd(c,d) = 1$, the vertical movement of $\mathbf{X}(0,t)$ can be compared with the imaginary part of $\phi_{c,d}(t) = \sum_{k \in A_{c,d}} \frac{e^{2\pi i k t}}{k}$, to which it converges as M tends to infinity or l to zero (i.e., $\mathbf{X}(s,0)$ to a smooth helix), with

$$t \in \begin{cases} [0,1/2] & \text{if } c \cdot d \text{ odd}, \\ [0,1] & \text{if } c \cdot d \text{ even}, \end{cases} \quad A_{c,d} = \begin{cases} \{n(nd+c)/2 \mid n \in \mathbb{Z}\} \cap \mathbb{N} & \text{if } c \cdot d \text{ odd}, \\ \{n(nd+c) \mid n \in \mathbb{Z}\} \cap \mathbb{N} & \text{if } c \cdot d \text{ even}. \end{cases} \tag{5}$$

Similarly, strong numerical evidence is given that, for a given M and as b approaches 1 (i.e., $\mathbf{X}(s,0)$ to a straight line), the stereographic projection of $\mathbf{X}(0,t)$ onto $\mathbb{C}$ tends to $\phi_M(t) = \sum_{k \in A_M} \frac{e^{2\pi i k^2 t}}{k^2}$, $t \in [0,1]$, where $A_M = \{1\} \cup \{nM \pm 1 \mid n \in \mathbb{N}\}$. Remark that, for $\theta_0 \neq 0$, $\psi_\theta(s,0)$ is quasi-periodic and becomes 2π-periodic when $b \to 1$. Thus, through a very formal computation if, instead of the NLS equation, one solves the initial value problem for the free Schrödinger equation $\psi_t = i\psi_{ss}$, for (4) with $b \approx 1$, then

$$\hat{\psi}_\theta(k,t) = e^{-ik^2 t}\hat{\psi}_\theta(k,0), \text{ with } \hat{\psi}_\theta(k,0) = \begin{cases} \frac{M}{2\pi} & \text{if } k \pm 1 = nM,\ n \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

Then, bearing in mind the Hasimoto transformation and (1)–(3), $\mathbf{X}(0,t)$ can be related to $\int_0^t \psi_\theta(s,\tilde{t})\,d\tilde{t}$, with $s = 0$, which computed using $\hat{\psi}_\theta(k,t)$ is ϕ_M up to a scaling factor. Nonetheless, the existence of ϕ and its variants in the evolution of $\mathbf{X}(0,t)$ has been proved rigorously by Banica and Vega recently in [1].

3. Conclusion

Thus, the appearance of Riemann's function (and its variants) in the evolution of polygonal curves indicates that the evolution of (1)–(2) for smooth curves is not stable. That is, as the number of sides M tends to infinity (or l to zero), the polygonal curve approaches a smooth curve; however, when measured in the right topology, the trajectory of a single particle located on it converges to a multifractal, unlike that of a smooth curve (to compare, see the link). Recall that Riemann's function satisfies the multifractal formalism proposed by Frisch and Parisi [6]. Therefore, these latest results also contribute to the debate, which is already more than a one-hundred-year-old, on the validity of the vortex filament equation as a simplified model for understanding fundamental but complex natural phenomena such as turbulence.

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